A kind of robust fault diagnosis algorithm to Lipschitz nonlinear system is proposed. The novel disturbances constraint condition of the nonlinear system is derived by group algebra method, and the novel constraint condition can meet the system stability performance. Besides, the defined robust performance index of fault diagnosis observer guarantees the robust. Finally, the effectiveness of the algorithm proposed is proved in the simulations.

1. Introduction

The complicated systems as aircraft, missile systems, and control system [1–6] easily show faults; how to diagnose the fault is a difficult problem to handle. When the systems show fault and you cannot isolate it, the systems will be collapsed. Therefore, the fault diagnosis and fault tolerant technology are meaningful to enhance the system performances. And fault diagnosis technology is foundation of the fault tolerance; in another way, fault tolerance is realized by the fault estimations information. The original control laws will be regulated by the fault estimation information, so the system reliability will be improved, particularly the missile control system. In this paper, we aim at the missile attitude control system. Robust fault diagnosis methods are useful ways to solve the systems with the disturbances and they are also proved effective in systems applications.

The robust fault diagnosis observer is designed based on unknown input observer theory in [7–9], and state estimation errors decouple from disturbances. Most of papers make assumptions that the disturbances constraint condition is known; this assumption limits the algorithm applications.

Dealing with the deficiency on assumption that the disturbance is norm bounded, a novel constraint condition for disturbance is designed. Besides, the defined robust performance index of fault diagnosis observer guarantees the robust. Furthermore, the threshold is designed in fault decision section.

2. Problem Statement

Consider the system uncertainty and unknown input disturbances, system state-space model:

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t) + h(x(t), u(t)) + g_0(x(t), u(t), d(t), t) + f(t),$$

$$y(t) = Cx(t).$$

$E_1, E_2, F_1,$ and $F_2$ are known matrix. $\Delta A$ and $\Delta B$ are model errors.

Therefore, we can get another form as

$$\dot{x}(t) = Ax(t) + Bu(t) + h(x(t), u(t)) + g(x(t), u(t), d(t), t) + f(t),$$

where $x(0) = x_0$ is initial value, $x(t) \in \mathbb{R}^n$ is state, $u(t) \in \mathbb{R}^p$ and $y(t) \in \mathbb{R}^q$ are system input and output, $g \in \mathbb{R}^n$ is unknown input, nonlinear functions $g$ and $h$ are continuous.
3. Stability Analysis

**Theorem 1.** There exists \( M \geq 1, \omega < 0, t \geq 0 \) to make nonlinear system (2) and (3) with external disturbances hold stable, when the constraint condition of external disturbances is

\[
\| g(x(t), u(t), d(t), t) \| < \left( \beta_1(t) - \beta_0(t) \right) - \frac{\omega \| x(t) \|}{M}
\]

and is defined as \( \beta_0(t) = \| Bu(t) \|, \beta_1(t) = \| h(x(t), u(t)) \| \).

**Proof.** Matrix \( A \) is Hurwitz matrix and \( A \) can generate asymptotic convergence linear semigroup \( \zeta_t \). Therefore, there exists \( M \geq 1, \omega < 0, t \geq 0 \) such that the inequality holds:

\[
\| \zeta_t \| \leq M \exp(\omega t).
\]

Fault-free mode, state-space description of system (2) and (3) with external disturbances is

\[
x(t) = x(0) + \int_0^t \left[ Ax(\tau) + Bu(\tau) + h(x(\tau), u(\tau)) 
\right. \\
\left. + g(x(\tau), u(\tau), d(\tau), t) \right] d\tau.
\]

Therefore, \( \exists \) stable linear semigroup \( \zeta_t \) makes (8) hold:

\[
x(t) = \zeta_t x(0) + \int_0^t \left[ Bu(\tau) + h(x(\tau), u(\tau)) 
\right. \\
\left. + g(x(\tau), u(\tau), d(\tau), t) \right] d\tau.
\]

Apply the 2-norm to formula (8):

\[
\| x(t) \| = \left\| \zeta_t x(0) + \int_0^t \left[ Bu(\tau) + h(x(\tau), u(\tau)) 
\right. \\
\left. + g(x(\tau), u(\tau), d(\tau), t) \right] d\tau \right\|.
\]

