Research Article

Optimal Control Problem Investigation for Linear Time-Invariant Systems of Fractional Order with Lumped Parameters Described by Equations with Riemann-Liouville Derivative

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Received 29 November 2015; Accepted 3 May 2016

Academic Editor: Francisco Gordillo

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This paper studies two optimal control problems for linear time-invariant systems of fractional order with lumped parameters whose dynamics is described by equations which contain Riemann-Liouville derivative. The first problem is to find control with minimal norm and the second one is to find control with minimal control time at given restriction for control norm. The problem setting with nonlocal initial conditions is considered which differs from other known settings for integer-order systems and fractional-order systems described in terms of equations with Caputo derivative. Admissible controls are allowed to belong to the class of functions which are p -integrable on half segment. The basic investigation approach is the moment method. The correctness and solvability of moment problem are validated for considered problem setting for the system of arbitrary dimension. It is shown that corresponding conditions are analogous to those derived for systems which are described in terms of equations with Caputo derivative. For several particular cases of one- and two-dimensional systems the posed problems are solved explicitly. The dependencies of basic values from derivative index and control time are analyzed. The comparison is performed of obtained results with known results for analogous integer-order systems and fractional-order systems which are described by equations with Caputo derivative.

1. Introduction

Affairs of dynamics and control for fractional-order systems attract sufficient attention of modern research community. This field develops impetuously and is characterized by both of significant theoretical and very actual applied results. And this field is much wider than integer-order dynamics and contains some open problems concerning the foundations of fractional calculus (e.g., the problem of unified definition of fractional derivative and its interpretation). It is known that Caputo derivative and Riemann-Liouville derivative are the most popular definitions of fractional-order derivative. First of them is recognized by many researches as more “physical”: more realistic and similar to ordinary derivative. But this definition imposes rather essential requirements (differentiability) on function from which the fractional derivative is calculated. The Riemann-Liouville definition is

used frequently in theoretical investigations although it has a physical sense but less similar to ordinary derivative. Particularly, this derivative is nonzero for constant function and initial and boundary problems for equations with derivative of this kind require posing nonlocal conditions. On the other hand, for existence of Riemann-Liouville derivative from any function only summability of the function is required.

Today optimal control problems for fractional-order dynamical systems actively develop. Many interesting results are already obtained in this area for different types of fractional derivative [1]. So, up to recent times there are no theorems analogous to Pontryagin maximum principle. In 2014 an analogue of this principle was formulated and proved in [2] for dynamical system described by equations with Riemann-Liouville derivative. Later the formulation and proof of Pontryagin-like maximum principle were proposed in [3] for linear systems, described by equations with Caputo

derivative. Another effective and quite universal approach to the search of optimal control is moment method [4]. Based on these methods in [5–8] the approach was developed to investigation of optimal control problems for linear dynamic systems of fractional order with lumped parameters, described by equations with Caputo derivative. In [6, 8] this approach was generalized on systems with distributed parameters, described by diffusion-like equation with Caputo time-derivative.

In this paper the moment method is applied to investigation of optimal control problems for dynamical systems of fractional order with lumped parameters, described by equations with Riemann-Liouville derivative. The distinctive feature of problems considered is the posing of nonlocal initial conditions. The optimal control problem for multidimensional linear time-invariant system is considered in general form. Possibility of this problem reduction to moment problem is demonstrated. Then, correctness and solvability are analyzed for the last problem. Further, one-dimensional system of general view and single and double integrators is investigated in detail. Explicit solutions of optimal control problem are obtained and analyzed, including comparison with analogous integer-order systems and systems described by equations with Caputo derivative.

2. Problem Statement

Let the system state and control be defined on a half segment $(0, T]$, $T > 0$ by vector-functions $q(t) = (q_1(t), \dots, q_N(t))$ and $u(t) = (u_1(t), \dots, u_N(t))$ correspondingly.

We will consider dynamical systems, described by the following equation:

$${}_0D_t^{\alpha_i} q_i(t) = a_{ij} q_j(t) + b_{ij} u_j(t) + f_i(t), \quad (1)$$

where $f_i(t)$ are perturbations (known) and a_{ij} are known coefficients, $i, j = \overline{1, N}$. The repeated indices suppose summation. The fractional derivative of arbitrary order $\alpha_i \in (0, 1)$ from function $q_i(t)$ is comprehended in our investigation as left-side Riemann-Liouville derivative [9]:

$${}_0^{RL}D_t^{\alpha_i} q_i(t) = \frac{1}{\Gamma(1 - \alpha_i)} \frac{d}{dt} \int_0^t \frac{q_i(\tau) d\tau}{(t - \tau)^{\alpha_i}}. \quad (2)$$

Also we will compare obtained results with results described in [5, 6, 8] for system (1) with Caputo left-side derivative [9]:

$${}_0^CD_t^{\alpha_i} q_i(t) = \frac{1}{\Gamma(1 - \alpha_i)} \int_0^t \frac{dq_i(\tau)}{d\tau} \frac{d\tau}{(t - \tau)^{\alpha_i}}. \quad (3)$$

Definition 1. System (1) with fractional derivative operator in sense of Riemann-Liouville (Caputo) will be named as Riemann-Liouville (Caputo) system or RL-system (C-system).

