Hindawi
Journal of Control Science and Engineering
Volume 2017, Article ID 3949428, 8 pages
https://doi.org/10.1155/2017/3949428

Research Article

Synchronization of Coupled Harmonic Oscillators Using Quantized Sampled Position Data

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Received 27 February 2017; Revised 25 April 2017; Accepted 29 May 2017; Published 3 July 2017

Academic Editor: Wenwu Yu

For coupled harmonic oscillators (CHO), the paper studies the synchronization problem by using the quantized current sampled position data. The logarithmic quantizer is adopted here to quantize the information transmitted; thus the quantization error of the sampled position data can be illustrated as the uncertainty according to the sector bound property. On the basis of that, the synchronization problem is converted into the asymptotical stability of the subsystems and even the solving problem of characteristic equation. Some sufficient conditions ensuring the synchronization of CHO are obtained relating to coupling strength, sampling period, and quantizer parameter. The usefulness of the theoretical result is shown by an example at the end.

1. Introduction

Recently, as a special multiagent system, CHO has gotten much attention. By assuming that the undirected network is not connectivity, literature [1] shows that the CHO is also able to achieve synchronization. When the CHO is affected by controller loss and communication time-delays, an algorithm is proposed in [2] to ensure the synchronization of the system and a brief procedure of the convergence analysis for such algorithm is given. By using some special analysis tools, literature [3] shows that the CHO can be synchronized under the conditions of mild connectivity for the directed graph. A frequency dependent topology condition is given in [4] to synchronize the CHO. With the measurement noise, literature [5] gives an effective algorithm and the convergence analysis of the CHO for the directed network topologies with or without leaders. A sufficient condition for the synchronized oscillatory motions of CHO is given by the influence of noise for the strongly connected directed graph.

The control algorithms proposed in the above literature use both velocity information and position information. In some cases, each oscillator exchanges just the velocity information or the position information with its neighbors. The relating papers which design control algorithms just use velocity information or position information include [6–9]. Assuming that the network is connected, the sufficient conditions are given for the synchronized oscillatory motions by just using velocity information in brief [6]. The results of [6] are generalized to the sampled-data CHO affected by input signals loss in [7]. Literature [8] proposes a distributed control method just using the outdated position states. For both positive and negative coupling strengths, it derives some necessary and sufficient conditions (NSCs) for the synchronization of CHO. In [9], by adopting current and outdated position data, respectively, two distributed protocols are adopted for the synchronization of CHO.

On the other hand, as an important issue, data quantization has received extensive attention in multiagent systems. Theoretical results in the past two years include [10–16]. The consensus of multiagent systems is studied according to the robust learning control approach in [10]. For the undirected connected graphs, literature [11] obtains some strong results for the problem of expected time to convergence for quantized consensus. Under the influence of quantized input and disturbances, the distributed quantized H_∞ consensus is investigated in [12]. For the high-order multiagent systems, the quantized consensus is studied in [13], in which it assumed that only the first state can be measured. By adopting both uniform and logarithmic quantizers, some quantized consensus results are achieved for heterogeneous systems in [14]. By using probabilistic versus deterministic quantizers, the quantized consensus problem for multiagent systems is studied in [15]. Moreover, event-triggered consensus [16], averaging consensus [17], consensus tracking [18], metropolis consensus [19], containment control [20], and other consensus problems [21–23] affected by data quantization are discussed in recent papers.

Summarized above, the synchronization of CHO and quantized consensus for the general multiagent systems have received enough attention, respectively. However, there are few results relating to the synchronization of CHO affected by data quantization, which is the problem studied in our paper. In fact, this paper's main contribution is to extend the results of literature [9] to the synchronization problem for CHO by using the quantized position data. By adopting the logarithmic quantizer, the quantization error of the sampled position data can be illustrated as the uncertainty according to the sector bound property. With this setup, some sufficient conditions relating to coupling strength, sampling period, and quantizer parameter are obtained for synchronized CHO.

The organizational structure of the paper is as follows. Section 2 demonstrates the problem discussed and the CHO network; distributed protocol and logarithmic quantizer are illustrated there in detail. Some sufficient conditions for synchronization of CHO and the detailed proving process are shown in Section 3. The usefulness of the theoretical results is illustrated by a simulation example in Section 4. Section 5 gives some conclusions.