The simplification form of formula (9) is

\[
\Xi = Bu(\tau) + h(x(\tau), u(\tau)) \\
+ g(x(\tau), u(\tau), d(\tau), t).
\]

Initial value of state is \( x_0 = x(0) \), we set \( \alpha = \| x(0) \| \). From constraint of linear semigroup \( \zeta_t \) in formula (6) and norm
The basic principle, the inequality constraint condition of formula (9) should be satisfied as follows:

\[ \|x(t)\| \leq M a \exp(\omega t) + \int_0^t \|M \exp(\omega(t - \tau))\| \Xi d\tau. \] (11)

Multiplied by \( \exp(-\omega t) \) for two sides of formula (11):

\[ \|x(t)\| \exp(-\omega t) \leq Ma + \int_0^t \|M \exp(\omega(t - \tau))\| \Xi \|x(\tau)\| d\tau, \] (12)

\[ \|x(t)\| \exp(-\omega t) \leq Ma \]

By Gronwall lemma,

\[ \|x(t)\| \leq M a \exp \left( \int_0^t \omega + \frac{\|M\|}{\|x(\tau)\|} \|x(\tau)\| d\tau \right). \] (13)

For simplification, we define as follows:

\[ \vartheta(t) = \int_0^t \left( \omega + \frac{\|M\|}{\|x(\tau)\|} \right) d\tau. \] (14)

The system is stable for \( \forall t \to +\infty, \lim_{\tau \to +\infty} \vartheta(t) < |\epsilon| < +\infty \) when there exists finite constant \( |\epsilon| < \infty \). There exists nonlinear semigroup \( \pi_\tau \) such that state-space of system is described by the following from formula (9):

\[ x(t) = \pi_\tau x(0). \] (15)

From formulas (13) and (14):

\[ \|x(t)\| = \|\pi_\tau x(0)\| \leq M a \exp(\vartheta(t)). \] (16)

Therefore, the nonlinear semigroup \( \pi_\tau \) is stable when \( \vartheta(t) < 0 \); in other words, system (2) and (3) with external disturbances is stable. As a result, the system holds stable when the following condition is satisfied:

\[ \|\Xi\| < \frac{-\omega \|x(\tau)\|}{M}. \] (17)

Substitute the formula above into (10):

\[ Bu(t) + h(x(t), u(t)) + g(x(t), u(t), d(t), t) \]

\[ < \frac{-\omega \|x(\tau)\|}{M}. \] (18)

And then

\[ Bu(t) + h(x(t), u(t)) + g(x(t), u(t), d(t), t) \]

\[ > \|Bu(t)\| - \|h(x(t), u(t))\| \]

\[ + \|g(x(t), u(t), d(t), t)\|, \] (19)

where, \( u(t) \) and \( h(x(t), u(t)) \) are known. With definition \( \beta_0(t) = \|Bu(t)\|, \beta_1(t) = \|h(x(t), u(t))\| \).

System states estimation errors and observer residuals are

\[ e(t) = x(t) - \hat{x}(t), \]

\[ r(t) = y(t) - \hat{y}(t). \] (25)

Substitute into the formula above:

\[ \beta_0(t) - \beta_1(t) + \frac{\|g(x(t), u(t), d(t), t)\|}{M} \]

\[ < -\frac{\omega \|x(\tau)\|}{M}, \]

\[ \frac{\|g(x(t), u(t), d(t), t)\|}{M} < \frac{\beta_3(t) - \frac{\omega \|x(\tau)\|}{M}}{M}. \] (20)

The constraint condition of external disturbances satisfies the inequality

\[ \|g(x(t), u(t), d(t), t)\| < \frac{\beta_3(t) - \frac{\omega \|x(\tau)\|}{M}}{M}. \] (22)

The robustness performance index to external disturbances is defined as \( R(A) = \omega / M \).