The initial conditions for RL-system are determined in nonlocal form [9]:

$$\left[{}_0I_t^{1-\alpha_i} q_i(t) \right] \Big|_{t=0+} = s_i^0, \quad i = \overline{1, N}, \quad (4)$$

where

$${}_0I_t^{\alpha_i} q_i(t) = \frac{1}{\Gamma(\alpha_i)} \int_0^t \frac{q_i(\tau) d\tau}{(t - \tau)^{1-\alpha_i}} \quad (5)$$

is a left-side Riemann-Liouville integral of order α_i [9]. The substitution $t = 0+$ is comprehended in sense of limit of the expression in square brackets at $t \rightarrow 0+$. Such initial conditions differ from ordinary (local) form used in optimal control problems. Inherently, in this case no initial state is determined (defined by phase coordinates values in initial time point) but the value of some integral functional from phase coordinates. One of the first papers in which the optimal control problem with nonlocal initial conditions is considered was [10]. Later such statement was studied by many authors, particularly, in [2]. Initial conditions of type (4) can have a quite definite physical interpretation [11, 12]. For example, in some problems of viscoelasticity the stress and strain appear to be coupled by integrodifferential operator of fractional order. Then the initial condition of type (4) for one of these functions can be rewritten in local form for other functions [11].

The final state of system (1) is determined in ordinary form:

$$q_i(T) = q_i^T, \quad i = \overline{1, N}. \quad (6)$$

Let the control belong to the space $L^p(0, T]$, $1 < p < \infty$, with norm [4, 13]:

$$\|u(t)\| = \left(\int_0^T \sum_{i=1}^N |u_i(t)|^p dt \right)^{1/p}. \quad (7)$$

We will consider also the limit case when $p \rightarrow \infty$ and the control norm is determined as follows [4, 13]:

$$\|u(t)\| = \text{vrai max}_{t \in (0, T]} \max_i |u_i(t)|. \quad (8)$$

Let us believe that functions $q_i(t)$, $i = \overline{1, N}$ possessing all properties required for existence of solutions of equations studied further on are, in particular, summable.

We will study the two following statements of optimal control problem.

Problem 2 (OCP A). Find a control $u(t)$, $t \in (0, T]$, such that system (1) transfers in given final state (6) from the state determined by (4) and norm of control will be minimal with the assigned control time T .

Problem 3 (OCP B). Find a control $u(t)$, $t \in (0, T]$, such that system (1) transfers in given final state (6) from the state determined by (4) and control time T will be minimal provided $\|u\| \leq l$, $l > 0$, where l is the assigned constant.

3. The Moment Problem

3.1. Preliminaries. The classical l -problem of moments can be formulated in the following way.

Problem 4. Let us have a system of functions $g_i(t) \in L^{p'}(0, T]$, $i = \overline{1, N}$, $1 < p' < \infty$. Let us also have the assigned numbers c_i (called moments), $i = \overline{1, N}$ and $l > 0$. We should find function $u(t) \in L^p(0, T]$, $1 < p < \infty$, that satisfies the following conditions:

$$\int_0^T g_i(\tau) u(\tau) d\tau = c_i(T), \quad (9)$$

$$\|u(t)\| \leq l, \quad (10)$$

$$\frac{1}{p} + \frac{1}{p'} = 1. \quad (11)$$

In order for the moment problem (9) and (10) to be solvable, the existence of number $\lambda_N > 0$ and numbers $\xi_1^*, \dots, \xi_N^*$ that give a solution to the following equivalent conditional minimum problem is necessary and sufficient [4, 14, 15].

Problem 5. Find

$$\min_{\xi_1, \dots, \xi_N} \left(\int_0^T \left| \sum_{i=1}^N \xi_i g_i(t) \right|^{p'} dt \right)^{1/p'} = \left(\int_0^T \left| \sum_{i=1}^N \xi_i^* g_i(t) \right|^{p'} dt \right)^{1/p'} = \frac{1}{\lambda_N} \quad (12)$$

with additional condition

$$\sum_{i=1}^N \xi_i c_i = 1. \quad (13)$$

In case problem (12) and (13) is solvable the optimal control for OCP A will be given by the following expression [4]:

$$u(t) = \lambda_N^{p'} \left| \sum_{i=1}^N \xi_i^* g_i(t) \right|^{p'-1} \text{sign} \left[\sum_{i=1}^N \xi_i^* g_i(t) \right], \quad t \in (0, T]. \quad (14)$$

OCP B may be resolved by the following formula:

$$u(t) = l^{p'} \left| \sum_{i=1}^N \xi_i^* g_i(t) \right|^{p'-1} \text{sign} \left[\sum_{i=1}^N \xi_i^* g_i(t) \right], \quad t \in (0, T^*], \quad (15)$$

where T^* is the minimal nonnegative real root of equation

$$\lambda_N(T^*) = l. \quad (16)$$

For solvability of Problem 4 it is necessary and sufficient to satisfy one of two equivalent conditions [4]: (1) $\lambda_N > 0$ and (2) functions $g_i(t)$ are linearly independent. The single question is the possibility of correct posing of Problem 4, which depends on the existence of norm of functions $g_i(t)$ in space $L^{p'}(0, T]$ and the existence of at least one nonzero component in number set c_i .

Definition 6. The moment problem (9) and (10) is called correct if the norm of functions $g_i(t)$ is determined in space $L^{p'}(0, T]$, $1 < p' < \infty$, and there exists at least one nonzero component in number set c_i .

It is known that optimal control problems in form of Problems 2 and 3 can be reduced to Problem 4 in case of integer-order systems with lumped parameters [4]. It was shown in [5–8] that the same is valid for Caputo systems. Below we will demonstrate that for Riemann-Liouville systems Problems 2 and 3 also can be reduced to Problem 4.

