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Research Article

Synthesis of Decentralized Variable Gain Robust Controllers with Guaranteed $\mathscr{L}_2$ Gain Performance for a Class of Uncertain Large-Scale Interconnected Systems

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We consider a design problem of a decentralized variable gain robust controller with guaranteed $\mathscr{L}_2$ gain performance for a class of uncertain large-scale interconnected systems. For the uncertain large-scale interconnected system, the uncertainties and the interactions satisfy the matching condition. In this paper, we show that sufficient conditions for the existence of the proposed decentralized variable gain robust controller with guaranteed $\mathscr{L}_2$ gain performance are given in terms of linear matrix inequalities (LMIs). Finally, simple illustrative examples are shown.

1. Introduction

Due to the rapid development of industry in recent years, the controlled systems become more complex and such complex systems should be considered as large-scale interconnected systems (e.g., traffic systems and electric systems). As is well known, it is difficult to apply centralized control strategy to such systems because of physical constraints, calculation amount, and so on. Therefore, decentralized control problems for large-scale interconnected systems have been widely studied (see [1, 2] and references therein for details). The major problem of large-scale interconnected systems is how to deal with the interactions among subsystems.

On the other hand, robust control problem is one of the most important topics, and there is a large number of results for robust controller design and robust stability analysis (e.g., see [3, 4] and references therein). In particular, for uncertain linear systems, several quadratic stabilizing controllers have been suggested [5, 6] and the so-called robust $\mathscr{H}^\infty$ control problem has also been considered [7]. In most of the existing results, the proposed robust control systems have fixed gain controllers which are derived by considering worst case variations of uncertainties. In contrast with these, several design methods of variable gain controllers for uncertain continuous-time systems have been shown (e.g., [8–11]). These robust controllers are composed of a fixed gain controller and a variable gain one which are tuned by updating laws. In particular, in Oya and Hagino [11], the variable gain robust output feedback controller which achieves not only robust stability but also a specified $\mathscr{L}_2$ gain performance for a class of Lipschitz uncertain nonlinear systems has been proposed.

For large-scale interconnected systems with uncertainties, there are many existing results for decentralized robust control (e.g., [12–17]). In the work of Mao and Lin [12], for large-scale interconnected systems with unmodelled interaction, the aggregative derivation is tracked by using a model following technique with on-line improvement, and a sufficient condition for which the overall system when controlled by the completely decentralized control is asymptotically stable has been established. Furthermore, Gong [14] has proposed a decentralized robust controller which guarantees robust stability with prescribed degree of exponential convergence. Mukaidani et al. [15, 16] have also proposed decentralized guaranteed cost controllers for uncertain large-scale interconnected systems. In addition, we have suggested a decentralized variable gain robust controller which achieves not only robust stability but also satisfactory

transient behavior for a class of uncertain large-scale interconnected systems [18, 19].

In this paper, on the basis of the existing results [11, 18, 19], we propose a decentralized variable gain robust controller with guaranteed $\mathscr{L}_2$ gain performance for a class of uncertain large-scale interconnected systems; that is, this study is an extension of our previous studies in this field. For the uncertain large-scale interconnected systems, uncertainties and interactions satisfy the matching condition. The proposed decentralized robust controller consists of a fixed gain matrix and a variable one determined by a parameter adjustment law. In this paper, LMI-based sufficient conditions for the existence of the proposed decentralized variable gain robust controller are derived. To put it in the concrete, the decentralized variable gain robust controller, that is, parameter adjustment law, is designed so that the effects of uncertainties and interactions are reduced.

This paper is organized as follows. In Section 2, we show the notations and useful lemmas used in this paper. In Section 3, the class of uncertain large-scale interconnected systems under consideration is presented. Section 4 contains the main results; that is, LMI-based sufficient conditions for the existence of the proposed decentralized variable gain robust controller with guaranteed $\mathscr{L}_2$ gain performance are derived. Finally, simple illustrative examples are included to show the effectiveness of the proposed decentralized robust controller.

2. Preliminaries

In this section, notations and useful and well-known lemmas (see [11, 20, 21] for details) which are used in this paper are shown.

In this paper, the following notations are adopted. For a matrix $\mathscr{X}$, the inverse of matrix $\mathscr{X}$ and the transpose of one are denoted by $\mathscr{X}^{-1}$ and $\mathscr{X}^T$, respectively. Additionally, $H_e\{\mathscr{X}\}$ and I_n mean $\mathscr{X} + \mathscr{X}^T$ and n-dimensional identity matrix, respectively, and the notation $\operatorname{diag}(\mathscr{X}_1, \ldots, \mathscr{X}_{\mathscr{M}})$ represents a block diagonal matrix composed of matrices $\mathscr{X}_i$ for $i = 1, \ldots, \mathscr{M}$. For real symmetric matrices $\mathscr{X}$ and $\mathscr{Y}$, $\mathscr{X} > \mathscr{Y}$ (resp., $\mathscr{X} \geq \mathscr{Y}$), means that $\mathscr{X} - \mathscr{Y}$ is positive (resp., nonnegative) definite matrix. For a vector $\alpha \in \mathbb{R}^n$, $\|\alpha\|$ denotes standard Euclidian norm and, for a matrix $\mathscr{X}$, $\|\mathscr{X}\|$ represents its induced norm. The symbols "$\triangleq$" and "$\star$" mean equality by definition and symmetric blocks in matrix inequalities, respectively. Besides, $\mathscr{L}_2[0, \infty)$ is $\mathscr{L}_2$-space (i.e., the collection of all square integrable functions) defined on $[0, \infty)$, and, for a signal $f(t) \in \mathscr{L}_2[0, \infty)$, $\|f(t)\|_{\mathscr{L}_2}$ denotes its $\mathscr{L}_2$-norm.

Lemma 1. *For arbitrary vectors λ and ξ and the matrices $\mathscr{G}$ and $\mathscr{H}$ which have appropriate dimensions, the following inequality holds:*

$$H_e\left\{\lambda^T \mathscr{G} \Delta(t) \mathscr{H} \xi\right\} \leq 2\left\|\mathscr{G}^T \lambda\right\| \|\mathscr{H} \xi\|, \tag{1}$$

where $\Delta(t)$ with appropriate dimension is a time-varying unknown matrix satisfying $\|\Delta(t)\| \leq 1.0$.

Lemma 2 (Schur complement). *For a given constant real symmetric matrix Θ, the following items are equivalent:*

(1) $\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^T & \Theta_{22} \end{pmatrix} > 0$;

(2) $\Theta_{11} > 0$ *and* $\Theta_{22} - \Theta_{12}^T \Theta_{11}^{-1} \Theta_{12} > 0$;

(3) $\Theta_{22} > 0$ *and* $\Theta_{11} - \Theta_{12} \Theta_{22}^{-1} \Theta_{12}^T > 0$.

3. Problem Formulation

Let us consider the uncertain large-scale interconnected system composed of $\mathscr{N}$ subsystems represented by

$$\begin{aligned} \frac{d}{dt} x_i(t) &= A_{ii}(t) x_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^{\mathscr{N}} A_{ij}(t) x_j(t) + B_i u_i(t) \\ &\quad + \Gamma_{x_i} \omega_i(t), \\ z_i(t) &= C_{ii} x_i(t) + \Gamma_{z_i} \omega_i(t), \end{aligned} \tag{2}$$

where $x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}^{m_i}$, $z_i(t) \in \mathbb{R}^{p_i}$, and $\omega_i \in \mathbb{R}^{q_i}$ $(i = 1, \ldots, \mathscr{N})$ are the vectors of the state, the control input, the controlled output, and the disturbance input for the ith subsystem, respectively. Besides, the disturbance input is assumed to be square integrable; that is, $\omega_i(t) \in \mathscr{L}_2[0, \infty)$. The matrices $A_{ii}(t)$ and $A_{ij}(t)$ in (2) are given by

$$\begin{aligned} A_{ii}(t) &= A_{ii} + B_i \Delta_{ii}(t) \mathscr{E}_{ii}, \\ A_{ij}(t) &= B_i \mathscr{D}_{ij} + B_i \Delta_{ij}(t) \mathscr{E}_{ij}; \end{aligned} \tag{3}$$

that is, the uncertainties and the interaction terms satisfy the matching condition [18, 19]. In (2) and (3), the matrices $A_{ii} \in \mathbb{R}^{n_i \times n_i}$, $A_{ij} \in \mathbb{R}^{n_i \times n_j}$, $B_i \in \mathbb{R}^{n_i \times m_i}$, $C_{ii} \in \mathbb{R}^{p_i \times n_i}$, $\Gamma_{x_i} \in \mathbb{R}^{n_i \times q_i}$, and $\Gamma_{z_i} \in \mathbb{R}^{p_i \times q_i}$ are known system parameters and the matrices $\mathscr{D}_{ij}$, $\mathscr{E}_{ii}$, and $\mathscr{E}_{ij}$ with appropriate dimensions represent the structure of interactions or uncertainties. Besides, the matrices $\Delta_{ii}(t) \in \mathbb{R}^{m_i \times r_i}$ and $\Delta_{ij}(t) \in \mathbb{R}^{m_i \times s_{ij}}$ denote unknown time-varying parameters satisfying the relations $\|\Delta_{ii}(t)\| \leq 1.0$ and $\|\Delta_{ij}(t)\| \leq 1.0$, respectively.