4. Robust Fault Diagnosis Algorithm

**Theorem 2.** The robust fault diagnosis observer (24) of system (2) and (3) with fault \( \beta_3 = \sup_{t \in [0,T]} \|f(t)\| \) and external disturbances is asymptotic convergence. Therefore, the robust performance index of observer satisfies the inequality as follows:

\[ -R(A - GC) > \lambda_1 \]

\[ + \left( \frac{\|g(x(\tau), u(\tau), d(\tau), \tau)\| + \beta_3}{\|e(\tau)\|} \right). \] (23)

**Proof.** Construct robust fault diagnosis observer as follows for missile pitching motion control system (2) and (3) with fault and external disturbances:

\[ \dot{x}(t) = A\bar{x}(t) + Bu(t) + h(x(t), u(t)) \]

\[ + G \left[ y(t) - \bar{y}(t) \right], \] (24)

\[ \bar{y}(t) = C\bar{x}(t). \]

System states estimation errors and observer residuals are

\[ e(t) = x(t) - \hat{x}(t), \]

\[ r(t) = y(t) - \hat{y}(t). \] (25)
And therefore,
\[ \dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = Ax(t) - A\hat{x}(t) + h(x(t), u(t)) - h(\hat{x}(t), u(t)) - g(x(t), u(t), d(t), t) + f(t), \]
(26)
\[ \dot{e}(t) = (A - GC) e(t) + h(x(t), u(t)) - h(\hat{x}(t), u(t)) + g(x(t), u(t), d(t), t) + f(t), \]
\[ r(t) = y(t) - \hat{y}(t) = Ce(t). \]

(35)

Therefore,
\[ r(t) = Ce(0) + C \int_{0}^{t} \left[ h(x(t), u(t)) - h(\hat{x}(t), u(t)) + g(x(t), u(t), d(t), t) \right] d\tau. \]
(36)

Apply the 2-norm to both sides of (27):
\[ \|e(t)\| = \left\| \exp \left[ (A - GC) t \right] e(0) \right\| + \int_{0}^{t} \left\| \exp \left[ (A - GC) (t - \tau) \right] \Psi d\tau \right\|. \]
(27)

(32)

The system matrix \( A - GC \) is Hurwitz matrix when system (26) is stable; therefore, it can generate a stable linear semigroup \( \xi_t \).

Consequently, there exists \( M \geq 1, \omega < 0, t \geq 0 \) such that \( \|\xi_t\| \leq M \exp(\omega t) \). Therefore, formula (28) fulfills the inequality as follows:
\[ \|e(t)\| \leq Mb \exp(\omega t) + \int_{0}^{t} M \exp(\omega (t - \tau)) \|\Psi\| d\tau. \]
(29)

Multiplier by \( \exp(-\omega t) \) for formula (29):
\[ \|e(t)\| \exp(-\omega t) \leq Mb + \int_{0}^{t} M \exp(-\omega \tau) \|\Psi\| d\tau, \]
\[ \|e(t)\| \exp(-\omega t) \leq Mb \]
(30)

\[ + \int_{0}^{t} \left\| Me(\tau) \exp(-\omega \tau) \Psi \right\| \frac{d\tau}{\|e(\tau)\|} d\tau. \]

Generally, the fault injected into the missile pitching motion control system is \( \beta_2 = \sup_{\tau \in [0,T]} \|f(\tau)\|: \)
\[ \|\Psi\| \leq \|h(x(\tau), u(\tau)) - h(\hat{x}(\tau), u(\tau))\| + \|f(\tau)\| \]
\[ + \|g(x(\tau), u(\tau), d(\tau), \tau)\| \]
\[ \leq \lambda_1 \|e(\tau)\| + \beta_2 \]
\[ + \|g(x(\tau), u(\tau), d(\tau), \tau)\| \]
\[ \|e(\tau)\| \leq Mb \exp \left\{ \int_{0}^{t} \frac{\omega + M \|\Psi\|}{\|e(\tau)\|} d\tau \right\}. \]