3.2. The Correctness and Solvability of Moment Problem for Multidimensional Fractional-Order System. The general solution of (1) in case of RL-system at $\alpha_1 = \dots = \alpha_N = \alpha$ can be represented by the following expression [9, §7.4.2]:

$$q(t) = e^{\mathbf{A}t} s^0 + \int_0^t e^{\mathbf{A}(t-\tau)} f(\tau) d\tau + \int_0^t e^{\mathbf{A}(t-\tau)} u(\tau) d\tau, \quad (17)$$

where $e^{\mathbf{A}t} = z^{\alpha-1} \sum_{k=0}^{\infty} \mathbf{A}^k (t^{\alpha k} / \Gamma((k+1)\alpha))$ is matrix α -exponent [9], $\mathbf{A}$ is a matrix of coefficients a_{ij} , and s^0 is the vector of values s_i^0 , defined by initial conditions (4). It is also known that general solution of (1) in case of Caputo system can be written as follows [9, §7.4.2]:

$$q(t) = q(0) + \int_0^t e^{\mathbf{A}(t-\tau)} [\mathbf{A}q(0) + f(\tau)] d\tau + \int_0^t e^{\mathbf{A}(t-\tau)} u(\tau) d\tau. \quad (18)$$

As in case of integer-order systems, solutions (17) and (18) at $t = T$ can be written in the form of expression (9) and, consequently, can be represented in the form of moment problem (in case of Caputo system it is demonstrated in detail in [5–8]). And in both of these solutions the components of matrix α -exponent act as functions $g_i(t)$. Other terms in expressions form the moments. Consequently, the moment problem for Riemann-Liouville system differs from the problem for analogous Caputo system only by expressions (and values) for the moments. So, the theorems concerning correctness and solvability of the moment problem, proved for Caputo systems [5–8], will be valid also for analogous Riemann-Liouville systems. Thus, if there exists at least one nonzero moment in the set c_i , $i = \overline{1, N}$, we have the following conditions for multidimensional Riemann-Liouville system:

- (1) In case of equal differentiation indices $\alpha_1 = \dots = \alpha_N = \alpha$ the moment problem of type (9) derived from (17) at $t = T$ will be correct and solvable for all α which satisfy the following condition:

$$\alpha > \frac{p' - 1}{p'}. \quad (19)$$

- (2) In case of system (1) at $a_{ij} = \delta_{i,j+1}$, $u(t) = (0, \dots, 0, u_N(t))$, and $f_i(t) = 0$, $i, j = \overline{1, N}$, the moment problem of type (9) derived from (17) at $t = T$ will be correct

and solvable for every $\alpha_1, \dots, \alpha_{N-1}$ and α_N which satisfy the following condition:

$$\alpha_N > \frac{p' - 1}{p'}. \quad (20)$$

3.3. The Moment Problem for One-Dimensional System of Fractional-Order: Problem Setting and Study. In case of $N = 1$ the solution of (1) with initial condition (4) can be written as follows [9, §4.1.1] (the subscripts are omitted):

$$q(t) = s^0 t^{\alpha-1} E_{\alpha,\alpha}(at^\alpha) + \int_0^t \frac{u(\tau) + f(\tau)}{(t-\tau)^{1-\alpha}} E_{\alpha,\alpha}[a(t-\tau)^\alpha] d\tau, \quad (21)$$

where $E_{\alpha,\alpha}(t)$ is two-parameter Mittag-Leffler function [9, §1.8]. By direct calculation one can obtain that at $t = T$ expression (21) with regard to (6) may be written over as (9) with the following symbols:

$$g(\tau) = \frac{E_{\alpha,\alpha}[a(T-\tau)^\alpha]}{(T-\tau)^{1-\alpha}}, \quad (22)$$

$$c(T) = q^T - s^0 T^{\alpha-1} E_{\alpha,\alpha}(aT^\alpha) - \int_0^T \frac{E_{\alpha,\alpha}[a(T-\tau)^\alpha]}{(T-\tau)^{1-\alpha}} f(\tau) d\tau. \quad (23)$$

Expression (22) is identical to analogous expression for one-dimensional Caputo system (formula (17) at $b = 1$ in [5]). So, expressions that result from (12) and (13) will match with analogous expressions obtained for the Caputo system [5, 6, 8] (the difference will appear only after substitution of expression (23) for the moment). Consequently, if we keep in mind the noted difference, we can use the solutions for Problem 5 obtained for Caputo system. As shown in [5, 6, 8], the analytical solution of Problem 4 for one-dimensional system can be obtained for arbitrary $p' > 1$ (if condition (19) is satisfied).

Let $f(t) = 0$. Using (13) we can reduce Problem 5 to simple integral calculation which can be carried out similarly to case of Caputo system.

Consider the case $u(t) \in L^\infty(0, T]$. Direct calculation by formula (12) with regard to (13) and (22) leads us to the following expression (which matches with analogous result for Caputo system [5, 6, 8]):

$$\lambda = \frac{|c(T)| a}{E_\alpha(aT^\alpha) - 1}, \quad (24)$$

where $E_\alpha(aT^\alpha) = E_{\alpha,1}(aT^\alpha)$. Using (24) we can obtain from (14) and (15) the solutions of OCP A and OCP B correspondingly:

$$u(t) = \frac{|c(T)| a}{E_\alpha(aT^\alpha) - 1}, \quad t \in (0, T]; \quad (25)$$

$$u(t) = l \operatorname{sign}(c(T^*)), \quad t \in (0, T^*],$$

where T^* may be received from (16) with (24). It is seen that controls (25) have not one switching point. It is similar to the system of order $[\alpha] + 1$ behaviour in accordance with Feldbaum's theorem about n intervals [16].

