Research Article

Nonuniform Polynomial Dichotomy for Noninvertible Linear Discrete-Time Systems in Banach Spaces

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We study three polynomial dichotomy concepts for linear discrete-time systems in Banach spaces. Our main objective is to give characterizations in terms of Lyapunov functions for nonuniform polynomial dichotomy of nonautonomous and noninvertible linear discrete-time systems.

1. Introduction

In recent years, an impressive progress has been made in the field of the asymptotic behaviors of solutions of evolution equations in finite- and infinite-dimensional spaces (see [1–8] and the references therein). Besides stability and instability, a special attention was devoted to the study of dichotomy of evolution equations. Since the existence problem of dichotomy of evolution equations is distinct compared to the studies devoted to stability and, respectively, to instability, dichotomy is a powerful tool when people analyze the asymptotic behavior of dynamical systems. The notion of (uniform) exponential dichotomy firstly introduced by Perron in [9] plays a central role in dynamics, particularly in the study of stable and unstable invariant manifolds, both for discrete and continuous time. In particular, there exist large classes of linear differential equations possessing exponential dichotomies. We refer to the books [10–12] for details and further references.

On the other hand, the notion of exponential dichotomy is too stringent for the dynamics and it is of considerable interest to look for more general types of dichotomic behaviors. The main reason is that from the point of view of ergodic theory almost all variational equations in a finite-dimensional space admit a nonuniform exponential dichotomy. Recently, a notion of nonuniform polynomial dichotomy was introduced independently by Barreira and Valls in [13] and Bento and Silva in [14] in somewhat distinct

forms, respectively, in the case of continuous and discrete-time systems. Rămneanu et al. had offered some integral properties for nonuniform polynomial dichotomy in [15]. In this case the rates of contraction and expansion vary polynomially.

In the spirit of the recent work of Popa et al. [16], this paper considers the general notion of nonuniform polynomial dichotomy for nonautonomous linear discrete-time systems in Banach spaces. The main objective is to give characterizations of nonuniform polynomial dichotomy in terms of Lyapunov functions for the general case of noninvertible linear discrete-time systems. Some simple examples are included to illustrate the connections between the dichotomy concepts considered in the present paper.

2. Notations and Preliminaries

Let X be a real or complex Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X . The norm on X and on $\mathcal{B}(X)$ will be denoted by $\|\cdot\|$. The identity operator on X is denoted by I . We denote $\Delta = \{(m, n) \in \mathbb{N}^2, m \geq n\}$ and $T = \{(m, r, n) \in \mathbb{N}^3, m \geq r \geq n\}$. If $T \in \mathcal{B}(X)$, then we will denote by $\text{Ker } T$ the kernel of T and by $\text{Im } T$ the range of T ; that is,

$$\text{Ker } T = \{x \in X : Tx = 0\}, \quad (1)$$

$$\text{Im } T = \{Tx : x \in X\}, \quad (2)$$

respectively.

In the present paper we consider the linear discrete-time system of difference equations:

$$x_{n+1} = A_n x_n, \quad n \in \mathbb{N}, \quad (3)$$

where $A : \mathbb{N} \rightarrow \mathcal{B}(X)$ is a sequence in $\mathcal{B}(X)$. If for every $n \in \mathbb{N}$ the operator (A_n) is invertible, then the linear discrete-time system (3) is called reversible. Then every solution $x = (x_n)$ of system (3) is given by

$$x_m = \mathcal{A}(m, n) x_n \quad (4)$$

for all $(m, n) \in \Delta$, where the mapping $\mathcal{A} : \Delta \rightarrow \mathcal{B}(X)$ is defined by

$$\mathcal{A}(m, n) := \begin{cases} I, & m = n, \\ A_{m-1} \cdots A_n, & m > n. \end{cases} \quad (5)$$

It is easy to see that $\mathcal{A}(m, r)\mathcal{A}(r, n) = \mathcal{A}(m, n)$, for all $(m, r, n) \in T$.

For the particular case when (3) is autonomous (i.e., $A_n = A \in \mathcal{B}(X)$ for all $n \in \mathbb{N}$), then $\mathcal{A}(m, n) = A^{m-n}$ for all $(m, n) \in \Delta$.

Definition 1 (see [16]). An application $P : \mathbb{N} \rightarrow \mathcal{B}(X)$ is said to be a projections sequence on X if

$$P_n^2 = P_n, \quad (6)$$

for every $n \in \mathbb{N}$.

