Research Article

A Robust Control Method for Q-S Synchronization between Different Dimensional Integer-Order and Fractional-Order Chaotic Systems

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A robust control approach is presented to study the problem of Q-S synchronization between Integer-order and fractional-order chaotic systems with different dimensions. Based on Laplace transformation and stability theory of linear integer-order dynamical systems, a new control law is proposed to guarantee the Q-S synchronization between n -dimensional integer-order master system and m -dimensional fractional-order slave system. This paper provides further contribution to the topic of Q-S chaos synchronization between integer-order and fractional-order systems and introduces a general control scheme that can be applied to wide classes of chaotic and hyperchaotic systems. Illustrative example and numerical simulations are used to show the effectiveness of the proposed method.

1. Introduction

Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. The advantages or the real objects of the fractional-order systems are that we have more degrees of freedom in the model and that a “memory” is included in the model [1–6].

Recently, more and more attentions were paid to synchronization of integer-order chaotic systems and fractional-order chaotic systems. Many control methods have been proposed and different synchronization types have been studied between integer-order and fractional-order chaotic system. For example, general control schemes have been described in [7, 8]. A sliding mode method has been designed in [9–11]. A synchronization method of a class of hyperchaotic systems is given in [12]. In [13], a nonlinear feedback control method has been introduced and some robust observer techniques have been used in [14, 15]. Also, complete synchronization and antisynchronization have been observed, for example,

in [16–18], and function projective synchronization has been studied in [19]. Until now, a variety of control schemes have been proposed to study the problem of chaos synchronization between different dimensional systems such as modified function projective synchronization [20], generalized matrix projective synchronization [21], generalized synchronization [22–24], inverse generalized synchronization [25], full state hybrid projective synchronization [26], Q-S synchronization [27], increased order synchronization [28, 29], and reduced order generalized synchronization [30]. Amongst all kinds of synchronization, Q-S synchronization has been extensively considered [31–39], due to its universality and its great potential applications in applied sciences and engineering.

Motivated by the above discussions, the main aim of this paper is to present constructive scheme to investigate Q-S synchronization between n -dimensional integer-order master system and m -dimensional fractional-order slave system in n -D. By using Laplace transformation, stability property of integer-order linear systems, and suitable fractional-order control law, a new criterion is derived to achieve the Q-S

chaos synchronization between n -dimensional integer-order master system and m -dimensional fractional-order slave system in n -D. The proposed control method is simple, efficient, and easy to implement in applications. The outline of the rest of this paper is organized as follows. First, Section 2 provides some basic concepts of fractional derivative. In Section 3, the problem of Q-S synchronization between integer-order master system and fractional-order slave system is formulated. Our main result is presented in Section 4. In Section 5, numerical example is used to verify the effectiveness and feasibility of the proposed control scheme. Finally, Section 6 is the brief conclusion.

2. Basic Concepts of Fractional Derivative

There are several definitions of a fractional derivative of order $\alpha > 0$ [40–42]. The two most commonly used definitions are the Riemann-Liouville and Caputo definitions. Each definition uses Riemann-Liouville fractional integration and derivatives of whole order. The difference between the two definitions is in the order of evaluation. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of the function $f(t)$ is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad \alpha > 0, t > 0. \quad (1)$$

Some properties of the operator J^α can be found, for example, in [43, 44]. We recall only the following, for $\mu \geq -1$, $\alpha, \beta \geq 0$, and $\gamma > -1$; we have

$$\begin{aligned} J^\alpha J^\beta f(t) &= J^{\alpha+\beta} f(t), \\ J^\alpha t^\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}. \end{aligned} \quad (2)$$

In this study, Caputo definition is used and the fractional derivative of $f(t)$ is defined as

$$\begin{aligned} {}^C D_t^\alpha f(t) &= J^{m-\alpha} \left(\frac{d^m}{dt^m} f(t) \right) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, \end{aligned} \quad (3)$$

for $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, $t > 0$. The fractional differential operator ${}^C D_t^\alpha$ is left-inverse (and not right-inverse) to the fractional integral operator J^α ; that is, ${}^C D_t^\alpha J^\alpha = I$, where I is the identity operator. The Laplace transform of the Caputo fractional derivative rule reads

$$\begin{aligned} L\{{}^C D_t^\alpha f(t)\} &= s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \\ &(\alpha > 0, n-1 < \alpha \leq n). \end{aligned} \quad (4)$$