In case of $u(t) \in L^p(0, T]$, $1 < p < \infty$, we can obtain the following from (12) and (13) subject to (22):

$$\lambda = \frac{|c(T)|}{F_T}, \quad (26)$$

where

$$F_T = \left[\int_0^T \left(\frac{E_{\alpha,\alpha}(a(T-t)^\alpha)}{(T-t)^{1-\alpha}} \right)^{p'} dt \right]^{1/p'}. \quad (27)$$

Taking into account (14) one can obtain further for OCP A the following:

$$u(t) = \frac{c(T)}{F_T^{p'}} \left(\frac{E_{\alpha,\alpha}(a(T-t)^\alpha)}{(T-t)^{1-\alpha}} \right)^{p'-1}. \quad (28)$$

For OCP B the solution can be expressed by the following formula:

$$u(t) = l^{p'} \left(\frac{E_{\alpha,\alpha}(a(T^*-t)^\alpha)}{(T^*-t)^{1-\alpha}} \right)^{p'-1} \operatorname{sign}(c(T^*)), \quad (29)$$

where T^* can be calculated from (16) using (26).

Let us now consider a single integrator, the special case of one-dimensional system at $a = 0$. Instead of (22) and (23) we will have more simple expressions written in terms of elementary functions:

$$g(\tau) = \frac{1}{\Gamma(\alpha)} \frac{1}{(T-\tau)^{1-\alpha}}, \quad (30)$$

$$c(T) = q^T - \frac{s^0 T^{\alpha-1}}{\Gamma(\alpha)}. \quad (31)$$

Taking into account formula (30) one can obtain the following from (12) and (13):

$$\lambda = \frac{|c(T)| \Gamma(\alpha) (p'(\alpha-1) + 1)^{1/p'}}{T^{\alpha-1+1/p'}}. \quad (32)$$

From formula (32) using (14) we can derive the following expression for control:

$$u(t) = \frac{c(T) (p'(\alpha-1) + 1) \Gamma(\alpha)}{T^{p'(\alpha-1)+1}} (T-t)^{(p'-1)(\alpha-1)}, \quad (33)$$

$$t \in (0, T].$$

Analogously, in case of OCP B we can derive the following using (15):

$$u(t) = l^{p'} \left(\frac{1}{\Gamma(\alpha)} \frac{1}{(T^*-t)^{1-\alpha}} \right)^{p'-1} \operatorname{sign}(c(T^*)), \quad (34)$$

$$t \in (0, T^*],$$

where T^* can be calculated from (16) using (32).

Note that expressions (32), (33), and (34) provide an explicit solution of optimal control problem for Riemann-Liouville single integrator at arbitrary $p' \geq 1$.

3.4. The Moment Problem for Double Integrator of Fractional Order: Problem Setting and Study. Two-dimensional system (1) at $f_1(t) = f_2(t) = 0$, $a_{11} = a_{21} = a_{22} = 0$, $a_{12} = 1$, $u_1(t) = 0$, and $u_2(t) = u(t)$ represents itself as a double integrator. Solution (17) subject to initial conditions (4) for this system can be written in the following form:

$$\begin{aligned} q_1(t) &= \frac{s_1^0 t^{\alpha_1-1}}{\Gamma(\alpha_1)} + \frac{s_2^0 t^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} \\ &\quad + \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_0^t \frac{u(\tau) d\tau}{(t-\tau)^{1-\alpha_1-\alpha_2}}, \\ q_2(t) &= \frac{s_2^0 t^{\alpha_2-1}}{\Gamma(\alpha_2)} + \frac{1}{\Gamma(\alpha_2)} \int_0^t \frac{u(\tau) d\tau}{(t-\tau)^{1-\alpha_2}}. \end{aligned} \quad (35)$$

It is clear that solutions (35) at $t = T$ can be written in form of expression (9) with the following functions $g_i(\tau)$ and moments:

$$g_1(\tau) = \frac{1}{\Gamma(\alpha_1+\alpha_2)} \frac{1}{(T-\tau)^{1-\alpha_1-\alpha_2}}, \quad (36)$$

$$g_2(\tau) = \frac{1}{\Gamma(\alpha_2)} \frac{1}{(T-\tau)^{1-\alpha_2}},$$

$$c_1(T) = q_1^T - \frac{s_2^0 T^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} - \frac{s_1^0 T^{\alpha_1-1}}{\Gamma(\alpha_1)}, \quad (37)$$

$$c_2(T) = q_2^T - \frac{s_2^0 T^{\alpha_2-1}}{\Gamma(\alpha_2)}. \quad (38)$$

The functions $g_{1,2}(\tau)$ defined by (36) are identical to the analogous functions obtained in [5] for Caputo double integrator. On the contrary, the moments defined by (37)-(38) differ, in general, from the moments obtained in [5] for Caputo double integrator. Consequently, as for one-dimensional system, the form of solution for moment problem will match with that for Caputo double integrator. So, we can use the solutions of OCP A and OCP B obtained in [5] taking into account the moments (37).

Firstly, consider the case when $u(t) \in L^\infty(0, T]$. It is shown [5] that in this case the minimization problem (12) and (13) reduces to the following algebraic equation:

$$\begin{aligned} &\frac{2\alpha_1}{(\alpha_1+\alpha_2)\Gamma(\alpha_2+1)} \left[\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_2)} \right]^{\alpha_2/\alpha_1} \\ &\cdot \left(\frac{c_2 \xi_2 - 1}{c_1} \right)^{-\alpha_2/\alpha_1-1} \frac{c_2 \xi_2 - 1 - \alpha_2/\alpha_1}{c_1} \xi_2^{\alpha_2/\alpha_1} \\ &= \frac{T^{\alpha_2}}{\Gamma(\alpha_2+1)} - \frac{c_2}{c_1} \frac{T^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}. \end{aligned} \quad (39)$$

There is no explicit solution of (39) that can be found at arbitrary α_1 and α_2 . In some particular cases the explicit solution exists [5], for example, in case of zero-second moment, $c_2(T) = 0$. Then the solutions of (12) and (13) lead us to the following expression:

$$\lambda_2 = \frac{2^{\alpha_1/\alpha_2} \Gamma(\alpha_1+\alpha_2+1)}{2^{\alpha_1/\alpha_2}-1} \frac{c_1}{T^{\alpha_1+\alpha_2}}. \quad (40)$$