Remark 2 (see [16]). If (P_n) is a projections sequence on X , then the mapping $Q : \mathbb{N} \rightarrow \mathcal{B}(X)$, $Q_n = I - P_n$, is also a projections sequence on X , which is called the complementary projections of (P_n) . One can easily see that $P_n Q_n = Q_n P_n = 0$, $\text{Ker } P_n = \text{Im } Q_n$, and $\text{Ker } Q_n = \text{Im } P_n$ for every $n \in \mathbb{N}$.

Definition 3 (see [16]). A projections sequence (P_n) is said to be invariant for system (3) if

$$A_n P_n = P_{n+1} A_n, \quad (7)$$

for all $n \in \mathbb{N}$.

Remark 4 (see [16]). The equality from Definition 3 also holds for the complementary projection (Q_n) and, as a consequence of the equality, we have that

$$\begin{aligned} \mathcal{A}(m, n) P_n &= P_m \mathcal{A}(m, n), \\ \mathcal{A}(m, n) Q_n &= Q_m \mathcal{A}(m, n) \end{aligned} \quad (8)$$

for all $(m, n) \in \Delta$.

Remark 5 (see [16]). If (P_n) is a projections sequence invariant for the reversible system (3) then $\mathcal{A}(m, n)$ is invertible for all $(m, n) \in \mathbb{N}^2$ and

$$\begin{aligned} \mathcal{A}(m, n)^{-1} P_m &= P_n \mathcal{A}(m, n)^{-1}, \\ \mathcal{A}(m, n)^{-1} Q_m &= Q_n \mathcal{A}(m, n)^{-1}, \end{aligned} \quad (9)$$

for all $(m, n) \in \Delta$.

Definition 6 (see [16]). Let (P_n) be a projections sequence which is invariant for system (3). One says that (P_n) is strongly invariant for system (3) if for every $(m, n) \in \Delta$ the linear operator $\mathcal{A}(m, n)$ is an isomorphism from $\text{Ker } P_n$ to $\text{Ker } P_m$.

Proposition 7 (see [16]). Let (P_n) be a projections sequence which is invariant for system (3). Suppose that for all $(m, n) \in \Delta$ the evolution operator $\mathcal{A}(m, n)$ is injective on $\text{Ker } P_n$. Then (P_n) is strongly invariant for system (3) if and only if $\text{Ker } P_m \subset \text{Im } \mathcal{A}(m, n)$ for all $(m, n) \in \Delta$.

In what follows, an example of an invariant projections sequence which is not strongly invariant is given.

Example 8. Let $X = \mathbb{R}^3$ with the norm

$$\|(x_1, x_2, x_3)\| = |x_1| + |x_2| + |x_3| \quad (10)$$

and let (3) be the discrete-time system defined by the sequence

$$\begin{aligned} A_n(x_1, x_2, x_3) &= \begin{cases} (x_1, 0, x_3), & n = 0, \\ \left(\left(\frac{n+1}{n+2} \right)^2 x_1, \left(\frac{n+1}{n+2} \right)^2 x_2, \left(\frac{n+2}{n+1} \right)^2 x_3 \right), & n \geq 1. \end{cases} \end{aligned} \quad (11)$$

It is easy to see that the sequence (P_n) defined by

$$P_n(x_1, x_2, x_3) = \begin{cases} (x_1, x_2, 0), & n = 0, \\ \left(x_1 + \frac{1}{(n+1)^2} x_2, 0, 0 \right), & n \geq 1, \end{cases} \quad (12)$$

is a projections sequence which is invariant for system (3). A simple calculus shows that

$$\mathcal{A}(m, n)(x_1, x_2, x_3) = \begin{cases} \left(\left(\frac{2}{m+1} \right)^2 x_1, 0, \left(\frac{m+1}{2} \right)^2 x_3 \right), & m > n = 0, \\ \left(\left(\frac{n+1}{m+1} \right)^2 x_1, \left(\frac{n+1}{m+1} \right)^2 x_2, \left(\frac{m+1}{n+1} \right)^2 x_3 \right), & m \geq n \geq 1 \text{ or } m = n = 0 \end{cases} \quad (13)$$

for all $(m, n) \in \Delta$. We can see that the evolution operator $\mathcal{A}(m, n)$ is injective on $\text{Ker } P_n$. The sequence (P_n) is not strongly invariant for (3) because $y = (1, -9, 0) \in \text{Ker } P_2$ and $y \notin \text{Im } \mathcal{A}(2, 0)$.