Particularly, when $\alpha \in (0, 1]$, we have $L\{{}^C D_t^\alpha f(t)\} = s^\alpha F(s) - s^{\alpha-1} f(0)$. The Laplace transform of the Riemann-Liouville fractional integral rule satisfies

$$L\{J^\alpha f(t)\} = s^{-\alpha} F(s), \quad (\alpha > 0). \quad (5)$$

Caputo fractional derivative appears more suitable to be treated by the Laplace transform technique in that it requires the knowledge of the (bounded) initial values of the function and of its integer derivatives of order $k = 1, 2, \dots, m-1$, in analogy with the case when $\alpha = n$.

3. Q-S Synchronization Problem for Integer-Order and Fractional-Order Chaotic System

We assume that the master chaotic systems can be considered in the following form:

$$\dot{X}(t) = AX(t) + F(X(t)), \quad (6)$$

where $X(t) \in \mathbb{R}^n$ is the state vector of the master system (6), $A \in \mathbb{R}^{n \times n}$ is a constant matrix, and $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a nonlinear vector function. Also, consider the slave chaotic system as

$${}^C D_t^p Y(t) = G(Y(t)) + U, \quad (7)$$

where $Y(t) \in \mathbb{R}^m$ is the state vector of the slave system (7), $G: \mathbb{R}^m \rightarrow \mathbb{R}^m$, p is a rational number between 0 and 1, ${}^C D_t^p$ is the Caputo fractional derivative of order p , and $U = (u_i)_{1 \leq i \leq m}$ is a vector controller. The problem of Q-S synchronization for the master system (6) and the slave system (7) in dimension n is to find the controller U such that the synchronization error,

$$e(t) = Q(Y(t)) - S(X(t)), \quad (8)$$

satisfies that

$$\lim_{t \rightarrow +\infty} \|e(t)\| = 0, \quad (9)$$

where $Q: \mathbb{R}^m \rightarrow \mathbb{R}^n$, $S: \mathbb{R}^n \rightarrow \mathbb{R}^n$ are continuously differentiable functions. We assume that $n < m$.

4. General Control Scheme

The integer-order derivative of the error system (8) can be derived as

$$\dot{e}(t) = DQ(Y(t)) \times \dot{Y}(t) - DS(X(t)) \times \dot{X}(t), \quad (10)$$

where $DQ(Y(t)) \in \mathbb{R}^{n \times m}$, $DS(X(t)) \in \mathbb{R}^{n \times n}$ are the Jacobian matrices of the functions Q and S , respectively,

$$DQ(Y(t)) = \begin{pmatrix} \frac{\partial Q_1}{\partial y_1} & \frac{\partial Q_1}{\partial y_2} & \dots & \frac{\partial Q_1}{\partial y_m} \\ \frac{\partial Q_2}{\partial y_1} & \frac{\partial Q_2}{\partial y_2} & \dots & \frac{\partial Q_2}{\partial y_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial Q_n}{\partial y_1} & \frac{\partial Q_n}{\partial y_2} & \dots & \frac{\partial Q_n}{\partial y_m} \end{pmatrix},$$

$$DS(X(t)) = \begin{pmatrix} \frac{\partial S_1}{\partial x_1} & \frac{\partial S_1}{\partial x_2} & \dots & \frac{\partial S_1}{\partial x_n} \\ \frac{\partial S_2}{\partial x_1} & \frac{\partial S_2}{\partial x_2} & \dots & \frac{\partial S_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial S_n}{\partial x_1} & \frac{\partial S_n}{\partial x_2} & \dots & \frac{\partial S_n}{\partial x_n} \end{pmatrix}. \quad (11)$$

Hence, we have the following result.