The solution of OCP A will be written as

$$\begin{aligned} u(t) &= - \frac{2^{\alpha_1/\alpha_2} \Gamma(\alpha_1+\alpha_2+1)}{2^{\alpha_1/\alpha_2}-1} \frac{c_1}{T^{\alpha_1+\alpha_2}} \\ &\cdot \text{sign} \left[\frac{(T-t)^{\alpha_1} - 2^{-\alpha_1/\alpha_2} T^{\alpha_1}}{c_1} \right], \quad t \in (0, T]. \end{aligned} \quad (41)$$

The OCP B explicit solution can be found at additional assumption $q_1^T = 0$:

$$u(t) = l \text{sign} \left[\frac{2^{-\alpha_1/\alpha_2} (T^*)^{\alpha_1} - (T^*-t)^{\alpha_1}}{q_1^0} \right], \quad t \in (0, T^*]. \quad (42)$$

The minimal control time T^* can be calculated based on (16) and (40):

$$T^* = \left(\frac{2^{\alpha_1/\alpha_2} \Gamma(\alpha_1+\alpha_2+1)}{2^{\alpha_1/\alpha_2}-1} \frac{|s_1^0|}{l \Gamma(\alpha_1)} \right)^{1/(1+\alpha_2)}. \quad (43)$$

In case of $u(t) \in L^2(0, T]$ calculations by (12) and (13) give an expression:

$$\lambda_2 = \frac{\sqrt{\gamma_1}}{\alpha_1 T^{\gamma_2/2}} Z^{1/2}, \quad (44)$$

where $\gamma_1 = \alpha_1 + 2\alpha_2 - 1$, $\gamma_2 = 2\alpha_1 + 2\alpha_2 - 1$, $\gamma_3 = 2\alpha_2 - 1$, and $Z = |\gamma_1 \gamma_2 c_1^2 \Gamma^2(\alpha_1+\alpha_2) - 2\gamma_2 \gamma_3 c_1 c_2 \Gamma(\alpha_1+\alpha_2) \Gamma(\alpha_2) T^{\alpha_1} + \gamma_1 \gamma_3 c_2^2 \Gamma^2(\alpha_2) T^{2\alpha_1}|$.

According to (14) the OCP A solution will be written as

$$\begin{aligned} u(t) &= \frac{\gamma_1 \gamma_3 c_2 \Gamma(\alpha_2) (\gamma_2 (T-t)^{\alpha_1} - \gamma_1 T^{\alpha_1})}{\alpha_1^2 T^{\gamma_1} (T-t)^{1-\alpha_2}} \left[\frac{\gamma_2 c_1 \Gamma(\alpha_1+\alpha_2)}{\gamma_3 c_2 \Gamma(\alpha_2) T^{\alpha_1}} \right. \\ &\cdot \left. \frac{\gamma_1 (T-t)^{\alpha_1} - \gamma_3 T^{\alpha_1}}{\gamma_2 (T-t)^{\alpha_1} - \gamma_1 T^{\alpha_1}} - 1 \right]. \end{aligned} \quad (45)$$

The OCP B solution can be derived from (15) and (16) and will be represented by the following expression:

$$u(t) = \frac{l^2 \gamma_3 c_2 \Gamma(\alpha_2) (T^*)^{\alpha_1} (\gamma_2 (T^* - t)^{\alpha_1} - \gamma_1 (T^*)^{\alpha_1})}{Z (T^* - t)^{1-\alpha_2}} \left[\frac{\gamma_2 c_1 \Gamma(\alpha_1 + \alpha_2)}{\gamma_3 c_2 \Gamma(\alpha_2) (T^*)^{\alpha_1}} \frac{\gamma_1 (T^* - t)^{\alpha_1} - \gamma_3 (T^*)^{\alpha_1}}{\gamma_2 (T^* - t)^{\alpha_1} - \gamma_1 (T^*)^{\alpha_1}} - 1 \right], \quad (46)$$

where T^* can be found as least positive real root of the equation

$$Z^{1/2} = \frac{\alpha_1 l}{\sqrt{\gamma_1}} T^{\gamma_2/2}. \quad (47)$$

Note also that for Caputo double integrator the case when $c_2(T) = 0$ corresponds to the case when $q_2^0 = q_2^T$ [5]. From formula (38) one can see that the condition $c_2(T) = 0$ is not so clear.

4. Qualitative Analysis of Results

In this chapter the obtained solutions of OCP are analyzed. It will be shown that at integer (equal to 1) values of differentiation indices these solutions reduce to the known results for corresponding integer-order systems. For double integrator analytical expressions derived for system phase trajectories in different modes.

4.1. Optimal Control Behaviour at Integer-Order Differentiation Indices. For one-dimensional system of first order one can obtain the solution of moment problem and explicit expressions for optimal controls. For example, in case of single integrator at $u(t) \in L^\infty(0, T]$ the OCP A solution is expressed by the following formula:

$$u^1(t) = \frac{q^T - q^0}{T}, \quad t \in (0, T]. \quad (48)$$

The OCP B solution gives the following [4]:

$$u^1(t) = l \operatorname{sign}(q^T - q^0), \quad t \in (0, T^*], \quad (49)$$

$$T_1^* = \frac{|q^T - q^0|}{l}.$$

It is easy to see that solution of OCP A defined by (33) (taking into account (31)) at $\alpha = 1$ is identical to expression (48). The same is true for OCP B solution: expressions (34) and (16) subject to (32) at $\alpha = 1$ give formulas (49).