Proposition 9 (see [16]). *If the projections sequence (P_n) is strongly invariant for system (3) then there exists $\mathcal{B} : \Delta \rightarrow \mathcal{B}(X)$ such that for all $(m, n) \in \Delta$ the evolution operator $\mathcal{B}(m, n)$ is an isomorphism from $\text{Ker } P_m$ to $\text{Ker } P_n$.*

Lemma 10 (see [16]). *The function $\mathcal{B} : \Delta \rightarrow \mathcal{B}(X)$ has the following properties:*

- (i) $\mathcal{A}(m, n)\mathcal{B}(m, n)Q_m = Q_m$,
- (ii) $\mathcal{B}(m, n)\mathcal{A}(m, n)Q_n = Q_n$,
- (iii) $\mathcal{B}(m, n)Q_m = Q_n\mathcal{B}(m, n)Q_m$,
- (iv) $Q_m = \mathcal{B}(m, m)Q_m = Q_m\mathcal{B}(m, m)Q_m$,

for all $(m, n) \in \Delta$.

Remark 11 (see [16]). If the projections sequence (P_n) is invariant for the reversible system (3) then

$$\mathcal{B}(m, n) = \mathcal{A}(m, n)^{-1} \quad (14)$$

for all $(m, n) \in \Delta$.

3. Nonuniform Polynomial Dichotomies

In this section we study some polynomial dichotomy concepts of linear discrete-time system (3) with respect to a projections sequence (P_n) invariant for (3).

Definition 12 (see [16]). One says that system (3) admits a nonuniform exponential dichotomy (n.e.d.) with respect to the projections sequence (P_n) , if there exist a constant $\alpha > 0$ and a sequence $H : \mathbb{N} \rightarrow [1, +\infty)$ such that the following properties hold:

$$\begin{aligned} e^{\alpha(m-n)} \|\mathcal{A}(m, n)x\| &\leq H(n) \|x\|, \\ e^{\alpha(m-n)} \|y\| &\leq H(m) \|\mathcal{A}(m, n)x\|, \end{aligned} \quad (15)$$

for all $(m, n) \in \Delta$ and all $(x, y) \in \text{Im } P_n \times \text{Ker } P_n$.

Definition 13. One says that system (3) admits a nonuniform polynomial dichotomy (n.p.d.) with respect to the projections sequence (P_n) , if there exist a constant $\alpha > 1$ and a sequence $D : \mathbb{N} \rightarrow [1, +\infty)$ such that the following properties hold:

$$\left(\frac{m+1}{n+1}\right)^\alpha \|\mathcal{A}(m, n)x\| \leq D(n) \|x\|, \quad (16)$$

$$\left(\frac{m+1}{n+1}\right)^\alpha \|y\| \leq D(m) \|\mathcal{A}(m, n)y\|, \quad (17)$$

for all $(m, n) \in \Delta$ and all $(x, y) \in \text{Im } P_n \times \text{Ker } P_n$.

In the following we have some particular cases of nonuniform polynomial dichotomy.

- (1) If $D(n) = D \geq 1$ for all $n \in \mathbb{N}$ then we say that system (3) is uniformly polynomially dichotomic (u.p.d.).
- (2) If $D(n) = D(n+1)^\beta$ with $D \geq 1$ and $\beta > 0$ then we say that system (3) is polynomially dichotomic (p.d.).

Remark 14. The linear discrete-time system (3) is nonuniformly polynomially dichotomic if and only if a constant $\alpha > 1$ and a sequence $D : \mathbb{N} \rightarrow [1, +\infty)$ exist such that

$$\begin{aligned} \left(\frac{m+1}{n+1}\right)^\alpha (\|\mathcal{A}(m, k)x\| + \|\mathcal{A}(n, k)y\|) \\ \leq D(n) \|\mathcal{A}(n, k)x\| + D(m) \|\mathcal{A}(m, k)y\| \end{aligned} \quad (18)$$

for all $(m, n, k) \in T$ and all $(x, y) \in \text{Im } P_n \times \text{Ker } P_n$.

Remark 15. It is obvious that if the linear discrete-time system (3) is nonuniformly exponentially dichotomic with $\alpha > 1$ then it is nonuniformly polynomially dichotomic. But the converse statement is not necessarily valid. This fact is illustrated by the following example.