Theorem 1. *There exists a suitable feedback gain matrix $C \in \mathbb{R}^{n \times n}$ to realize the Q-S synchronization between the master system (6) and the slave system (7) in n -D under the following controller:*

$$U = -G(Y(t)) + J^{1-P}(G(Y(t)) + V), \quad (12)$$

where the vector quantity $V = (V_i)_{1 \leq i \leq m}$ is defined as

$$(V_1, V_2, \dots, V_n)^T = -M^{-1}R, \quad (13)$$

$$(V_{n+1}, V_{n+2}, \dots, V_m)^T = 0, \quad (14)$$

where M^{-1} is the inverse of matrix M ,

$$M = \begin{pmatrix} \frac{\partial Q_1}{\partial y_1} & \frac{\partial Q_1}{\partial y_2} & \dots & \frac{\partial Q_1}{\partial y_n} \\ \frac{\partial Q_2}{\partial y_1} & \frac{\partial Q_2}{\partial y_2} & \dots & \frac{\partial Q_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial Q_n}{\partial y_1} & \frac{\partial Q_n}{\partial y_2} & \dots & \frac{\partial Q_n}{\partial y_n} \end{pmatrix}, \quad (15)$$

$$R = (C - A)e(t) + DQ(Y(t)) \times G(Y(t)) - DS(X(t)) \times (AX(t) + F(X(t))). \quad (16)$$

Proof. By inserting the control law described by (12) into (7), we can rewrite the slave system as follows:

$${}^C D_t^P Y(t) = J^{1-P}(G(Y(t)) + V). \quad (17)$$

Applying the Laplace transform to (17) and letting $F(s) = L(Y(t))$, we obtain

$$s^P F(s) - s^{P-1} Y(0) = s^{P-1} L(G(Y(t)) + V); \quad (18)$$

multiplying both the left-hand and right-hand sides of (18) by s^{1-P} and applying the inverse Laplace transform to the result, we obtain a new equation for the slave system

$$\dot{Y}(t) = G(Y(t)) + V. \quad (19)$$

Now, the error system (10) can be described as

$$\begin{aligned} \dot{e}(t) &= (A - C)e(t) + DQ(Y(t)) \times G(Y(t)) \\ &+ (C - A)(Q(Y(t)) - S(X(t))) - DS(X(t)) \\ &\times (AX(t) + F(X(t))) + DQ(Y(t)) \times V, \end{aligned} \quad (20)$$

FIGURE 1: The chaotic attractor of Lorenz system when $(\alpha, \beta, \gamma) = (10, 28, -8/3)$.

where $C \in \mathbb{R}^{n \times n}$ is a feedback gain matrix to be chosen. The error system (20) can be simplified as follows:

$$\dot{e}(t) = (A - C)e(t) + R + DQ(Y(t)) \times V, \quad (21)$$

where R was defined by (16). By using (14), the error system (21) can be written as

$$\dot{e}(t) = (A - C)e(t) + R + M \times (V_1, V_2, \dots, V_n)^T. \quad (22)$$

Hence, by substituting (13) into (22), we get

$$\dot{e}(t) = (A - C)e(t). \quad (23)$$

With respect to the asymptotic stability property of linear continuous-time dynamical systems, if the feedback gain matrix C is selected such that all eigenvalues of $A - C$ are strictly negative, it is immediate that all solutions of the error system (23) go to zero as $t \rightarrow \infty$. Therefore, systems (6) and (7) are globally Q-S synchronized in n -D. $\square$

5. Illustrative Example

In this section, to validate the Q-S synchronization method proposed in the previous section, we consider the Lorenz system as a master system and the fractional-order hyperchaotic Lorenz system as a slave system. The Lorenz system can be described as

$$\begin{aligned} \dot{x}_1 &= \alpha(x_2 - x_1), \\ \dot{x}_2 &= \beta x_1 - x_2 - x_1 x_3, \\ \dot{x}_3 &= \gamma x_3 + x_1 x_2, \end{aligned} \quad (24)$$

FIGURE 2: Chaotic attractor projections of fractional-order hyperchaotic Lorenz system when $p = 0.98$ and $(a, b, c, r) = (10, 8/3, 28, -1)$.

which has a chaotic attractor, for example, when $(\alpha, \beta, \gamma) = (10, 28, -8/3)$ [45]. Then the linear part $A = (a_{ij})_{3 \times 3}$ and the nonlinear part F of the Lorenz system (24) are given by

$$A = \begin{pmatrix} -\alpha & \alpha & 0 \\ \beta & -1 & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \quad (25)$$

$$F = \begin{pmatrix} 0 \\ -x_1 x_3 \\ x_1 x_2 \end{pmatrix}.$$

The Lorenz chaotic attractor is shown in Figure 1.