For double integrator of first order at arbitrary initial and final conditions the solutions of OCP A and OCP B lead to quadratic equation for ξ_2 [4]. Equation (39) reduces to that equation at $\alpha_1 = \alpha_2 = 1$. The solution of OCP A for double integrator of first order at $q_2^0 = q_2^T = 0$ for $u(t) \in L^\infty(0, T]$ is given by the following formula [4]:

$$u^2(t) = \frac{4(q_1^T - q_1^0)}{T^2} \operatorname{sign}\left(\frac{T/2 - t}{q_1^T - q_1^0}\right). \quad (50)$$

By direct calculation one can obtain that formula (41) (i.e., OCP A solution for Riemann-Liouville double integrator)

subject to (37) at $q_2^0 = q_2^T = 0$ and $\alpha_1 = \alpha_2 = 1$ reduces to (50).

Consider OCP B in case of $q_1^T = 0$. Then the solution of the problem for double integrator of first order will be given by the following formulas:

$$T_2^* = 2\sqrt{\frac{q_1^0}{l}}, \quad (51)$$

$$u^2(t) = l \operatorname{sign}\left(\frac{t}{q_1^0} - \frac{1}{\sqrt{lq_1^0}}\right), \quad t \in (0, T_2^*].$$

Expressions (42) and (43) at $\alpha_1 = \alpha_2 = 1$ transform into formulas (51), which can be proved by corresponding substitution.

Thus, as for Caputo systems [5, 6, 8], all the results obtained for single and double integrators of fractional order reduce to corresponding formulas for integer-order systems when differentiation indices are equated to 1.

4.2. Investigation of Qualitative Dynamics for Double Integrator. For two-dimensional systems the analysis of qualitative dynamics is interesting itself. We will calculate below the boundary trajectories for double integrator and its trajectories corresponding to optimal control mode. Consider $u(t) \in L^\infty(0, T]$.

Definition 7. The boundary trajectories of some system are the phase trajectories corresponding to boundary values of control $u(t) = \pm l$.

In case of differentiation indices equal to 1 the boundary trajectories of some system represent the boundaries of integral vortex for differential inclusion corresponding to this system [17]. This manifold bounds the phase space region which contains all admissible trajectories of the system. In case of fractional-order systems (Caputo and Riemann-Liouville) this property is also valid caused by comparison theorem [18, 19].

Substituting the boundary values of control $u(t) = \pm l$ to solutions (35) one can obtain the following explicit expressions for boundary trajectories:

$$q_1^{\pm l}(t) = \frac{s_1^0 t^{\alpha_1 - 1}}{\Gamma(\alpha_1)} + \frac{s_2^0 t^{\alpha_1 + \alpha_2 - 1}}{\Gamma(\alpha_1 + \alpha_2)} \pm \frac{lt^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1 + \alpha_2 + 1)}, \quad (52)$$

$$q_2^{\pm l}(t) = \frac{s_2^0 t^{\alpha_2 - 1}}{\Gamma(\alpha_2)} \pm \frac{lt^{\alpha_2}}{\Gamma(\alpha_2 + 1)}.$$

It is seen that expressions (52) at arbitrary α_1 and α_2 do not allow eliminating time and obtaining the explicit dependency

$q_2^{\pm l}(q_1^{\pm l})$ as opposed to double integrator of integer order [17, §35] and Caputo double integrator [5]. Nevertheless, in some particular cases the time can be eliminated from expressions (52). For example, in case of $s_2^0 = 0$ one can calculate directly that these formulas reduce to the following equation:

$$q_1^{\pm l} = \left(\frac{\Gamma(\alpha_2 + 1) q_2^{\pm l}}{\pm l} \right)^{\alpha_1/\alpha_2} \cdot \left[\frac{s_1^0}{\Gamma(\alpha_1)} \left(\frac{\Gamma(\alpha_2 + 1) q_2^{\pm l}}{\pm l} \right)^{-1/\alpha_2} + \frac{\Gamma(\alpha_2 + 1) q_2^{\pm l}}{\Gamma(\alpha_1 + \alpha_2 + 1)} \right]. \quad (53)$$

Analogously, in case of $\alpha_2 = 1$ it is possible to eliminate time from (52) and to derive the following equation:

$$q_1^{\pm l} = \frac{1}{\Gamma(\alpha_1)} \left(\frac{q_2^{\pm l} - s_2^0}{\pm l} \right)^{\alpha_1 - 1} \cdot \left[s_1^0 \pm \frac{q_2^{\pm l} - s_2^0}{l} \frac{s_2^0}{\alpha_1} + \left(\frac{q_2^{\pm l} - s_2^0}{l} \right)^2 \frac{1}{\alpha_1(\alpha_1 + 1)} \right]. \quad (54)$$

Note that expressions (52) at $\alpha_1 = \alpha_2 = 1$ reduce to the quadratic equation for integral vortex boundaries of differential inclusion corresponding to the double integrator of integer order [17, §35].