Example 16. Let $X = \mathbb{R}^2$ and $A : \mathbb{N} \rightarrow \mathcal{B}(\mathbb{R}^2)$ defined by

$$A_n(x_1, x_2) = \left(\frac{n^2 + 1}{(n+1)^2 + 1} x_1, \frac{(n+1)^2 + 1}{n^2 + 1} x_2 \right) \quad (19)$$

for all $(n, x_1, x_2) \in \mathbb{N} \times \mathbb{R}^2$. Let us consider the projections sequences $P, Q : \mathbb{N} \rightarrow \mathcal{B}(\mathbb{R}^2)$ defined by

$$\begin{aligned} P_n(x_1, x_2) &= (x_1, 0), \\ Q_n(x_1, x_2) &= (0, x_2) \end{aligned} \quad (20)$$

for all $n \in \mathbb{N}$ and all $x = (x_1, x_2) \in X$. We have that

$$\begin{aligned} \mathcal{A}(m, n)P_n(x_1, x_2) &= \left(\frac{n^2 + 1}{m^2 + 1} x_1, 0 \right), \\ \mathcal{A}(m, n)Q_n(x_1, x_2) &= \left(0, \frac{m^2 + 1}{n^2 + 1} x_2 \right). \end{aligned} \quad (21)$$

Then

$$\begin{aligned} \left(\frac{m+1}{n+1}\right)^2 (\|\mathcal{A}(m, n)P_n x\| + \|Q_n x\|) \\ = \frac{(m+1)^2}{m^2 + 1} \frac{n^2 + 1}{(n+1)^2} \|P_n x\| + \left(\frac{m+1}{n+1}\right)^2 \|Q_n x\| \\ \leq 2 \left(\|P_n x\| + \frac{m^2 + 1}{n^2 + 1} \|Q_n x\| \right) \\ \leq 2^{n+1} \|P_n x\| + 2^{m+1} \|\mathcal{A}(m, n)Q_n x\| \end{aligned} \quad (22)$$

for all $(m, n, x) \in \Delta \times \mathbb{R}^2$. Thus Definition 13 is satisfied for $D(n) = 2^{n+1}$ and $\alpha = 2$; hence system (3) is nonuniformly polynomially dichotomic.

On the other hand, if we suppose that system (3) is nonuniformly exponentially dichotomic, then there exist $\alpha > 0$ and a sequence $H : \mathbb{N} \rightarrow [1, +\infty)$ such that

$$e^{\alpha(m-n)} \cdot \frac{n^2 + 1}{m^2 + 1} \leq H(n) \quad (23)$$

for all $(m, n) \in \Delta$. In particular, for $n = 0$, we obtain

$$e^{\alpha m} \leq H(0) (m^2 + 1), \quad (24)$$

which is absurd for $m \rightarrow \infty$. Hence system (3) is not nonuniformly exponentially dichotomic.

Remark 17. It is obvious that

$$(\text{u.p.d.}) \implies (\text{p.d.}) \implies (\text{n.p.d.}). \quad (25)$$

The following two examples show that the converse implications between these dichotomy concepts are not valid.

Example 18. Let $X = \mathbb{R}^2$ and $A : \mathbb{N} \rightarrow \mathcal{B}(\mathbb{R}^2)$ defined by

$$A_n(x_1, x_2) = \left(\frac{(n+1)^2 u(n)}{(n+2)^2 u(n+1)} x_1, \frac{(n+2)^2 u(n+1)}{(n+1)^2 u(n)} x_2 \right) \quad (26)$$

for all $(n, x_1, x_2) \in \mathbb{N} \times \mathbb{R}^2$. Let $P_n(x_1, x_2) = (x_1, 0)$ and $Q_n(x_1, x_2) = (0, x_2)$ for all $n \in \mathbb{N}$ and all $x = (x_1, x_2) \in X$. Consider $u : \mathbb{N} \rightarrow [1, +\infty)$,

$$u(n) = \begin{cases} e^{n^2}, & n = 3k, \\ e^4, & n = 3k+1, \\ 1, & n = 3k+2, \end{cases} \quad k \in \mathbb{N} \quad (27)$$

for every $n \in \mathbb{N}$.