Since the ith subsystem is given by (2), we find that the overall system can be written as

$$\begin{aligned} \frac{d}{dt} x(t) &= \mathscr{A}(t) x(t) + \mathscr{B} u(t) + \Gamma_x \omega(t), \\ z(t) &= \mathscr{C} x(t) + \Gamma_z \omega(t), \end{aligned} \tag{4}$$

where $x(t) = (x_1^T(t), \ldots, x_{\mathscr{N}}^T(t))^T$, $u(t) = (u_1^T(t), \ldots, u_{\mathscr{N}}^T(t))^T$, $z(t) = (z_1^T(t), \ldots, z_{\mathscr{N}}^T(t))^T$, and $\omega(t) = (\omega_1^T(t), \ldots, \omega_{\mathscr{N}}^T(t))^T$ are the state, the control input, the controlled output, and the disturbance input of the overall system. Besides, the matrices

$\mathscr{A}(t) \in \mathbb{R}^{n\times n}$, $\mathscr{B} \in \mathbb{R}^{n\times m}$, $\mathscr{C} \in \mathbb{R}^{p\times n}$, $\Gamma_x \in \mathbb{R}^{n\times q}$, and $\Gamma_z \in \mathbb{R}^{p\times q}$ are given by

$$\mathscr{A}(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) & \cdots & A_{1\mathscr{N}}(t) \\ A_{21}(t) & A_{22}(t) & \cdots & A_{2\mathscr{N}}(t) \\ \vdots & \vdots & \ddots & \vdots \\ A_{\mathscr{N}1}(t) & A_{\mathscr{N}2}(t) & \cdots & A_{\mathscr{N}\mathscr{N}}(t) \end{pmatrix},$$

$$\mathscr{B} = \operatorname{diag}(B_1, B_2, \ldots, B_{\mathscr{N}}), \tag{5}$$

$$\mathscr{C} = \operatorname{diag}(C_{11}, C_{22}, \ldots, C_{\mathscr{N}\mathscr{N}}),$$

$$\Gamma_x = \operatorname{diag}(\Gamma_{x_1}, \Gamma_{x_2}, \ldots, \Gamma_{x_{\mathscr{N}}}),$$

$$\Gamma_z = \operatorname{diag}(\Gamma_{z_1}, \Gamma_{z_2}, \ldots, \Gamma_{z_{\mathscr{N}}}),$$

where n, m, p, and q are given by $n = \sum_{i=1}^{\mathscr{N}} n_i$, $m = \sum_{i=1}^{\mathscr{N}} m_i$, $p = \sum_{i=1}^{\mathscr{N}} p_i$, and $q = \sum_{i=1}^{\mathscr{N}} q_i$, respectively.

Now for the ith subsystem of (2), we define the following control input [18, 19]:

$$u_i(t) \triangleq F_i x_i(t) + \mathscr{L}_i(x_i, t) x_i(t). \tag{6}$$

In (6), $F_i \in \mathbb{R}^{m_i\times n_i}$ and $\mathscr{L}_i(x_i, t) \in \mathbb{R}^{m_i\times n_i}$ are the fixed compensation gain matrix and the variable one for the ith subsystem of (2). From (2), (3), and (6), the following closed-loop subsystem can be derived:

$$\begin{aligned} \frac{d}{dt}x_i(t) &= (A_{ii} + B_i F_i) x_i(t) + B_i \Delta_{ii} \mathscr{E}_{ii} x_i(t) \\ &\quad + B_i \sum_{\substack{j=1 \\ j\neq i}}^{\mathscr{N}} (\mathscr{D}_{ij} + \Delta_{ij}\mathscr{E}_{ij}) x_j(t) \\ &\quad + B_i \mathscr{L}_i(x_i, t) x_i(t) + \Gamma_{x_i}\omega_i(t). \end{aligned} \tag{7}$$

Now we will give a definition of the decentralized variable gain robust control with guaranteed $\mathscr{L}_2$ gain performance.

Definition 3. For the uncertain large-scale interconnected system of (2), the control input of (6) is said to be a decentralized variable gain robust control with guaranteed $\mathscr{L}_2$ gain performance $\gamma^* > 0$ if the resultant closed-loop system of (7) is internally stable, and $\mathscr{H}_\infty$-norm of the transfer function from the disturbance input $\omega(t)$ to the controlled output $z(t)$ is less than or equal to a positive constant γ^*.

By using symmetric positive definite matrices $\mathscr{P}_i \in \mathbb{R}^{n_i\times n_i}$, we consider the following quadratic function:

$$\mathscr{V}(x, t) \triangleq \sum_{i=1}^{\mathscr{N}} \mathscr{V}_i(x_i, t), \tag{8}$$

where $\mathscr{V}_i(x_i, t)$ is

$$\mathscr{V}_i(x_i, t) \triangleq x_i^T(t) \mathscr{P}_i x_i(t). \tag{9}$$

Besides, we introduce the following Hamiltonian:

$$\begin{aligned} \mathscr{H}(x, t) &\triangleq \frac{d}{dt}\mathscr{V}(t) \\ &\quad + \sum_{i=1}^{\mathscr{N}} \left\{ z_i^T(t) z_i(t) - (\gamma_i^*)^2 \omega_i^T(t) \omega_i(t) \right\}. \end{aligned} \tag{10}$$

Then we have the following lemma for the decentralized variable gain robust control with guaranteed $\mathscr{L}_2$ gain performance $\gamma^* > 0$ for the uncertain large-scale interconnected system of (2) and the control input of (6).

Lemma 4. *Consider the uncertain large-scale interconnected system of (2) and the control input of (6).*

For the quadratic function $\mathscr{V}(x, t)$ and the signals $z(t)$ and $\omega(t)$, if there exist symmetric positive definite matrices $\mathscr{P}_i$ ($i = 1, \ldots, \mathscr{N}$) and positive scalars γ^* which satisfy the inequality

$$\mathscr{H}(x, t) < 0, \tag{11}$$

then the control input of (6) is a decentralized variable gain robust control with guaranteed $\mathscr{L}_2$ gain performance γ^*, where γ^* is given by

$$\gamma^* = \max_i \gamma_i^* \quad (i = 1, \ldots, \mathscr{N}). \tag{12}$$

Proof. By integrating both sides of the inequality of (11) from 0 to ∞ with $x_i(0) = 0$, we obtain the following inequality from $\mathscr{V}(x, 0) = 0$. Consider

$$\begin{aligned} &\mathscr{V}(x, \infty) \\ &+ \sum_{i=1}^{\mathscr{N}} \left\{ \int_0^\infty z_i^T(t) z_i(t)\, dt - (\gamma_i^*)^2 \int_0^\infty \omega_i^T(t) \omega_i(t)\, dt \right\} \\ &< 0. \end{aligned} \tag{13}$$

We see from the inequality of (13) that the uncertain closed-loop subsystem of (7) is robustly stable (internally stable). Namely, robust stability of the uncertain closed-loop subsystem with $\omega(t) = 0$ is guaranteed and the $\mathscr{H}_\infty$-norm of the transfer function from the disturbance input $\omega(t)$ to the controlled output $z(t)$ is less than a positive constant γ^* because the inequality of (13) means the following relation:

$$\|z(t)\|_{\mathscr{L}_2} < \gamma^* \|\omega(t)\|_{\mathscr{L}_2}. \tag{14}$$

Thus the proof of Lemma 4 is accomplished. □

From the above discussion, our design objective in this paper is to determine the decentralized variable gain robust control input of (6) such that the overall system achieves not only internal stability but also guaranteed $\mathscr{L}_2$ gain performance $\gamma^* > 0$. That is to derive the symmetric positive definite matrices $\mathscr{P}_i \in \mathbb{R}^{n_i\times n_i}$, positive constants γ^*, the fixed gain matrices $F_i \in \mathbb{R}^{m_i\times n_i}$, and the variable one $\mathscr{L}_i(x_i, t) \in$

$\mathbb{R}^{m_i \times n_i}$ satisfying the inequality of (11) for all admissible uncertainties $\Delta_{ii}(t) \in \mathbb{R}^{m_i \times r_i}$ and $\Delta_{ij}(t) \in \mathbb{R}^{m_i \times s_{ij}}$ and the disturbance input $\omega_i(t) \in \mathscr{L}_2[0, \infty)$.