Notation. The n-dimensional Euclidean space is denoted by $\mathbb{R}^n$. $\mathbb{N}$ ($\mathbb{R}^+$) indicates the positive integers (real numbers) set. We denote by $|c|$ the module of complex number c. A^{-1} and A^T are the inverse and the transpose of matrix A, respectively. $\mathbf{0}_n$ and 1_n denote the vectors with dimension $n \times 1$ whose elements are all zeros or ones, respectively. For a multiagent system, especially for CHO, information exchange between agents can be modeled by a network or graph [24]. Let graph $\mathcal{G}$ be with node set $\Omega = \{1, \ldots, M\}$ and edge set $\Upsilon \subseteq \Omega \times \Omega$. If there exists at least one node having a directed path to every other node, the digraph $\mathcal{G}$ is called containing a directed spanning tree (DST). The adjacency matrix $\mathcal{A} = (a_{ij})_{M\times M}$ satisfies $a_{ij} > 0$ if $(j, i) \in \Upsilon$ and $a_{ij} = 0$ if $(j, i) \notin \Upsilon$. $\mathcal{M}_i = \{j \mid a_{ij} > 0, j \notin i\}$, $\forall i \in \{1, \ldots, M\}$ is a set including all the neighbor nodes of node i. We use $L = (l_{ij})_{M\times M}$ to denote the Laplacian matrix, whose elements satisfy $l_{ij} = -a_{ij}$ $(i \neq j)$ and $l_{ii} = \sum_{k=1,k\neq i}^{M} a_{ik}$ [25]. $0 = |\eta_1| \leq |\eta_2| \leq \cdots \leq |\eta_M|$ represent the eigenvalues of L when the digraph $\mathcal{G}$ contains a DST. $\mathrm{diag}(a_1, \ldots, a_M)$ indicates the diagonal matrix with diagonal elements $a_1, \ldots, a_M$.

2. Problem Formulation

For continuous-time CHO, we illustrate the synchronization problem and protocol by using the quantized sampled position data in this section.

For a positive constant h, the synchronization problem of the following CHO network composed by M nodes is discussed:

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= -hx_i(t) + u_i(t), \quad i \in \{1, \ldots, M\} \end{aligned} \tag{1}$$

with position states $x_i(t)$, velocity states $v_i(t)$, and control input $u_i(t)$ for node i.

Assuming that the information between nodes is transmitted over the network, then it must be quantized before transmission. A protocol using the quantized value of the current relative sampled position data is adopted in this paper:

$$\begin{aligned} u_i(t) = \gamma \sum_{j\in\mathcal{M}_i} a_{ij} \left(q\left(x_i(t_k)\right) - q\left(x_j(t_k)\right) \right), \\ t \in [t_k, t_{k+1}), \; i \in \{1, \ldots, M\}, \; k \in \mathbb{N}, \end{aligned} \tag{2}$$

in which coupling strength $\gamma > 0$ is to be designed and $x_i(t_k)$, $i \in \{1, \ldots, M\}$, denote the sampled position at time instants $t_k = kT, k \in \mathbb{N}$, with $t_0 = 0$. The quantizer used here is a general logarithmic one defined as [26, 27]

$$q(\chi) = \begin{cases} \kappa_w & \text{if } \chi \in \left[\dfrac{1}{1+\delta}\kappa_w, \dfrac{1}{1-\delta}\kappa_w \right), \; \chi > 0 \\ 0 & \text{if } \chi = 0 \\ -q(-\chi) & \text{if } \chi < 0 \end{cases} \tag{3}$$

for any $\chi \in \mathbb{R}$, with $w \in \mathbb{N} \cup \{0\}$, $\delta \in (0, 1)$, $\kappa_{w+1} = \rho\kappa_w$, $v_0 > 0$, and $\rho = (1 - \delta)/(1 + \delta) \in (0, 1)$.

The aim of our paper is designing suitable γ and T such that the CHO (1) under the control input (2) achieves synchronization; that is,

$$\begin{aligned} \lim_{t\to\infty} \left\| x_i(t) - x_j(t) \right\| = 0, \\ \lim_{t\to\infty} \left\| v_i(t) - v_j(t) \right\| = 0 \end{aligned} \tag{4}$$

for any $i, j \in \{1, \ldots, M\}$ $(i \neq j)$.