Thus we have that the following inequalities hold:

$$\begin{aligned} & \left(\frac{m+1}{n+1} \right)^2 \| \mathcal{A}(m, n) P_n x \| \\ &= \frac{(m+1)^2 (n+1)^2 u(n)}{(n+1)^2 (m+1)^2 u(m)} \| P_n x \| \leq u(n) \| P_n x \| \\ &\leq D(n) \| P_n x \|, \\ & \left(\frac{m+1}{n+1} \right)^2 \| Q_n x \| \leq D(m) \frac{(m+1)^2 u(m)}{(n+1)^2 u(n)} \| Q_n x \| \\ &= D(m) \| \mathcal{A}(m, n) Q_n x \| \end{aligned} \quad (28)$$

for all $(m, n, x) \in \Delta \times X$, where $D(n) = e^{4n^2}$. Hence system (3) is nonuniformly polynomially dichotomic.

On the other hand, if we suppose that system (3) is polynomially dichotomic, then there exist constants $D \geq 1$, $\alpha > 1$, and $\beta > 0$ such that

$$\left(\frac{m+1}{n+1} \right)^\alpha \frac{(n+1)^2 u(n)}{(m+1)^2 u(m)} \leq D(n+1)^\beta \quad (29)$$

for all $(m, n) \in \Delta$. In particular, for $m = 3k+1$ and $n = 3k$, we obtain

$$\left(1 + \frac{1}{3k+1} \right)^\alpha \left(\frac{3k+1}{3k+2} \right)^2 \leq D \frac{e^4}{e^{9k^2}} (3k+1)^\beta, \quad (30)$$

which is absurd for $k \rightarrow +\infty$. Hence system (3) is not polynomially dichotomic.

Example 19. Let $X = \mathbb{R}^2$ and $A : \mathbb{N} \rightarrow \mathcal{B}(\mathbb{R}^2)$ defined by

$$A_n(x_1, x_2) = \left(\frac{\varphi_n}{\varphi_{n+1}} x_1, \frac{\phi_{n+1}}{\phi_n} x_2 \right) \quad (31)$$

for all $(n, x_1, x_2) \in \mathbb{N} \times \mathbb{R}^2$, where the sequences $\varphi, \phi : \mathbb{N} \rightarrow \mathbb{R}$ are given by

$$\begin{aligned} \varphi_n &= (n+1)^2 \left(n+1 - n \sin \frac{n\pi}{2} \right), \\ \phi_n &= n^2 \left(n+1 + n \sin \frac{n\pi}{2} \right). \end{aligned} \quad (32)$$

Let us consider the projections sequences $P, Q : \mathbb{N} \rightarrow \mathcal{B}(\mathbb{R}^2)$ defined by

$$\begin{aligned} P_n(x_1, x_2) &= (x_1, 0), \\ Q_n(x_1, x_2) &= (0, x_2) \end{aligned} \quad (33)$$

for all $n \in \mathbb{N}$ and all $x = (x_1, x_2) \in X$. We have that

$$\begin{aligned} & \mathcal{A}(m, n) P_n(x_1, x_2) \\ &= \left(\frac{(n+1)^2 (n+1 - n \sin(n\pi/2))}{(m+1)^2 (m+1 - m \sin(m\pi/2))} x_1, 0 \right), \\ & \mathcal{A}(m, n) Q_n(x_1, x_2) \\ &= \left(0, \frac{m^2 (m+1 + m \sin(m\pi/2))}{n^2 (n+1 + n \sin(n\pi/2))} x_2 \right). \end{aligned} \quad (34)$$

Then

$$\begin{aligned} & (\| \mathcal{A}(m, n) P_n x \| + \| Q_n x \|) \\ &\leq \left(\frac{n+1}{m+1} \right)^2 (2n+1) \| P_n x \| \\ &\quad + \frac{2(n+1)m^2}{(2n+1)n^2} \left(\frac{n+1}{m+1} \right)^2 \| Q_n x \| \\ &\leq 2(n+1) \left(\frac{n+1}{m+1} \right)^2 (\| P_n x \| + \| \mathcal{A}(m, n) Q_n x \|) \end{aligned} \quad (35)$$

for all $(m, n, x) \in \Delta \times \mathbb{R}^2$. Thus system (3) is polynomially dichotomic for $D = \alpha = 2$ and $\beta = 1$.

On the other hand, if we suppose that system (3) is uniformly polynomially dichotomic, then there exist $D \geq 1$ and $\alpha > 1$ such that

$$\left(\frac{m+1}{n+1} \right)^\alpha \leq D \frac{m^2 (m+1 + m \sin(m\pi/2))}{n^2 (n+1 + n \sin(n\pi/2))} \quad (36)$$

for all $(m, n) \in \Delta$. In particular, for $m = 4k + 3$ and $n = 4k + 1$, $k \in \mathbb{N}$, we obtain that

$$\left(\frac{4k+4}{4k+2}\right)^\alpha \leq D \frac{(4k+3)^2}{(4k+1)^2(8k+3)}, \quad (37)$$

which is false for $k \rightarrow \infty$. Hence system (3) is not uniformly polynomially dichotomic.