4. Decentralized Variable Gain Controllers

In this section, we show a design method of the decentralized variable gain robust controller with guaranteed $\mathscr{L}_2$ gain performance.

The following theorem shows sufficient conditions for the existence of the proposed decentralized control system.

Theorem 5. *Let one consider the large-scale interconnected system of (2) and the control input of (6).*

By using symmetric positive definite matrices $\mathscr{Y}_i \in \mathbb{R}^{n_i \times n_i}$, the matrices $\mathscr{W}_i \in \mathbb{R}^{m_i \times n_i}$, and positive scalars ϵ_i and γ_i which satisfy the LMIs,

$$\left(\begin{array}{c|c|c} H_e\{A_{ii}\mathscr{Y}_i + B_i\mathscr{W}_i\} & \Gamma_{x_i} + \mathscr{Y}_i C_{ii}^T \Gamma_{z_i} & \Lambda(\mathscr{Y}_i) \\ \hline \star & \Gamma_{z_i}^T \Gamma_{z_i} - \gamma_i I_{q_i} & 0 \\ \hline \star & \star & -\Omega(\epsilon_i) \end{array}\right) < 0, \tag{15}$$

the fixed gain matrix $F_i \in \mathbb{R}^{m_i \times n_i}$ and the variable one $\mathscr{L}_i(x_i, t) \in \mathbb{R}^{m_i \times n_i}$ are determined as

$$F_i = \mathscr{W}_i \mathscr{Y}_i^{-1},$$
$$\mathscr{L}_i(x_i, t) \triangleq \begin{cases} -\dfrac{\zeta_i(x_i, t) + \eta_i(x_i, t)}{\left\| B_i^T \mathscr{P}_i x_i(t) \right\|^2} B_i^T \mathscr{P}_i \\ \mathscr{L}_i(x_i, t_\epsilon). \end{cases} \tag{16}$$

In (15) and (16), matrices $\Lambda(\mathscr{Y}_i)$ and $\Omega(\epsilon_i)$ and scalar functions $\zeta(x_i, t)$ and $\eta(x_i, t)$ are given by

$$\begin{aligned} \Lambda(\mathscr{Y}_i) &\triangleq \left(\mathscr{Y}_i C_{ii}^T \;\; \mathscr{Y}_i \mathscr{D}_{1i}^T \;\; \mathscr{Y}_i \mathscr{E}_{1i}^T \;\; \mathscr{Y}_i \mathscr{D}_{2i}^T \;\; \mathscr{Y}_i \mathscr{E}_{2i}^T \;\; \cdots \;\; \mathscr{Y}_i \mathscr{D}_{i-1i}^T \;\; \mathscr{Y}_i \mathscr{E}_{i-1i}^T \;\; \mathscr{Y}_i \mathscr{D}_{i+1i}^T \;\; \mathscr{Y}_i \mathscr{E}_{i+1i}^T \;\; \cdots \;\; \mathscr{Y}_i \mathscr{D}_{\mathscr{N}i}^T \;\; \mathscr{Y}_i \mathscr{E}_{\mathscr{N}i}^T\right), \\ \Omega(\epsilon_i) &\triangleq \operatorname{diag}\left(I_{p_i}, \epsilon_1 I_{m_1}, \epsilon_1 I_{s_1 i}, \epsilon_2 I_{m_2}, \epsilon_2 I_{s_2 i}, \ldots, \epsilon_{i-1} I_{m_{i-1}}, \epsilon_{i-1} I_{s_{i-1} i}, \epsilon_{i+1} I_{m_{i+1}}, \epsilon_{i+1} I_{s_{i+1} i}, \ldots, \epsilon_{\mathscr{N}} I_{m_{\mathscr{N}}}, \epsilon_{\mathscr{N}} I_{s_{\mathscr{N}} i}\right), \\ \zeta_i(x_i, t) &= \left\| B_i^T \mathscr{P}_i x_i(t) \right\| \left\| \mathscr{E}_{ii} x_i(t) \right\|, \\ \eta_i(x_i, t) &= \epsilon_i (\mathscr{N} - 1) \left\| B_i^T \mathscr{P}_i x_i(t) \right\|^2. \end{aligned} \tag{17}$$

Note that t_ϵ in (16) is given by $t_\epsilon = \lim_{\epsilon > 0, \epsilon \to 0}(t - \epsilon)$ [8].

Then the control input of (6) is the decentralized variable gain robust control with guaranteed $\mathscr{L}_2$ gain performance γ^*.

Proof. In order to prove Theorem 5, we consider the quadratic function $\mathscr{V}(x, t)$ of (8), the Hamiltonian $\mathscr{H}(x, t)$ of (10), and the inequality of (11).

For the quadratic function $\mathscr{V}_i(x_i, t)$ of (9), its time derivative along the trajectory of the resultant closed-loop subsystem of (7) can be computed as

$$\begin{aligned} \frac{d}{dt}\mathscr{V}_i(x_i, t) &= x_i^T(t)\left[H_e\left\{(A_{ii} + B_i F_i)^T \mathscr{P}_i\right\}\right] x_i \\ &\quad + H_e\left\{x_i^T(t) \mathscr{P}_i B_i \Delta_{ii}(t) \mathscr{E}_{ii} x_i(t)\right\} \\ &\quad + H_e\left\{x_i^T(t) \mathscr{P}_i B_i \sum_{\substack{j=1 \\ j \neq i}}^{\mathscr{N}} \left(\mathscr{D}_{ij} + \Delta_{ij}\mathscr{E}_{ij}\right) x_j(t)\right\} \\ &\quad + H_e\left\{x_i^T(t) \mathscr{P}_i B_i \mathscr{L}_i(x_i, t) x_i(t)\right\} \\ &\quad + H_e\left\{x_i^T(t) \mathscr{P}_i \Gamma_{x_i} \omega_i(t)\right\}. \end{aligned} \tag{18}$$

Besides, by using Lemma 1 and the well-known inequality

$$2\alpha^T \beta \leq \delta \alpha^T \alpha + \frac{1}{\delta}\beta^T \beta \tag{19}$$

for any vectors α and β with appropriate dimensions and a positive scalar δ, we have the following relation for the function $\mathscr{V}_i(x_i, t)$:

$$\begin{aligned} &\frac{d}{dt}\mathscr{V}_i(x_i, t) \\ &\quad \leq x_i^T(t)\left[H_e\left\{(A_{ii} + B_i F_i)^T \mathscr{P}_i\right\}\right] x_i(t) \\ &\qquad + 2\left\| B_i^T \mathscr{P}_i x_i(t) \right\| \left\| \mathscr{E}_{ii} x_i(t) \right\| \\ &\qquad + 2\epsilon_i (\mathscr{N} - 1) x_i^T(t) \mathscr{P}_i B_i B_i^T \mathscr{P}_i x_i(t) \\ &\qquad + \frac{1}{\epsilon_i} \sum_{\substack{j=1 \\ j \neq i}}^{\mathscr{N}} x_j^T(t)\left(\mathscr{D}_{ij}^T \mathscr{D}_{ij} + \mathscr{E}_{ij}^T \mathscr{E}_{ij}\right) x_j(t) \\ &\qquad + H_e\left\{x_i^T(t) \mathscr{P}_i B_i \mathscr{L}_i(x_i, t) x_i(t)\right\} \\ &\qquad + H_e\left\{x_i^T(t) \mathscr{P}_i \Gamma_{x_i} \omega_i(t)\right\}. \end{aligned} \tag{20}$$