3. The Sufficient Conditions for Synchronization

For CHO (1) with protocol (2), the synchronization problem is studied in this section. The sufficient conditions for synchronization are pursued on the basis of the following two lemmas.

Lemma 1 (see [28]). *The fact that the graph $\mathcal{G}$ contains a DST is equal to the fact that the matrix L has a simple zero eigenvalue and other eigenvalues having positive real parts. Furthermore, 1_M is the right eigenvector relating to zero eigenvalue of L. Moreover, we can always get a suitable ϕ, which is left eigenvector relating to zero eigenvalue of L, satisfying $\phi^T L = 0$ and $\phi^T 1_M = 1$.*

Lemma 2 (see [29]). *Let $p(s) = s^2 + (\varpi_1 + \mathbf{i}\sigma_1)s + \varpi_0 + \mathbf{i}\sigma_0$ be a complex-coefficient polynomial with real constants $\varpi_0, \sigma_0, \varpi_1$, and σ_1, and then the asymptotical stability of $p(s)$ is equivalent to $\varpi_1 > 0$ and $\varpi_1\sigma_1\sigma_0 + \varpi_1^2\varpi_0 - \sigma_0^2 > 0$.*

By using the above lemmas, the following theorem is given to discuss the synchronization of CHO (1) under protocol (2), which is the main result of this paper.

Theorem 3. *If the graph $\mathscr{G}$ contains a DST and the initial states satisfy $\phi^T x(0) = \phi^T v(0) = 0$ with ϕ defined in Lemma 1, then network (1) under protocol (2) can achieve synchronization if γ and T are selected satisfying*

$$\gamma \notin \left[\frac{h}{\max_{i\in\{2,\dots,M\}} \operatorname{Re}\left(\tilde{\xi}_i\right)}, \frac{h}{\min_{i\in\{2,\dots,M\}} \operatorname{Re}\left(\bar{\xi}_i\right)}\right], \tag{5a}$$

$$T \notin \left\{\frac{l\pi}{\sqrt{h}} \mid l \in \mathbb{N}\right\}, \tag{5b}$$

$$\gamma < \frac{h \operatorname{Re}\left(\tilde{\xi}_{i_{\max}}\right)}{\left|\tilde{\xi}_{i_{\max}}\right|^2}, \tag{5c}$$

$$T \in \Gamma_T\left(\max_{i\in\{2,\dots,M\}, l\in\mathbb{N}} \arctan\sqrt{e_{il}}\right), \tag{5d}$$

where $\tilde{\xi}_i$ and $\bar{\xi}_i, i \in \{2,\dots,M\}$ are, respectively, the nonzero eigenvalues of $L(I_M + \overline{\Lambda})$ and $L(I_M - \overline{\Lambda})$ with $\overline{\Lambda} = \delta I_M$, $i_{\max} = \arg\max_{i\in\{2,\dots,M\}}|\tilde{\xi}_i|$, $\Gamma_T(\varphi) \triangleq \{T \mid \tan^2(\sqrt{h}T/2) > \tan^2\varphi\}$, for any $\varphi \in [0, \pi/2)$, and

$$e_{il} = \frac{\left|h - \gamma\xi_i^l\right|^2 b_{il}^2}{a_{il}b_{il}^2 + a_{il}^2 c_{il}} \tag{6}$$

for any $i \in \{2,\dots,M\}, l \in \mathbb{N}$ with $a_{il} = h\gamma\operatorname{Re}(\xi_i^l) - \gamma^2|\xi_i^l|^2$, $b_{il} = h\gamma\operatorname{Im}(\xi_i^l)$, and $c_{il} = h^2 - h\gamma\operatorname{Re}(\xi_i^l)$, in which ξ_i^l denote possible values for $\xi_i(t_k), k \in \mathbb{N}$, defined by (10). Moreover, let $x(t) = (x_1(t),\dots,x_M(t))^T$ and $v(t) = (v_1(t),\dots,v_M(t))^T$, and we obtain

$$\begin{aligned} \lim_{t\to\infty} x_i(t) &= \cos\left(\sqrt{h}t\right)\phi^T x(0) \\ &\quad + \frac{1}{\sqrt{h}}\sin\left(\sqrt{h}t\right)\phi^T v(0) = 0, \\ \lim_{t\to\infty} v_i(t) &= -\sqrt{h}\sin\left(\sqrt{h}t\right)\phi^T x(0) \\ &\quad + \cos\left(\sqrt{h}t\right)\phi^T v(0) = 0 \end{aligned} \tag{7}$$

for all $i \in \{1,\dots,M\}$.