Consequently, the fault diagnosis observer (24) is asymptotic convergence when the following formula holds:
\[ \omega + \frac{M \|\Psi\|}{\|e(\tau)\|} < 0, \]
\[ \omega M > \lambda_1 + \left( \frac{\|g(x(\tau), u(\tau), d(\tau), \tau)\| + \beta_2}{\|e(\tau)\|} \right). \]

The performance index \( \Re(A - GC) \) of observer (24) satisfies the following constraint condition from stability theory:
\[ - \Re(A - GC) > \lambda_1 \]
\[ + \left( \frac{\|g(x(\tau), u(\tau), d(\tau), \tau)\| + \beta_2}{\|e(\tau)\|} \right). \]

Consequently, the fault diagnosis observer is asymptotic convergence when formula (33) holds. And then, fault diagnosis observer can be realized by robust performance index proposed. Consider
\[ \lambda_1 \left[ (A - GC) + (A - GC)^T \right] \]
\[ < -2 \left( \lambda_1 + \left( \frac{\|g(x(\tau), u(\tau), d(\tau), \tau)\| + \beta_2}{\|e(\tau)\|} \right) \right), \]
(34)

where \( \lambda_{\text{max}}(\cdot) \) and \( \lambda_{\text{min}}(\cdot) \) represent the maximum and arbitrary eigenvalue for matrix (\( \cdot \)); the gain matrix \( G \) of the observer can be solved by pole assignment when the robust performance index \( \Re(A - GC) \) is given.

**5. Adaptive Threshold Design**

Usually, compare the residuals with threshold to diagnose fault.

The states estimation errors are
\[ e(t) = e(0) + \int_{0}^{t} [h(x(\tau), u(\tau)) - h(\hat{x}(\tau), u(\tau))] \]
\[ + g(x(\tau), u(\tau), d(\tau), \tau) \] \] d\tau.
(35)

Therefore,
\[ r(t) = Ce(0) + C \int_{0}^{t} [h(x(\tau), u(\tau)) - h(\hat{x}(\tau), u(\tau))] \]
\[ - h(\hat{x}(\tau), u(\tau)) + g(x(\tau), u(\tau), d(\tau), \tau) \] d\tau.
(36)
Apply the 2-norm for formula (36):

$$\|r(t)\| = \left\| Ce(0) + C \int_0^t [h(x(\tau), u(\tau)) - h(\bar{x}(\tau), u(\tau))] d\tau \right\|$$

(37)

with definition $b_c = \|C\|$ and

$$\delta(h) = h(x(\tau), u(\tau)) - h(\bar{x}(\tau), u(\tau)).$$

(38)

Therefore,

$$\|r(t)\| \leq b_c$$

$$\leq b_c \left| \int_0^t [h(\bar{x}(\tau), u(\tau)) + g(x(\tau), u(\tau), d(\tau), \tau)] d\tau \right|.$$  

(39)

We can get from (38)

$$\|\delta(h)\| \leq \lambda_1 \|x(\tau) - \bar{x}(\tau)\| = \lambda_1 \|e(\tau)\|$$

$$= \frac{\lambda_1}{b_c} \|r(\tau)\|. $$

(40)

As a result,

$$\|r(t)\| \leq b_c$$

$$\leq b_c \left[ \int_0^t \|e(\tau)\| d\tau + b_c \|g(x(\tau), u(\tau), d(\tau), \tau)\| d\tau \right].$$

(41)

We can get from (41)

$$\|r(t)\| \leq b_c \left( \int_0^t \lambda_1 b_c \|e(\tau)\| d\tau \right)$$

$$\leq b_c \left[ \int_0^t \lambda_1 b_c \|e(\tau)\| d\tau + \int_0^t [\beta_1(\tau) - \beta_0(\tau)] d\tau \right].$$

(42)

where maximum tolerant values of estimation errors $e_{\text{max}}$ and the system stable value $x_\text{sta}$ are known:

$$e_{\text{max}} = \|e_{\text{max}}(t)\|,$$

$$\lim_{t \to \infty} \|x(t)\| \to x_\text{sta}. $$

(43)