Firstly, we consider the case of $B_i^T \mathscr{P}_i x_i(t) \neq 0$. In this case, substituting the variable gain matrix of (16) into (20) and some algebraic manipulations derive the following inequality:

$$\frac{d}{dt}\mathscr{V}_i(x_i,t) \le x_i^T(t)\left[H_e\left\{(A_{ii}+B_iF_i)^T\mathscr{P}_i\right\}\right]x_i(t) + \frac{1}{\epsilon_i}\sum_{\substack{j=1\\ j\neq i}}^{\mathscr{N}} x_j^T(t)\left(\mathscr{D}_{ij}^T\mathscr{D}_{ij}+\mathscr{E}_{ij}^T\mathscr{E}_{ij}\right)x_j(t) + H_e\left\{x_i^T(t)\mathscr{P}_i\Gamma_{x_i}\omega_i(t)\right\}. \quad (21)$$

Additionally, one can see from (2) that the following relation holds:

$$z_i^T(t)z_i(t)-(\gamma_i^*)^2\omega_i^T(t)\omega_i(t) = x_i^T(t)C_{ii}^TC_{ii}x_i(t)+H_e\left\{x_i^T(t)C_{ii}\Gamma_{z_i}\omega_i(t)\right\} + \omega_i^T(t)\left(\Gamma_{z_i}^T\Gamma_{z_i}-\gamma_iI_{q_i}\right)\omega_i(t), \quad (22)$$

where $(\gamma_i^*)^2 \triangleq \gamma_i$. Therefore, from (8), (10), (21), and (22), we can obtain the following relation for the Hamiltonian $\mathscr{H}(x,t)$:

$$\begin{aligned}\mathscr{H}(x,t) \le{}& \sum_{i=1}^{\mathscr{N}} x_i^T(t)\left[H_e\left\{(A_{ii}+B_iF_i)^T\mathscr{P}_i\right\}\right]x_i(t) \\ &+\sum_{i=1}^{\mathscr{N}}\frac{1}{\epsilon_i}\sum_{\substack{j=1\\ j\neq i}}^{\mathscr{N}} x_j^T(t)\left(\mathscr{D}_{ij}^T\mathscr{D}_{ij}+\mathscr{E}_{ij}^T\mathscr{E}_{ij}\right)x_j(t) \\ &+\sum_{i=1}^{\mathscr{N}} H_e\left\{x_i^T(t)\left(\mathscr{P}_i\Gamma_{x_i}+C_{ii}^T\Gamma_{z_i}\right)\omega_i(t)\right\} \\ &+\sum_{i=1}^{\mathscr{N}} x_i^T(t)C_{ii}^TC_{ii}x_i(t) \\ &+\sum_{i=1}^{\mathscr{N}}\omega_i^T(t)\left(\Gamma_{z_i}^T\Gamma_{z_i}-\gamma_iI_{q_i}\right)\omega_i(t).\end{aligned} \quad (23)$$

The inequality of (23) can also be rewritten as

$$\begin{aligned}\mathscr{H}(x,t) \le{}& \sum_{i=1}^{\mathscr{N}} x_i^T(t)\left[H_e\left\{(A_{ii}+B_iF_i)^T\mathscr{P}_i\right\}\right]x_i(t) \\ &+\sum_{i=1}^{\mathscr{N}} x_i^T(t)\left\{\sum_{\substack{j=1\\ j\neq i}}^{\mathscr{N}}\frac{1}{\epsilon_j}\left(\mathscr{D}_{ji}^T\mathscr{D}_{ji}+\mathscr{E}_{ji}^T\mathscr{E}_{ji}\right)\right\}x_i(t) \\ &+\sum_{i=1}^{\mathscr{N}} H_e\left\{x_i^T(t)\mathscr{P}_i\Gamma_{x_i}\omega_i(t)\right\}+\sum_{i=1}^{\mathscr{N}} x_i^T(t)C_{ii}^TC_{ii}x_i(t) \\ &+\sum_{i=1}^{\mathscr{N}} H_e\left\{x_i^T(t)C_{ii}^T\Gamma_{z_i}\omega_i(t)\right\}+\sum_{i=1}^{\mathscr{N}}\omega_i^T(t)\left(\Gamma_{z_i}^T\Gamma_{z_i}\right. \\ &\left.-\gamma_iI_{q_i}\right)\omega_i(t) = \sum_{i=1}^{\mathscr{N}} x_i^T(t)\left[H_e\left\{(A_{ii}+B_iF_i)^T\mathscr{P}_i\right\}\right. \\ &\left.+C_{ii}^TC_{ii}+\sum_{\substack{j=1\\ j\neq i}}^{\mathscr{N}}\frac{1}{\epsilon_j}\left(\mathscr{D}_{ji}^T\mathscr{D}_{ji}+\mathscr{E}_{ji}^T\mathscr{E}_{ji}\right)\right]x_i(t) \\ &+\sum_{i=1}^{\mathscr{N}} H_e\left\{x_i^T(t)\left(\mathscr{P}_i\Gamma_{x_i}+C_{ii}^T\Gamma_{z_i}\right)\omega_i(t)\right\}+\sum_{i=1}^{\mathscr{N}}\omega_i^T(t) \\ &\cdot\left(\Gamma_{z_i}^T\Gamma_{z_i}-\gamma_iI_{q_i}\right)\omega_i(t).\end{aligned} \quad (24)$$

Furthermore, some algebraic manipulations for (24) give the following inequality:

$$\mathscr{H}(x,t) \le \sum_{i=1}^{\mathscr{N}}\begin{pmatrix}x_i(t)\\ \omega_i(t)\end{pmatrix}^T\Psi_i(\mathscr{P}_i,\epsilon_i,\gamma_i)\begin{pmatrix}x_i(t)\\ \omega_i(t)\end{pmatrix}, \quad (25)$$

where $\Psi_i(\mathscr{P}_i,\epsilon_i,\gamma_i)$ is given by

$$\begin{aligned}\Psi_i(\mathscr{P}_i,\epsilon_i,\gamma_i) &\triangleq \left(\begin{array}{c|c}\Upsilon_i(\mathscr{P}_i,\epsilon_i) & \mathscr{P}_i\Gamma_{x_i}+C_{ii}^T\Gamma_{z_i}\\ \hline \star & \Gamma_{z_i}^T\Gamma_{z_i}-\gamma_iI_{q_i}\end{array}\right), \\ \Upsilon_i(\mathscr{P}_i,\epsilon_i) &\triangleq H_e\left\{(A_{ii}+B_iF_i)^T\mathscr{P}_i\right\}+C_{ii}^TC_{ii} \\ &\quad+\sum_{i=1}^{\mathscr{N}}\frac{1}{\epsilon_i}\left(\mathscr{D}_{ji}^T\mathscr{D}_{ji}+\mathscr{E}_{ji}^T\mathscr{E}_{ji}\right).\end{aligned} \quad (26)$$

Hence, if the matrix inequality

$$\Psi_i(\mathscr{P}_i,\epsilon_i,\gamma_i) < 0 \quad (27)$$

holds, then the inequality of (11) for the Hamiltonian is satisfied.

Next we consider the case of $B_i^T\mathscr{P}_ix_i(t) = 0$. In this case, one can see from (18) and (22) and the definition of the control input of (6) and the variable gain matrix (16) that if the matrix inequality of (27) holds, then the inequality of (11) is also satisfied.