In order to calculate the phase trajectories of Riemann-Liouville double integrator in optimal control mode we will substitute control (41) into solutions (35). Then the following motion laws can be obtained:

$$q_1(t) = \frac{s_1^0 t^{\alpha_1 - 1}}{\Gamma(\alpha_1)} + \frac{s_2^0 t^{\alpha_1 + \alpha_2 - 1}}{\Gamma(\alpha_1 + \alpha_2)} - \frac{2^{\alpha_1/\alpha_2}}{2^{\alpha_1/\alpha_2} - 1} \cdot \frac{c_1(T)}{T^{\alpha_1 + \alpha_2}} \left[t^{\alpha_1 + \alpha_2} - 2(t - t')^{\alpha_1 + \alpha_2} \Theta(t - t') \right],$$

$$q_2(t) = \frac{s_2^0 t^{\alpha_2 - 1}}{\Gamma(\alpha_2)} - \frac{2^{\alpha_1/\alpha_2}}{2^{\alpha_1/\alpha_2} - 1} \frac{c_1(T)}{T^{\alpha_1 + \alpha_2}} \cdot \frac{\Gamma(\alpha_1 + \alpha_2 + 1)}{\Gamma(\alpha_2 + 1)} \left[t^{\alpha_2} - 2(t - t')^{\alpha_2} \Theta(t - t') \right], \quad (55)$$

where t' is the switching point of control (41) and $\Theta(t)$ is the Heaviside's function. Formulas (55) are obtained for control (41) which is valid in case of $c_2 = 0$. Consequently, these expressions allow some simplification.

5. Computational Results and Analysis

In this chapter the computational results are represented and analyzed (including comparison with analogous Caputo systems). Calculations and its visualization were performed

FIGURE 1: The dependencies for norm of control from α at $p' = 1$ (logarithmic scale used for ordinates).

FIGURE 2: The dependencies for norm of control from α at $p' = 2$ (logarithmic scale used for ordinates).

in MatLab 7.9. Mittag-Leffler function values were calculated using special procedure [20]. Integrals were calculated by Gauss-Kronrod method [21].

5.1. Single Integrator. We will consider the task of system transfer from any state with $s^0 \neq 0$ in the final state $q^T = 0$.

Figures 1–3 show the dependencies of control norm from differentiation index calculated by formula (32) at $T = 100$ and different values s^0 for $p' = 1$, $p' = 2$, and $p' = 3/2$ correspondingly (dash-dot lines). Also the analogous dependencies are shown for Caputo single integrator calculated by formulas from [5, 8] at different q^0 for the same values of p' (solid lines). It is seen that curves for Riemann-Liouville and Caputo single integrator differ from each other qualitatively but converge to the same point at $\alpha = 1$. In addition, the control norm increases with s^0 or q^0 growing for both of integrator types.

In Figure 4 the dependencies of control norm from control time T are shown at different values of differentiation index and $s^0 = 1$ (for Riemann-Liouville integrator) and $q^0 = 1$ (for Caputo integrator). The solid lines correspond to $p' = 1$, the dotted lines correspond to $p' = 3/2$, and the dash-dotted lines correspond to $p' = 2$. It is clear that all of these curves decrease monotonically which correspond to the case of Caputo integrator [8] and integrator of integer order [4]. And the norm of control increases with growing of α and p' .

FIGURE 3: The dependencies for norm of control from α at $p' = 3/2$ (logarithmic scale used for ordinates).

FIGURE 4: The dependencies for norm of control from T . Logarithmic scale used for ordinates.

Consider now the minimal control time T^* dependency from differentiation index α . In case of $s^0 \neq 0$ and $q^T = 0$ (16) subject to formula (32) can be solved explicitly and have only one real root:

$$T^* = \left(\frac{|s^0|}{l} \right)^{p'} (p'(\alpha - 1) + 1). \quad (56)$$

In Figure 5 the computational results are shown for the obtained formula used at different p' at $l = 1$ and $s^0 = 1$ (dash-dotted lines). Also the similar dependencies (calculated by formulas from [8]) are represented for Caputo integrator at $l = 1$ and $q^0 = 1$ (solid lines). As can be seen from Figure 5 the dependency at $p' = 1$ is linear for both of integrator types. In case of $p' > 1$ the dependency for Caputo integrator becomes nonlinear.

5.2. One-Dimensional System of General Type. Consider (as above) the task of system transfer from any state with $s^0 \neq 0$ to the state $q^T = 0$.

Figures 6 and 7 represent the dependencies of control norm from differentiation index at $p' = 1$ and $p' = 2$ calculated by formulas (24) and (26) correspondingly (dash-dotted lines). The analogical dependencies for Caputo one-dimensional system are also shown (solid lines) calculated by formulas from [5, 8]. The curves were calculated for different values of parameter a and the following values of other parameters: $T = 2$, $b = 1$, $s^0 = 1$ (RL-system), and $q^0 = 1$

FIGURE 5: The dependencies for minimal control time from α .

FIGURE 6: The dependencies for control norm from α at $p' = 1$ and different values of parameter a .

FIGURE 7: The dependencies for control norm from α at $p' = 2$ and different values of parameter a .

(C-system). It is seen from Figures 6 and 7 that difference in derivative type has qualitatively effect on behaviour of dependencies. This behaviour changes at $a < 1$ and $a = 1$. In case of $p' = 1$ the curves for C-system decrease and curves for RL-system increase at $a < 1$ (analogically to single integrator). At $a = 1$ the curves for both of systems increase. For $p' = 2$ the dependencies increase for both of system types.

The dependencies of control norm from control time are similar, in general, for single integrator and are not shown here.

The dependencies of minimal control time from α at different values of parameter a are shown in Figures 8 and 9 for $p' = 1$ and $p' = 2$ correspondingly. The following values of other parameters were chosen: $l = 2$, $s^0 = 1$ (RL-system),

FIGURE 8: The dependencies for minimal control time from α at $p' = 1$ and different values of parameter a .

FIGURE 9: The dependencies for minimal control time from α at $p' = 2$ and different values of parameter a .

and $q^0 = 1$ (C-system). Dash-dotted lines correspond to RL-system and solid lines correspond to C-system. It is seen that in case of RL-system the curves have the quasi-linear trend and differ qualitatively from the curves for C-system.

5.3. Double Integrator. As in case of one-dimensional systems, we will consider the task of system transfer from any state with nonzero values of phase coordinates into the origin at phase plane: we will suppose that $s_1^0 \neq 0$ and $q_1^T = q_2^T = s_2^0 = 0$.