It can be obtained from (42) that

$$\|r(t)\| \leq b_c \left[ \int_0^t \lambda_1 b_c \|e_{\text{max}}(\tau)\| d\tau \right]$$

$$\left. + b_c \int_0^t [\beta_1(\tau) - \beta_0(\tau)] d\tau \right].$$

(44)

As a result, the adaptive threshold of the fault diagnosis observer designed is

$$J_0(r(t)) = b_c + \left( \lambda_1 b_c \|e_{\text{max}}(\tau)\| - \frac{\omega x_{\text{max}}}{M} \right) t$$

$$+ b_c \int_0^t [\beta_1(\tau) - \beta_0(\tau)] d\tau.$$ 

(45)

6. Simulation

In order to verify the effectiveness of the algorithm proposed, the following simulations are performed.

6.1. Simulation Parameters. The differential equations of missile pitching motion control system are represented as follows [6].

The pitching moment: $M_x = \varphi_1(ma, \alpha) + \varphi_2(ma, \alpha)\delta_x + \varphi_3(\alpha, \omega_3, v)$. The missile aerodynamic parameters are as follows: $X(ma, \alpha)$, $Y(ma, \alpha)$, $\varphi_1(ma, \alpha)$, $\varphi_2(ma, \alpha)$, and $\varphi_3(\alpha, \omega_3, v)$. The force situation of missile with different flight states is different.

The missile empty weight is 230 kg, $z$-axis rotational inertia is $I_z = 247.26$ kg m$^2$, and generator impulse thrust of missile attitude control system is $p = 2200$ N. The initial location of missile in inertial coordinates system is $x_0 = 8550$ m, $y = 11600$ m; missile initial velocity is $v = 300$ m/s, and trajectory pointing angle is $\theta = 0.536$ rad.

The system disturbances are $d_i(t) = 0.01 \sin(t), i = 1, 2, \ldots, 7$. The constraint condition of the external disturbances can be derived by Section 3.

6.2. Performance Analysis for the Fault Diagnosis Algorithm. The supreme of missile external disturbances in the considered period under the Simulink condition from Theorem 1 is $\|g\|_{\text{max}} = 9.69507 \times 10^3$. Therefore, not all of the disturbances can satisfy norm bounded constraint conditions and the prior constraint condition on external disturbances restricts the generality of fault diagnosis applications. The maximum value of estimation errors is defined as

$$e_{\text{max}} = [40 0.01 2 20 20 0.02 0.02].$$

(46)

Therefore, $\|e\|_{\text{max}} = 1.202 \times 10^3$. From Theorem 2,

$$\lambda_1 (\mathbf{A} - \mathbf{GC}) + (\mathbf{A} - \mathbf{GC})^T < -806.5786.$$ 

(47)

The fault diagnosis observer designed is asymptotic convergence robust fault diagnosis observer when the poles are placed at the left plane of $-806.5786$. It should be noticed that the gain matrix $\mathbf{G}$ is not unique because of the poles selected; in this paper, the poles are settled on

$$p = [-850 + 5i, -850 - 5i, -880 + 6i, -880 - 6i,$$

$$-900 + 10i, -900 - 10i, -1000].$$

(48)

The gain matrix $\mathbf{G}$ of the observer can be get through the pole assignment.

As a result, the missile trajectory character is depicted in Figure 3 and substitute the gain matrix $\mathbf{G}$ into the observer (26) and then the residual effect is depicted in Figure 4 by Simulink. Without loss of generality, just take the 3rd channel residual of fault diagnosis as an example to research and assume the missile attitude control system.

Residual is asymptotic convergence and therefore it has the robustness to disturbances from Figure 4. The good convergence of residual illustrates that the algorithm proposed is effective. Furthermore, the smooth residual curve also
illustrates that the disturbances constraint condition which can satisfy the system stability is reasonable and the defined robust performance index is practicable.

7. Conclusion

In this paper, novel disturbances constraint condition is derived to improve the limitation that external disturbance is norm bounded. And then, the novel constraint condition can meet the system stability. Besides, the defined robust performance index of fault diagnosis observer guarantees the robust. In decision-making unit, adaptive threshold is designed. Finally, simulation results show the effectiveness of the algorithm proposed.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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