Finally, we consider the matrix inequality of (27). By introducing the matrices $\mathscr{Y}_i \triangleq \mathscr{P}_i^{-1}$ and $\mathscr{W}_i \triangleq F_i\mathscr{Y}_i$ and pre- and postmultiplying both sides of the matrix inequality of (27) by $\mathrm{diag}(\mathscr{Y}_i, I_{q_i})$, we have the following inequality:

$$\Phi_i(\mathscr{Y}_i,\mathscr{W}_i,\epsilon_i,\gamma_i) = \left(\begin{array}{c|c}\Xi_i(\mathscr{Y}_i,\mathscr{W}_i,\epsilon_i) & \Gamma_{x_i}+\mathscr{Y}_iC_{ii}^T\Gamma_{z_i}\\ \hline \star & \Gamma_{z_i}^T\Gamma_{z_i}-\gamma_iI_{q_i}\end{array}\right) < 0, \quad (28)$$

where $\Xi_i(\mathscr{Y}_i,\mathscr{W}_i,\epsilon_i) \in \mathbb{R}^{n_i\times n_i}$ is given by

$$\begin{aligned}\Xi_i(\mathscr{Y}_i,\mathscr{W}_i,\epsilon_i) &\triangleq H_e\left\{A_{ii}\mathscr{Y}_i+B_i\mathscr{W}_i\right\}+\mathscr{Y}_iC_{ii}^TC_{ii}\mathscr{Y}_i \\ &\quad+\sum_{i=1}^{\mathscr{N}}\frac{1}{\epsilon_i}\mathscr{Y}_i\left(\mathscr{D}_{ji}^T\mathscr{D}_{ji}+\mathscr{E}_{ji}^T\mathscr{E}_{ji}\right)\mathscr{Y}_i.\end{aligned} \quad (29)$$

Thus by applying Lemma 2 (Schur complement) to (28) we find that the matrix inequalities of (28) are equivalent to the LMIs of (15). In the LMIs of (15), scalar variables $\epsilon_i > 0$ and $\gamma_i > 0$ can arbitrarily be selected. Therefore, we find that the LMIs of (15) are always feasible; that is, there always exists the solution of the LMIs of (15). Namely, by solving the LMIs of (15), the fixed gain matrix is determined as $F_i = \mathscr{W}_i \mathscr{Y}_i^{-1}$ and the variable one is given by (16) and the proposed control input of (6) becomes a decentralized variable gain robust control with guaranteed $\mathscr{L}_2$ gain performance γ^* of (12). Therefore, the proof of Theorem 5 is accomplished. □

Remark 6. Although the uncertainties included in the large-scale interconnected system of (2) are structured ones, the proposed design method can also be applied to the systems with the parameter structured uncertainties [19]. Besides, in [19], the nominal system is introduced so as to generate the desired trajectory of the state and the control input. The proposed design method in this paper can be easily extended to such control problem.

Remark 7. The proposed decentralized variable gain robust controller can be obtained by solving LMIs of (15). Since LMIs of (15) define convex solution sets of $(\mathscr{Y}_i, \mathscr{W}_i, \epsilon_i, \gamma_i)$, thus various efficient convex optimization algorithms can be applied to test whether these LMIs are solvable and to generate particular solutions [22, 23]. In addition, these solutions parametrize the set of decentralized variable gain robust controllers with $\mathscr{L}_2$ gain performance. Namely, one can see that the result in Theorem 5 can easily be extended to the decentralized variable gain robust controller with suboptimal $\mathscr{L}_2$ gain performance (see Appendices for details).

Remark 8. The decentralized robust controller synthesis proposed in this paper is adaptable when some assumptions are satisfied. Namely, if the matching condition for uncertainties and interactions is satisfied, then the proposed decentralized variable gain robust controller is applicable; that is, the LMIs of (15) are always feasible (see [19]). On the other hand, for decentralized robust controllers with fixed gain matrices based on the existing results [14–16], the size of LMIs to be solved is $2n_i + \sum_{j=1,j\neq i}^{\mathscr{N}} n_j + 2p_i + q_i + r_i + \sum_{j=1,j\neq i}^{\mathscr{N}} s_{ij}$. However, the size of LMIs in the proposed design is equal to $n_i + 2q_i + \sum_{j=1,j\neq i}^{\mathscr{N}} (n_j + s_{ij})$. Moreover, the number of variables for LMIs of (15) is smaller than that of the decentralized robust controllers with fixed gain matrices. Therefore, one can see that the proposed decentralized robust controller design method is very useful (see also Remark 5 in [19]).

5. Numerical Examples

In order to demonstrate the efficiency of the proposed robust controller, we have run a simple example. The control problems considered here are not necessary practical. However, the simulation results stated below illustrate the distinct feature of the proposed decentralized robust controller.

In this example, we consider the uncertain large-scale interconnected systems consisting of three two-dimensional subsystems; that is, $\mathscr{N} = 3$. The system parameters are given as follows:

$$A_{11} = \begin{pmatrix} -1.0 & 1.0 \\ 0.0 & 1.0 \end{pmatrix},$$

$$A_{22} = \begin{pmatrix} 0.0 & 1.0 \\ -1.0 & -1.0 \end{pmatrix},$$

$$A_{33} = \begin{pmatrix} 1.0 & 0.0 \\ -1.0 & -3.0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0.0 \\ 1.0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 1.0 \\ 1.0 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 1.0 \\ 0.0 \end{pmatrix},$$

$$\mathscr{E}_{11}^T = \begin{pmatrix} 1.0 \\ 0.0 \end{pmatrix},$$

$$\mathscr{E}_{22}^T = \begin{pmatrix} 2.0 \\ 1.0 \end{pmatrix},$$

$$\mathscr{E}_{33}^T = \begin{pmatrix} 2.0 \\ 2.0 \end{pmatrix},$$

$$\mathscr{D}_{12}^T = \begin{pmatrix} 1.0 \\ 2.0 \end{pmatrix},$$

$$\mathscr{D}_{13}^T = \begin{pmatrix} 2.0 \\ 1.0 \end{pmatrix},$$

$$\mathscr{D}_{21}^T = \begin{pmatrix} 1.0 \\ 0.0 \end{pmatrix},$$

$$\mathscr{D}_{23}^T = \begin{pmatrix} 1.0 \\ 1.0 \end{pmatrix},$$

$$\mathscr{D}_{31}^T = \begin{pmatrix} 2.0 \\ 1.0 \end{pmatrix},$$

$$\mathscr{D}_{32}^T = \begin{pmatrix} 0.0 \\ 2.0 \end{pmatrix},$$

$$\mathscr{E}_{12}^T = \begin{pmatrix} 2.0 \\ 1.0 \end{pmatrix},$$

$$\mathscr{E}_{13}^T = \begin{pmatrix} 2.0 \\ 2.0 \end{pmatrix},$$

$$\mathscr{E}_{21}^T = \begin{pmatrix} 1.0 \\ 0.0 \end{pmatrix},$$

$$\mathscr{E}_{23}^T = \begin{pmatrix} 0.0 \\ 3.0 \end{pmatrix},$$

$$\mathscr{E}_{31}^T = \begin{pmatrix} 1.0 \\ 2.0 \end{pmatrix},$$

$$\mathscr{E}_{32}^T = \begin{pmatrix} 3.0 \\ 1.0 \end{pmatrix},$$

$$\Gamma_{x_1} = \begin{pmatrix} 1.0 \\ 1.0 \end{pmatrix},$$

$$\Gamma_{x_2} = \begin{pmatrix} 1.0 \\ 0.0 \end{pmatrix},$$

$$\Gamma_{x_3} = \begin{pmatrix} 1.0 \\ 2.0 \end{pmatrix},$$

$$C_{11}^T = \begin{pmatrix} 1.0 \\ 0.0 \end{pmatrix},$$

$$C_{22}^T = \begin{pmatrix} 1.0 \\ 1.0 \end{pmatrix},$$

$$C_{33}^T = \begin{pmatrix} 1.0 \\ 1.0 \end{pmatrix},$$

$$\Gamma_{z_1} = 1.0,$$

$$\Gamma_{z_2} = 1.0,$$

$$\Gamma_{z_3} = 1.0. \tag{30}$$

Firstly, by using Theorem 5, we design the proposed decentralized variable gain robust controller. By solving LMIs of (15), we have positive definite matrices $\mathscr{Y}_i \in \mathbb{R}^{2\times 2}$, matrices $\mathscr{W}_i \in \mathbb{R}^{1\times 2}$, and positive scalars ϵ_i and γ_i given by

$$\mathscr{Y}_1 = \begin{pmatrix} 6.7486\times 10^{-1} & -6.7790\times 10^{-1} \\ \star & 1.8774 \end{pmatrix},$$

$$\mathscr{Y}_2 = \begin{pmatrix} 3.9813\times 10^{-1} & 8.2896\times 10^{-2} \\ \star & 7.9385\times 10^{-1} \end{pmatrix},$$

$$\mathscr{Y}_3 = \begin{pmatrix} 1.5567 & -7.9700\times 10^{-1} \\ \star & 1.2887 \end{pmatrix},$$

$$\mathscr{W}_1^T = \begin{pmatrix} -5.8074\times 10^{-1} \\ -7.9480 \end{pmatrix},$$

$$\mathscr{W}_2^T = \begin{pmatrix} -2.9910 \\ -2.8262 \end{pmatrix},$$

$$\mathscr{W}_3^T = \begin{pmatrix} -7.3316 \\ -7.0692\times 10^{-1} \end{pmatrix},$$