Proposition 20. *The reversible system (3) is nonuniformly polynomially dichotomic if and only if there exist a projections sequence (P_n) invariant for (3), a constant $\alpha > 1$, and a sequence $D : \mathbb{N} \rightarrow [1, +\infty)$ such that*

$$\|\mathcal{A}(m, n)x\| \leq D(n) \left(\frac{n+1}{m+1}\right)^\alpha \|x\|, \quad (38)$$

$$\|\mathcal{A}(m, n)^{-1}y\| \leq D(m) \left(\frac{n+1}{m+1}\right)^\alpha \|y\|, \quad (39)$$

for all $(m, n) \in \Delta$ and all $(x, y) \in \text{Im } P_n \times \text{Ker } P_m$.

Proof. It is sufficient to prove (17) $\Leftrightarrow$ (39).

Necessity: if relation (17) holds then for all $(m, n, y) \in \Delta \times \text{Ker } P_m$ we have

$$\begin{aligned} \|\mathcal{A}(m, n)^{-1}y\| &= \|\mathcal{A}(m, n)^{-1}Q_m y\| \\ &= \|Q_n \mathcal{A}(m, n)^{-1}Q_m y\| \\ &\leq D(m) \left(\frac{n+1}{m+1}\right)^\alpha \|\mathcal{A}(m, n)Q_n \mathcal{A}(m, n)^{-1}Q_m y\| \\ &= D(m) \left(\frac{n+1}{m+1}\right)^\alpha \|Q_m y\| \\ &= D(m) \left(\frac{n+1}{m+1}\right)^\alpha \|y\|. \end{aligned} \quad (40)$$

Sufficiency: from (39) it results that for all $(m, n, y) \in \Delta \times \text{Ker } P_m$ we have

$$\begin{aligned} \|y\| &= \|Q_n y\| = \|\mathcal{A}(m, n)^{-1}Q_m \mathcal{A}(m, n)Q_n y\| \\ &\leq D(m) \left(\frac{n+1}{m+1}\right)^\alpha \|Q_m \mathcal{A}(m, n)Q_n y\| \\ &= D(m) \left(\frac{n+1}{m+1}\right)^\alpha \|\mathcal{A}(m, n)Q_n y\|. \end{aligned} \quad (41)$$

□

A characterization of nonuniform polynomial dichotomy property with respect to strongly invariant projections sequence is given by the following.

Theorem 21. *Let (P_n) be a projections sequence which is strongly invariant for system (3). Then (3) is nonuniformly polynomially dichotomic with respect to (P_n) if and only if there exist a constant $\alpha > 1$ and a sequence $D : \mathbb{N} \rightarrow [1, +\infty)$ such that*

$$\|\mathcal{A}(m, n)x\| \leq D(n) \left(\frac{n+1}{m+1}\right)^\alpha \|x\|, \quad (42)$$

$$\|\mathcal{B}(m, n)y\| \leq D(m) \left(\frac{n+1}{m+1}\right)^\alpha \|y\|, \quad (43)$$

for all $(m, n) \in \Delta$ and all $(x, y) \in \text{Im } P_n \times \text{Ker } P_m$.

Proof. It is sufficient to prove (17) $\Leftrightarrow$ (43).

Necessity: from relation (17) and Lemma 10, we have

$$\begin{aligned} \|\mathcal{B}(m, n)y\| &= \|\mathcal{B}(m, n)Q_m y\| \\ &= \|Q_n \mathcal{B}(m, n)Q_m y\| \\ &\leq D(m) \left(\frac{n+1}{m+1}\right)^\alpha \|\mathcal{A}(m, n)Q_n \mathcal{B}(m, n)Q_m y\| \\ &= D(m) \left(\frac{n+1}{m+1}\right)^\alpha \|Q_m \mathcal{A}(m, n)\mathcal{B}(m, n)Q_m y\| \\ &= D(m) \left(\frac{n+1}{m+1}\right)^\alpha \|Q_m y\| \\ &= D(m) \left(\frac{n+1}{m+1}\right)^\alpha \|y\| \end{aligned} \quad (44)$$

for all $(m, n, y) \in \Delta \times \text{Ker } P_m$.