The dependencies of control norm from one of differentiation indices for Riemann-Liouville double integrator at several fixed values of other indexes are shown in Figures 10 and 11 (dash-dotted lines). The curves were calculated by expression (40) for $p' = 1$, $T = 100$, and $s_1^0 = 1$. Also the similar dependencies for Caputo double integrator are shown at these figures calculated for the same values of parameters p' and T and for $q_1^0 = 1$ (solid lines). The dependencies for $p' = 2$ are similar and not demonstrated here. It is seen from Figure 10 that curves corresponding to different systems (RL- and C-systems) at identical values of α_2 converge into the same point at $\alpha_1 = 1$. In Figure 11 the distance between different curves for C- and RL-systems decreases when α_1 increases.

The dependencies of control norm from control time for double integrator are analogous, generally, to that in case of one-dimensional systems and are not demonstrated here.

Let us consider the dependency of minimal control time from differentiation indices α_1 and α_2 . In case of $p' = 1$ this

FIGURE 10: The dependencies for control norm from α_1 at $p' = 1$ and several values of α_2 .

FIGURE 11: The dependencies for control norm from α_2 at $p' = 1$ and several values of α_1 .

dependency is given by formula (43). In case of $p' = 2$ with additional assumption $q_1^T = q_2^T = s_2^0 = 0$ the explicit formula also can be found for such dependency (as solution of (16) subject to (40)):

$$T^* = \left(\frac{\sqrt{2\alpha_1 + 2\alpha_2 + 1} (\alpha_1 + 2\alpha_2 + 1) \Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1 + 1)} \cdot \frac{|s_1^0|}{l} \right)^{2/(2\alpha_2 + 1)}. \quad (57)$$

In Figures 12–15 the dependencies are shown for minimal control time from one of differentiation indices at fixed value of other indices. The curves are calculated at $l = 2$ for $p' = 1$ (Figures 12 and 13) and $p' = 2$ (Figures 14 and 15). Dash-dotted lines correspond to Riemann-Liouville double integrator and are calculated for $s_1^0 = 1$. Solid lines correspond to Caputo double integrator and are calculated for $q_1^0 = 1$. As seen from these figures, the dependencies for RL- and C-systems differ qualitatively from each other.

Let us now analyze the qualitative dynamics of Riemann-Liouville double integrator, its boundary trajectories, and phase trajectories in optimal control mode. Note that last type of trajectories in case of Riemann-Liouville double integrator will not have an obvious initial point at phase plane since the initial condition (4) defines not the system state but the value of some integral functional at initial time. The system state in

FIGURE 12: The dependencies for minimal control time from α_1 at different values of α_2 for $p' = 1$. Logarithmic scale used for ordinates.

FIGURE 15: The dependencies for minimal control time from α_2 at different values of α_1 for $p' = 2$. Logarithmic scale used for ordinates.

FIGURE 13: The dependencies for minimal control time from α_2 at different values of α_1 for $p' = 1$. Logarithmic scale used for ordinates.

FIGURE 16: The boundary trajectories for double integrator of fractional order at $\alpha_1 = \alpha_2 = 0.5$ and $l = 1$.

FIGURE 14: The dependencies for minimal control time from α_1 at different values of α_2 for $p' = 2$. Logarithmic scale used for ordinates.

FIGURE 17: The boundary trajectories for double integrator of fractional order at $\alpha_1 = \alpha_2 = 0.5$ and $l = 1$.

this case can be undefined as follows from solutions (35) and expressions (55).

The boundary trajectories for studied system are represented in Figure 16 for Riemann-Liouville double integrator (dash-dotted lines) and for analogous Caputo system (solid lines). For the last system calculations were performed using formulas from [5]. It is clear that trajectories for both of systems are very close to each other. In Figure 17 the fragment of trajectories is shown at enlarged scale.

The phase trajectories of Riemann-Liouville double integrator in optimal control mode were calculated using expressions (55). In Figure 18 the results are shown for different

FIGURE 18: The phase trajectories in optimal control mode for Riemann-Liouville double integrator.

FIGURE 19: The phase trajectories in optimal control mode for Caputo double integrator.

FIGURE 20: The phase trajectories in optimal control mode for Riemann-Liouville and Caputo double integrator in case of $\alpha_1 = 1$.

values of differentiation indices, $T = 100$ and $s_1^0 = 1$. In Figure 19 the phase trajectories of Caputo double integrator in optimal control mode are represented which were calculated by formulas from [5] at $T = 100$ and $q_1^0 = 1$. It is clear from Figures 18 and 19 that both of systems demonstrate the overcontrolling effect. In case of $\alpha_1 = 1$ this effect does not appear and phase trajectories coincide for both of systems studied (Figure 20).

6. Summary

In this paper the optimal control problem is investigated for linear dynamical systems of fractional order described by equations with Riemann-Liouville derivative. The problem reduced to the moment problem. For the last problem the correctness and solvability conditions derived. Some examples of one- and two-dimensional systems were considered and explicit solutions of corresponding optimal control problems were found. The properties of these solutions were analyzed and compared with analogous integer-order systems and fractional-order systems, described by equations with Caputo derivative.

Some computational results are demonstrated that show the dependency of control norm from differentiation indices and control time and the dependency of minimal control time from differentiation indices. Qualitative dynamics of double integrator are analyzed and show that boundary trajectories of Riemann-Liouville system of this type are very close to

the same trajectories for corresponding Caputo system. It is shown also that the overcontrolling effect appears in optimal control mode of system studied.

Obtained results can be useful in investigation of optimal control problems for fractional-order systems and in design of particular control systems.

Competing Interests

The authors declare that they have no competing interests.

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