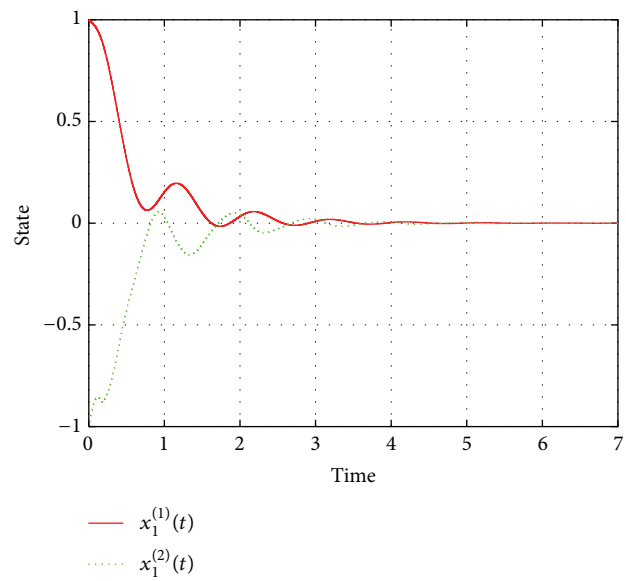

Figure 1: Time histories of $x_1(t)$.

$$\epsilon_1 = 3.8370,$$

$$\epsilon_2 = 7.2402,$$

$$\epsilon_3 = 2.4049,$$

$$\gamma_1 = 3.8259,$$

$$\gamma_2 = 3.4318,$$

$$\gamma_3 = 4.5212. \tag{31}$$

Thus the fixed gain matrices $F_i \in \mathbb{R}^{1\times 2}$ can be computed as

$$F_1 = \begin{pmatrix} -8.0232 & -7.1306 \end{pmatrix},$$

$$F_2 = \begin{pmatrix} -6.9218 & -2.8373 \end{pmatrix}, \tag{32}$$

$$F_3 = \begin{pmatrix} -7.3027 & -5.0650 \end{pmatrix}.$$

Furthermore, the positive scalars $\gamma_i^* = \sqrt{\gamma_i}$ can be obtained as

$$\gamma_1^* = 1.9560,$$

$$\gamma_2^* = 1.8525, \tag{33}$$

$$\gamma_3^* = 2.1263.$$

Therefore, the guaranteed $\mathscr{L}_2$ gain performance γ^* of (12) for the proposed controller is given by

$$\gamma^* = 2.1263. \tag{34}$$

The simulation result of this numerical example is shown in Figures 1–4. In this example, the initial value of the

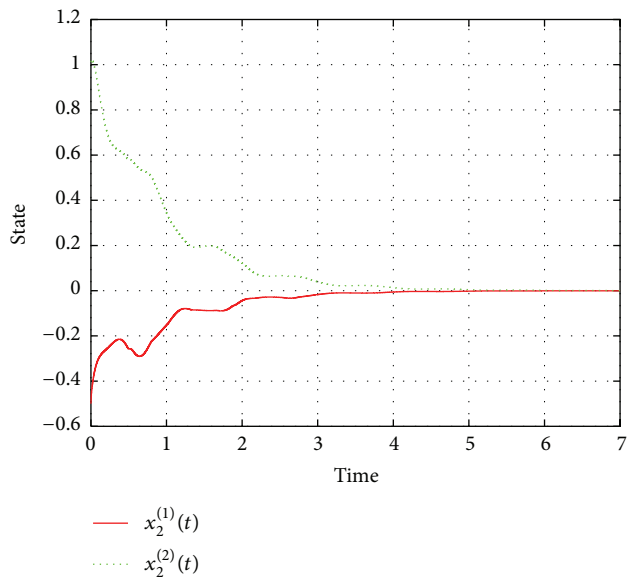

FIGURE 2: Time histories of $x_2(t)$.

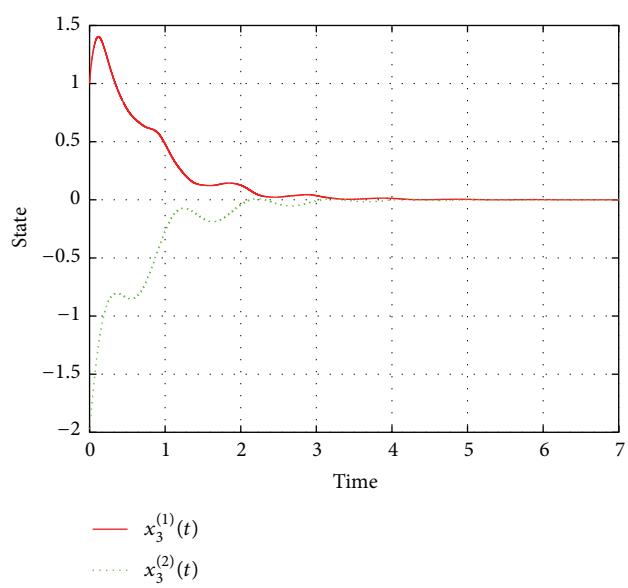

FIGURE 3: Time histories of $x_3(t)$.

uncertain large-scale system with system parameters of (30) is selected as follows:

$$x(0) = \left(1.0 \quad -1.0 \,\middle|\, -0.5 \quad 1.0 \,\middle|\, 1.0 \quad -2.0 \right)^T. \tag{35}$$

In these figures, $x_i^{(l)}(t)$ denotes the lth element of the state $x_i(t)$ for the ith subsystem, respectively. Furthermore, unknown parameters and disturbance input are given as

$$\begin{aligned} \Delta_{ii}(t) &= \cos(5\pi t), \\ \Delta_{ij}(t) &= -\sin(2\pi t), \\ \omega_i(t) &= 2.0\exp(-t)\cos(5\pi t). \end{aligned} \tag{36}$$

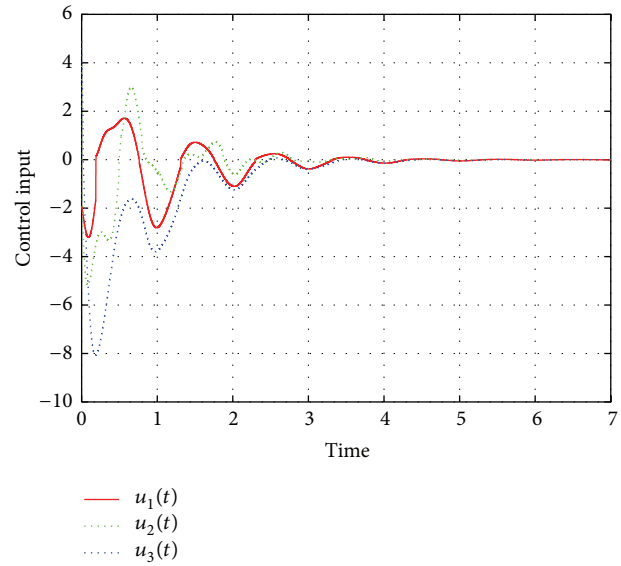

FIGURE 4: Time histories of $u_i(t)$.

From these figures, the proposed decentralized variable gain controller stabilizes the uncertain large-scale systems with system parameters of (30) in spite of uncertainties, interactions, and disturbance inputs. Therefore, the effectiveness of the proposed decentralized robust control system has been shown.

6. Conclusions

In this paper, on the basis of the result of [11, 18, 19], we have proposed a decentralized variable gain robust controller with guaranteed $\mathscr{L}_2$ gain performance for a large-scale interconnected system with uncertainties. For the uncertain large-scale interconnected systems, we have presented an LMI-based design method of the proposed decentralized variable gain robust controller. In addition, the effectiveness of the proposed decentralized robust controller has been shown by simple numerical examples. One can easily see that the result in this paper is an extension of the existing results [18, 19].

In the future research, we will extend the proposed controller synthesis to such a broad class of systems as large-scale systems with general uncertainties, uncertain large-scale systems with time delays, and so on.

Appendices

In this appendix, a decentralized variable gain robust controller with suboptimal $\mathscr{L}_2$ gain performance and the conventional fixed gain decentralized robust controller with $\mathscr{L}_2$ gain performance are presented.