Sufficiency: from relation (43) and Lemma 10, we obtain

$$\begin{aligned} \|y\| &= \|Q_n y\| = \|\mathcal{B}(m, n)\mathcal{A}(m, n)Q_n y\| \\ &= \|\mathcal{B}(m, n)Q_m \mathcal{A}(m, n)Q_n y\| \\ &\leq D(m) \left(\frac{n+1}{m+1}\right)^\alpha \|\mathcal{A}(m, n)Q_n y\| \\ &= D(m) \left(\frac{n+1}{m+1}\right)^\alpha \|\mathcal{A}(m, n)y\| \end{aligned} \quad (45)$$

for all $(m, n, y) \in \Delta \times \text{Ker } P_n$. □

Remark 22. Proposition 20 and Theorem 21 are some versions of the classical nonuniform exponential dichotomy theorems due to Popa et al. [16], for nonuniform polynomial dichotomy of the linear discrete-time system (3).

4. Lyapunov Functions and Nonuniform Polynomial Dichotomies

Let (3) be a linear discrete-time system on a Banach space X and let (P_n) be a projections sequence which is invariant for (3).

Definition 23. One says that $L : \Delta \times X \rightarrow \mathbb{R}_+$ is a Lyapunov function for system (3) with respect to projections sequence (P_n) if there exists a constant $l > 1$ such that the following properties hold:

$$\begin{aligned} L(m+p, n, x) + \sum_{k=m}^{m+p-1} \frac{1}{k+1} \left(\frac{k+1}{n+1}\right)^l \|\mathcal{A}(k, n)x\| \\ \leq L(m, n, x) \end{aligned} \quad (46)$$

for all $(m, n, x) \in \Delta \times \text{Im } P_n$ and $p \in \mathbb{N}_+$ and

$$\begin{aligned} L(m, n+p, y) + \sum_{k=n}^{n+p-1} (k+1) \left(\frac{m+1}{k+1}\right)^l \|\mathcal{A}(k, n)y\| \\ \leq L(m, n, y) \end{aligned} \quad (47)$$

for all $(m, n, x) \in \Delta \times \text{Ker } P_n$ and $p \in \mathbb{N}_+$.

Theorem 24. *If the linear discrete-time system (3) is nonuniformly polynomially dichotomic with respect to the projections sequence (P_n) invariant for (3), then there are $L : \Delta \times X \rightarrow \mathbb{R}_+$ a Lyapunov function for (3) and a sequence $\beta : \mathbb{N} \rightarrow [1, +\infty)$ such that*

$$\begin{aligned} L(m, n, x) + L(m, n, y) \\ \leq \beta(n) \|x\| + \beta(m) \|\mathcal{A}(m, n) y\|, \end{aligned} \quad (48)$$

for all $(m, n) \in \Delta$ and all $(x, y) \in \text{Im } P_n \times \text{Ker } P_n$.

Proof. Suppose that system (3) is nonuniformly polynomially dichotomic with respect to the projections sequence (P_n) . We consider the application $L : \Delta \times X \rightarrow \mathbb{R}_+$ by

$$\begin{aligned} L(m, n, x) = \sum_{k=m}^{\infty} \frac{1}{k+1} \left(\frac{k+1}{n+1} \right)^l \|\mathcal{A}(m, n) P_n x\| \\ + \sum_{k=n}^m (k+1) \left(\frac{m+1}{k+1} \right)^l \|\mathcal{A}(k, n) Q_n x\|, \end{aligned} \quad (49)$$

where $l \in (1, \alpha)$ and α is given by Definition 13. With a simple verification, we have that L is a Lyapunov function for system (3) with respect to projections sequence (P_n) .