The controlled fractional-order hyperchaotic Lorenz system can be described by the following nonlinear fractional-order ODE:

$$D^p y_1 = a(y_2 - y_1) + y_4 + u_1,$$

$$D^p y_2 = cy_1 - y_2 - y_1 y_3 + u_2,$$

$$D^p y_3 = -by_3 + y_1 y_2 + u_3,$$

$$D^p y_4 = ry_4 - y_2 y_3 + u_4,$$

(26)

where u_i , $i = 1, 2, 3, 4$, are the controllers. This system, as shown in [46], exhibits hyperchaotic behaviors when $p = 0.98$ and $(a, b, c, r) = (10, 8/3, 28, -1)$. Chaotic attractor projections of the fractional-order hyperchaotic Lorenz system are shown in Figure 2.

According to Q-S synchronization control technique proposed in the previous section, the functions $Q: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ and $S: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ are selected as follows:

$$Q(y_1, y_2, y_3, y_4) = \left(y_1 + \frac{1}{3}y_1^3 + y_4, y_2 + 2y_4, y_3 + 3y_4 \right)^T, \quad (27)$$

$$S(x_1, x_2, x_3) = (x_1, x_2, x_3 + x_1)^T.$$

Then,

$$\mathbf{DQ}(y_1, y_2, y_3, y_4) = \begin{pmatrix} 1 + y_1^2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}, \quad (28)$$

$$\mathbf{DS}(x_1, x_2, x_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

It is easy to show that if we choose the feedback gain matrix C as

$$C = \begin{pmatrix} 0 & 10 & 0 \\ 28 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (29)$$

then the eigenvalues of the matrix $A - C$ are strictly negative. Then the controllers u_1, u_2, u_3 , and u_4 can be designed as follows:

$$\begin{aligned} u_1 &= J^{0.02} \left[-\alpha e_1(t) \right. \\ &\quad \left. - \frac{1}{1 + y_1^2} (ry_4 - y_2y_3 - \alpha(x_2 - x_1)) \right] - a(y_2 \\ &\quad - y_1) - y_4, \\ u_2 &= -cy_1 + y_2 + y_1y_3 + J^{0.02} [-e_2(t) - 2ry_4 + 2y_2y_3 \\ &\quad + \beta x_1 - x_2 - x_1x_3], \\ u_3 &= by_3 - y_1y_2 + J^{0.02} [-\gamma e_3(t) - 3ry_4 + 3y_2y_3 \\ &\quad + \alpha(x_2 - x_1) + \gamma x_3 + x_1x_2], \\ u_4 &= -ry_4 + y_2y_3 + J^{0.02} [ry_4 - y_2y_3], \end{aligned} \quad (30)$$

and the error system can be written as

$$\begin{aligned} \dot{e}_1 &= -10e_1, \\ \dot{e}_2 &= -e_2, \\ \dot{e}_3 &= -\frac{8}{3}e_3. \end{aligned} \quad (31)$$

Therefore, according to Theorem 1, systems (24) and (26) are globally Q-S synchronized in 3D. The numerical simulations of the error functions evolution are shown in Figure 3.

6. Conclusion

In this paper, new control scheme, based on fractional control law, Laplace transformation, and stability theory of linear integer-order dynamical system, for Q-S synchronization was presented between n -dimensional integer-order master

FIGURE 3: Time evolution of Q-S synchronization errors between systems (24) and (26).

system and m -dimensional fractional-order slave system. To observe Q-S synchronization with respect to dimension n , the synchronization criterion was obtained via controlling the linear part of the master system. Numerical example and simulations result were used to verify the effectiveness of the proposed control method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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