A. Suboptimal Guaranteed $\mathscr{L}_2$ Gain Performance

In this section, we show a design method of a decentralized variable gain robust controller with suboptimal $\mathscr{L}_2$ gain performance.

Since the LMIs of (15) define a convex solution set, we consider minimizing the parameter γ_i because our interest is in establishing $\mathscr{L}_2$ gain performance. Furthermore, in the LMIs of (15), γ_i has no correlation with γ_j $(j \neq i)$. Thus our design problem can be replaced with the following constrained convex optimization problem (see [22, 23]):

$$\begin{aligned} &\underset{\mathscr{Y}_i>0,\mathscr{W}_i,\epsilon_i>0,\gamma_i>0}{\text{Minimize}} \quad [\gamma_i] \\ &\text{subject to} \quad (15). \end{aligned} \tag{A.1}$$

If the optimal solution $\mathscr{Y}_i > 0$, $\mathscr{W}_i, \epsilon_i > 0$, and $\gamma_i > 0$ of the constrained optimization problem of (A.1) is obtained, then the control input of (6) with the fixed gain matrix F_i and variable one $\mathscr{L}_i(x_i, t)$ of (16) is the decentralized variable gain robust control with suboptimal $\mathscr{L}_2$ gain performance γ^* of (12).

Consequently, the following theorem for the decentralized variable gain robust controller with suboptimal $\mathscr{L}_2$ gain performance is obtained.

Theorem A.1. *Let one consider the uncertain large-scale interconnected system composed of $\mathscr{N}$ subsystems of (2) and the control input of (6).*

The control input of (6) is the decentralized variable gain robust control with suboptimal $\mathscr{L}_2$ gain performance γ^* of (12) provided that the constrained convex optimization problem of (A.1) is feasible.

One can easily see that although the solution of LMIs of (15) is not unique, the decentralized robust controller obtained by solving the convex optimization problem of (A.1) is unique.

Note that the constrained optimization problem of (A.1) can be solved by software such as MATLAB's LMI Control Toolbox and Scilab's LMITOOL.

B. An LMI-Based Design Method for the Conventional Fixed Gain Decentralized Robust Controller with $\mathscr{L}_2$ Gain Performance

In this section, the conventional fixed gain decentralized robust controller with $\mathscr{L}_2$ gain performance is presented.

Consider the uncertain large-scale interconnected system composed of $\mathscr{N}$ subsystems of (2). For the ith subsystem of (2), we define the following control input:

$$u_i(t) = K_i x_i(t), \tag{B.1}$$

where $K_i \in \mathbb{R}^{m\times n}$ is the fixed gain matrix for the ith subsystem of (2). From (2), (3), and (B.1), the closed-loop subsystem of (B.2) can be obtained as

$$\begin{aligned} \frac{d}{dt}x_i(t) &= \left(A_{ii} + B_i K_i\right) x_i(t) + B_i \Delta_{ii} \mathscr{E}_{ii} x_i(t) \\ &\quad + B_i \sum_{\substack{j=1\\ j\neq i}}^{\mathscr{N}} \left(\mathscr{D}_{ij} + \Delta_{ij}\mathscr{E}_{ij}\right) x_j(t) + \Gamma_{x_i}\omega_i(t). \end{aligned} \tag{B.2}$$

Namely, the control objective in this section is to design the fixed gain matrices $K_i \in \mathbb{R}^{m\times n}$ of (B.1) such that the overall closed-loop system composed of subsystems of (B.2) achieves not only internal stability but also guaranteed $\mathscr{L}_2$ gain performance $\gamma^* > 0$.

In order to design the fixed gain matrices $K_i \in \mathbb{R}^{m\times n}$, the quadratic function $\mathscr{V}(x,t)$ of (8) and the Hamiltonian $\mathscr{H}(x,t)$ of (10) are defined. Additionally, we consider the inequality of (11). The time derivative of the quadratic function $\mathscr{V}_i(x_i,t)$ of (9) along the trajectory of the closed-loop subsystem of (B.2) can be written as

$$\begin{aligned} &\frac{d}{dt}\mathscr{V}_i(x_i,t) \\ &\quad = x_i^T(t)\left[H_e\left\{\left(A_{ii} + B_i K_i\right)^T \mathscr{P}_i\right\}\right] x_i(t) \\ &\qquad + H_e\left\{x_i^T(t)\mathscr{P}_i B_i \Delta_{ii}(t)\mathscr{E}_{ii}x_i(t)\right\} \\ &\qquad + H_e\left\{x_i^T(t)\mathscr{P}_i B_i \sum_{\substack{j=1\\ j\neq i}}^{\mathscr{N}} \mathscr{D}_{ij}x_j(t)\right\} \\ &\qquad + H_e\left\{x_i^T(t)\mathscr{P}_i B_i \sum_{\substack{j=1\\ j\neq i}}^{\mathscr{N}} \Delta_{ij}(t)\mathscr{E}_{ij}x_j(t)\right\} \\ &\qquad + H_e\left\{x_i^T(t)\mathscr{P}_i \Gamma_{x_i}\omega_i(t)\right\}. \end{aligned} \tag{B.3}$$

Besides, by using the well-known inequality of (19), we have the following relation for the function $\mathscr{V}_i(x_i,t)$:

$$\begin{aligned} \frac{d}{dt}\mathscr{V}_i(x_i,t) &\leq x_i^T(t)\left[H_e\left\{\left(A_{ii} + B_i K_i\right)^T \mathscr{P}_i\right\}\right] x_i(t) \\ &\quad + \epsilon_i x_i^T(t)\mathscr{P}_i B_i B_i^T \mathscr{P}_i x_i(t) \\ &\quad + \frac{1}{\epsilon_i} x_i^T(t)\mathscr{E}_{ii}^T\mathscr{E}_{ii}x_i(t) \\ &\quad + 2x_i^T(t)\mathscr{P}_i B_i \sum_{\substack{j=1\\ j\neq i}}^{\mathscr{N}} \mathscr{D}_{ij}x_j(t) \\ &\quad + \sum_{\substack{j=1\\ j\neq i}}^{\mathscr{N}} \epsilon_{ij} x_i^T(t)\mathscr{P}_i B_i B_i^T \mathscr{P}_i x_i(t) \\ &\quad + \sum_{\substack{j=1\\ j\neq i}}^{\mathscr{N}} \frac{1}{\epsilon_{ij}} x_j^T(t)\mathscr{E}_{ij}^T\mathscr{E}_{ij}x_j(t) \\ &\quad + H_e\left\{x_i^T(t)\mathscr{P}_i\Gamma_{x_i}\omega_i(t)\right\}, \end{aligned} \tag{B.4}$$

where ϵ_i and ϵ_{ij} are positive constants. Moreover, one can see that the following relation holds:

$$\sum_{i=1}^{\mathcal{N}} \left\{ \sum_{\substack{j=1\\ j\neq i}}^{\mathcal{N}} \left(x_j^T(t)\, x_j(t) - x_i^T(t)\, x_i(t) \right) \right\} = 0. \tag{B.5}$$

Therefore from (B.4) and (22) we find the following relation for the Hamiltonian $\mathcal{H}(x,t)$:

$$\begin{aligned}
\mathcal{H}(x,t) \leq \sum_{i=1}^{\mathcal{N}} \Bigg\{ & x_i^T(t) \left[H_e \left\{ (A_{ii} + B_i K_i)^T \mathcal{P}_i \right\} \right] x_i(t) \\
& + \epsilon_i x_i^T(t)\, \mathcal{P}_i B_i B_i^T \mathcal{P}_i x_i(t) + \frac{1}{\epsilon_i} x_i^T(t)\, \mathcal{E}_{ii}^T \mathcal{E}_{ii} x_i(t) \\
& + 2 x_i^T(t)\, \mathcal{P}_i B_i \sum_{\substack{j=1\\ j\neq i}}^{\mathcal{N}} \mathcal{D}_{ij} x_j(t) \\
& + \sum_{\substack{j=1\\ j\neq i}}^{\mathcal{N}} \epsilon_{ij} x_i^T(t)\, \mathcal{P}_i B_i B_i^T \mathcal{P}_i x_i(t) \\
& + \sum_{\substack{j=1\\ j\neq i}}^{\mathcal{N}} \frac{1}{\epsilon_{ij}} x_j^T(t)\, \mathcal{E}_{ij}^T \mathcal{E}_{ij} x_j(t) \\
& + H_e \left\{ x_i^T(t)\, \mathcal{P}_i \Gamma_{x_i} \omega_i(t) \right\} + x_i^T(t)\, C_{ii}^T C_{ii} x_i(t) \\
& + 2 x_i^T(t)\, C_{ii} \Gamma_{z_i} \omega_i(t) + \omega_i^T(t) \left(\Gamma_{z_i}^T \Gamma_{z_i} - \gamma_i I_{q_i} \right) \\
& - \sum_{\substack{j=1\\ j\neq i}}^{\mathcal{N}} \left(x_j^T(t)\, x_j(t) - x_i^T(t)\, x_i(t) \right) \Bigg\}.
\end{aligned} \tag{B.6}$$