For $x \in \text{Im } P_n$ we have that

$$\begin{aligned} L(m, n, x) &= \sum_{k=m}^{\infty} \frac{1}{k+1} \left(\frac{k+1}{n+1} \right)^l \|\mathcal{A}(m, n) x\| \\ &\leq \sum_{k=m}^{\infty} \frac{D(n)}{k+1} \left(\frac{k+1}{n+1} \right)^l \left(\frac{n+1}{k+1} \right)^{\alpha} \|x\| = D(n) \\ &\quad \cdot (n+1)^{\alpha-l} \|x\| \sum_{k=m}^{\infty} \frac{1}{(k+1)^{\alpha-l+1}} = D(n) (n+1)^{\alpha-l} \\ &\quad \cdot \|x\| \left(\frac{1}{(m+1)^{\alpha-l+1}} + \sum_{k=m+1}^{\infty} \frac{1}{(k+1)^{\alpha-l+1}} \right) \leq D(n) \quad (50) \\ &\quad \cdot (n+1)^{\alpha-l} \|x\| \\ &\quad \cdot \left(\frac{1}{(m+1)^{\alpha-l+1}} + \frac{1}{\alpha-l} \frac{1}{(m+1)^{\alpha-l+1}} \right) \leq D(n) \\ &\quad \cdot \left(\frac{n+1}{m+1} \right)^{\alpha-l} \left(1 + \frac{1}{\alpha-l} \right) \|x\| \leq D(n) \\ &\quad \cdot \frac{\alpha-l+1}{\alpha-l} \|x\| \leq \beta(n) \|x\|, \end{aligned}$$

where $\beta(n) = ((\alpha-l+1)/(\alpha-l))D(n)(n+1)^2$ for all $n \in \mathbb{N}$.

On the other hand, for $y \in \text{Ker } P_n$ we have that

$$\begin{aligned} L(m, n, y) &= \sum_{k=n}^m (k+1) \left(\frac{m+1}{k+1} \right)^l \|\mathcal{A}(k, n) y\| \\ &\leq \sum_{k=n}^m D(m) (k+1) \left(\frac{m+1}{k+1} \right)^l \left(\frac{k+1}{m+1} \right)^{\alpha} \\ &\quad \cdot \|\mathcal{A}(m, n) y\| = D(m) (m+1)^{l-\alpha} \|\mathcal{A}(m, n) y\| \quad (51) \\ &\quad \cdot \sum_{k=n}^m (k+1)^{\alpha-l+1} \leq D(m) (m-n+1) (m+1) \\ &\quad \cdot \|\mathcal{A}(m, n) y\| \leq D(m) (m+1)^2 \|\mathcal{A}(m, n) y\| \\ &\leq \beta(m) \|\mathcal{A}(m, n) y\|. \end{aligned}$$

Thus, the proof is completed. $\square$

Theorem 25. *If there exist a Lyapunov function L with $l > 2$ for system (3) and a sequence $\beta : \mathbb{N} \rightarrow [1, +\infty)$ such that relation (48) holds for all $(m, n) \in \Delta$ and all $(x, y) \in \text{Im } P_n \times \text{Ker } P_n$, then system (3) is nonuniformly polynomially dichotomic.*

Proof. According to the hypothesis, from (46) and (48) for $x \in \text{Im } P_n$ we have that

$$\begin{aligned} \sum_{k=m}^{m+p-1} \frac{1}{k+1} \left(\frac{k+1}{n+1} \right)^l \|\mathcal{A}(k, n) x\| &\leq L(m, n, x) \\ &\leq \beta(n) \|x\|. \end{aligned} \quad (52)$$

If we consider $k = m$ then

$$\begin{aligned} \|\mathcal{A}(m, n) x\| &\leq \beta(n) (m+1) \left(\frac{n+1}{m+1} \right)^l \|x\| \\ &= \beta(n) (n+1) \left(\frac{n+1}{m+1} \right)^{l-1} \|x\| \quad (53) \\ &= D(n) \left(\frac{n+1}{m+1} \right)^{l-1} \|x\|. \end{aligned}$$

In a similar way, from (47) and (48) for $y \in \text{Ker } P_n$ we have that

$$\begin{aligned} \sum_{k=n}^{n+p-1} (k+1) \left(\frac{m+1}{k+1} \right)^l \|\mathcal{A}(k, n) y\| &\leq L(m, n, y) \\ &\leq \beta(m) \|\mathcal{A}(m, n) y\|. \end{aligned} \quad (54)$$

Considering $k = n$, then

$$\begin{aligned} \|y\| &\leq \frac{\beta(m)}{n+1} \left(\frac{n+1}{m+1} \right)^l \|\mathcal{A}(m, n) y\| \\ &\leq \beta(m) \left(\frac{n+1}{m+1} \right)^{l-1} \|\mathcal{A}(m, n) y\| \quad (55) \\ &\leq D(m) \left(\frac{n+1}{m+1} \right)^{l-1} \|\mathcal{A}(m, n) y\|. \end{aligned}$$

Thus Definition 13 is satisfied for $D(n) = \beta(n)(n + 1)$ and $\alpha = l - 1$; hence system (3) is nonuniformly polynomial dichotomic. $\square$

Conflict of Interests

The author declares that there is no conflict of interests.

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