Furthermore, some algebraic manipulations for (B.6) give the following inequality:

$$\begin{aligned}
& \mathcal{H}(x,t) \\
& \leq \begin{pmatrix} x_i(t) \\ x_1(t) \\ \vdots \\ x_{\mathcal{N}}(t) \\ \omega_i(t) \end{pmatrix}^T \Phi\left(\mathcal{P}_i, \epsilon_i, \epsilon_{ij}, \gamma_i \right) \begin{pmatrix} x_i(t) \\ x_1(t) \\ \vdots \\ x_{\mathcal{N}} \\ \omega_i(t) \end{pmatrix},
\end{aligned} \tag{B.7}$$

where $\Phi(\mathcal{P}_i, \epsilon_i, \epsilon_{ij}, \gamma_i)$ is given by

$$\begin{aligned}
& \Phi\left(\mathcal{P}_i, \epsilon_i, \epsilon_{ij}, \gamma_i \right) \\
& \triangleq \left(\begin{array}{c|c|c} \Theta_i\left(\mathcal{P}_i, \epsilon_i, \epsilon_{ij} \right) & \Pi_i\left(\mathcal{P}_i \right) & \mathcal{P}_i \Gamma_{x_i} \\ \hline \star & \mathcal{Q}_i & 0 \\ \hline \star & \star & \Gamma_{z_i}^T \Gamma_{z_i} - \gamma_i I_{q_i} \end{array} \right), \\
& \Theta_i\left(\mathcal{P}_i, \epsilon_i, \epsilon_{ij} \right) \triangleq H_e \left\{ (A_{ii} + B_i K_i)^T \mathcal{P}_i \right\} \\
& \quad + \epsilon_i \mathcal{P}_i B_i B_i^T \mathcal{P}_i + \frac{1}{\epsilon_i} \mathcal{E}_{ii}^T \mathcal{E}_{ii} + \sum_{\substack{j=1\\ j\neq i}}^{\mathcal{N}} \epsilon_{ij} \mathcal{P}_i B_i B_i^T \mathcal{P}_i \\
& \quad + C_{ii}^T C_{ii} + (\mathcal{N} - 1) I_n, \\
& \Pi_i\left(\mathcal{P}_i \right) \triangleq \left(\mathcal{P}_i B_i \mathcal{D}_{i1} \quad \mathcal{P}_i B_i \mathcal{D}_{i2} \quad \cdots \quad \mathcal{P}_i B_i \mathcal{D}_{i\mathcal{N}} \right), \\
& \mathcal{Q}_i \triangleq \operatorname{diag}\Bigg(-I_{n_1} + \frac{1}{\epsilon_{i1}} \mathcal{E}_{i1}^T \mathcal{E}_{i1}, -I_{n_2} + \frac{1}{\epsilon_{i2}} \mathcal{E}_{i2}^T \mathcal{E}_{i2}, \ldots, \\
& \quad - I_{n_{\mathcal{N}}} + \frac{1}{\epsilon_{i\mathcal{N}}} \mathcal{E}_{i\mathcal{N}}^T \mathcal{E}_{i\mathcal{N}} \Bigg).
\end{aligned} \tag{B.8}$$

Hence, if the following inequality holds, then

$$\Phi\left(\mathcal{P}_i, \epsilon_i, \epsilon_{ij}, \gamma_i \right) < 0. \tag{B.9}$$

Additionally, by using $\mathcal{S}_i \triangleq \mathcal{P}_i^{-1}$ and $\mathcal{W}_i \triangleq K_i \mathcal{S}_i$ and pre- and postmultiplying both sides of the matrix inequality of (B.9) by $\operatorname{diag}(\mathcal{S}_i, I_{n_i^\star}, I_{q_i})$, one can obtain

$$\left(\begin{array}{c|c|c} \Theta_i^\star\left(\mathcal{S}_i, \mathcal{W}_i, \epsilon_i, \epsilon_{ij} \right) & \Pi_i^\star & \Gamma_{x_i} \\ \hline \star & \mathcal{Q}_i & 0 \\ \hline \star & \star & \Gamma_{z_i}^T \Gamma_{z_i} - \gamma_i I_{q_i} \end{array} \right) < 0, \tag{B.10}$$

$$\begin{aligned}
& \Theta_i^\star\left(\mathcal{S}_i, \mathcal{W}_i, \epsilon_i, \epsilon_{ij} \right) \\
& \quad \triangleq H_e \left\{ A_{ii}^T \mathcal{S}_i + B_i \mathcal{W}_i \right\} + \epsilon_i B_i B_i^T + \frac{1}{\epsilon_i} \mathcal{S}_i \mathcal{E}_{ii}^T \mathcal{E}_{ii} \mathcal{S}_i \\
& \quad + \sum_{\substack{j=1\\ j\neq i}}^{\mathcal{N}} \epsilon_{ij} B_i B_i^T + \mathcal{S}_i C_{ii}^T C_{ii} \mathcal{S}_i + (\mathcal{N} - 1) \mathcal{S}_i \mathcal{S}_i,
\end{aligned} \tag{B.11}$$

$$\Pi_i^\star \triangleq \left(B_i \mathcal{D}_{i1} \quad B_i \mathcal{D}_{i2} \quad \cdots \quad B_i \mathcal{D}_{i\mathcal{N}} \right). \tag{B.12}$$

Moreover, by applying Lemma 2 (Schur complement) to (B.10), we obtain

$$\left(\begin{array}{c|c|c|c} \Theta_i^{\star\star}\left(\mathcal{S}_i, \mathcal{W}_i, \epsilon_i, \epsilon_{ij}\right) & \Pi_i^{\star} & \Gamma_{x_i} & \Lambda_i\left(\mathcal{S}_i\right) \\ \hline \star & -I_{n_i^{\star}} & 0 & 0 \\ \hline \star & \star & \Gamma_{z_i}^T \Gamma_{z_i} - \gamma_i I_{q_i} & 0 \\ \hline \star & \star & \star & \Omega_i^{\star}\left(\epsilon_i, \epsilon_{ij}\right) \end{array}\right) < 0, \tag{B.13}$$

$$\begin{aligned} \Theta_i^{\star\star}\left(\mathcal{S}_i, \mathcal{W}_i, \epsilon_i, \epsilon_{ij}\right) &\triangleq H_e\left\{A_{ii}\mathcal{S}_i + B_i\mathcal{W}_i\right\} + \epsilon_i B_i B_i^T \\ &\quad + \sum_{\substack{j=1 \\ j\neq i}}^{\mathcal{N}} \epsilon_{ij} B_i B_i^T, \\ \Lambda_i\left(\mathcal{S}_i\right) &\triangleq \left(\mathcal{S}_i C_{ii}^T \;\; \mathcal{S}_i \;\; \mathcal{S}_i \mathcal{E}_{ii}^T \;\; \mathcal{E}_{i1}^T \;\; \mathcal{E}_{i2}^T \;\; \cdots \;\; \mathcal{E}_{i\mathcal{N}}^T\right), \\ \Omega_i^{\star}\left(\epsilon_i, \epsilon_{ij}\right) &\triangleq -\operatorname{diag}\left(I_{p_i}, \frac{1}{\mathcal{N}-1} I_{n_i}, \epsilon_i I_{r_i}, \epsilon_{i1} I_{s_{i1}}, \ldots, \epsilon_{i\mathcal{N}} I_{i\mathcal{N}}\right). \end{aligned} \tag{B.14}$$

Namely, by solving the LMIs of (B.13), the fixed gain matrix is determined as $K_i = \mathcal{W}_i \mathcal{S}_i^{-1}$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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