Proof. By the property of logarithmic quantizer, we get $q(x_i(t_k)) = (1+\varepsilon_i(t_k))x_i(t_k)$ with $|\varepsilon_i(t_k)| \le \delta, \forall k \in \mathbb{N}\cup\{0\}, i \in \{1,\dots,M\}$. For any $t \in \mathbb{R}^+ \cup \{0\}$, let $z_i(t) = (x_i(t), v_i(t))^T$, $i \in \{1,\dots,M\}$ and $z(t) = (z_1^T(t),\dots,z_M^T(t))^T$; then network (1) under protocol (2) can be rewritten as

$$\begin{aligned} \dot{z}(t) &= (I_M \otimes A) z(t) + \gamma (L \otimes B) \begin{bmatrix} q(x_1(t_k)) \\ 0 \\ q(x_2(t_k)) \\ 0 \\ \vdots \\ q(x_M(t_k)) \\ 0 \end{bmatrix} \\ &= (I_M \otimes A) z(t) + \gamma (L \otimes B) \\ &\quad \cdot \begin{bmatrix} 1 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 0 \end{bmatrix} \begin{bmatrix} (1+\varepsilon_1(t_k)) x_1(t_k) \\ (1+\varepsilon_1(t_k)) v_1(t_k) \\ \vdots \\ (1+\varepsilon_M(t_k)) x_M(t_k) \\ (1+\varepsilon_M(t_k)) v_M(t_k) \end{bmatrix} \\ &= (I_M \otimes A) z(t) + \gamma (L \otimes B)(I_M \otimes D) \\ &\quad \cdot ((I_M + \Lambda(t_k)) \otimes I_2) z(t_k) = (I_M \otimes A) z(t) \\ &\quad + \gamma (L(I_M + \Lambda(t_k)) \otimes B) z(t_k), \\ &\qquad t \in [t_k, t_{k+1}),\ k \in \mathbb{N}\cup\{0\}, \end{aligned} \tag{8}$$

where $A = \left[\begin{smallmatrix} 0 & 1 \\ -h & 0 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$, $D = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, and $\Lambda(t_k) = \operatorname{diag}\{\varepsilon_1(t_k),\dots,\varepsilon_M(t_k)\}$.

For the matrix $L(I_M + \Lambda(t_k)), \forall k \in \mathbb{N}\cup\{0\}$, the following claim holds by Lemma 1.

Claim. If the graph $\mathscr{G}$ contains a DST, then the matrix $L(I_M + \Lambda(t_k)), \forall k \in \mathbb{N}\cup\{0\}$, has a simple zero eigenvalue and other eigenvalues having positive real parts. Furthermore, $[1/(1+\varepsilon_1(t_k)), 1/(1+\varepsilon_1(t_k)),\dots,1/(1+\varepsilon_M(t_k))]^T$ and ϕ defined in Lemma 1 are the right eigenvector and left eigenvector, respectively, relating to zero eigenvalue of $L(I_M + \Lambda(t_k))$.

Based on the above claim, for any $k \in \mathbb{N}\cup\{0\}$, there must exist matrices $Q(t_k) = (p_1(t_k),\dots,p_M(t_k)) \in \mathbb{R}^{M\times M}$ and $Q^{-1}(t_k) = (q_1(t_k),\dots,q_M(t_k))^T \in \mathbb{R}^{M\times M}$ with $p_1(t_k) = [1/(1+\varepsilon_1(t_k)), 1/(1+\varepsilon_1(t_k)),\dots,1/(1+\varepsilon_M(t_k))]^T$ and $q_1(t_k) = \phi$ such that

$$\begin{aligned} J(t_k) &= Q^{-1}(t_k) L(I_M + \Lambda(t_k)) Q(t_k) \\ &= \begin{pmatrix} 0 & 0_{M-1}^T \\ 0_{M-1} & \widehat{L}(t_k) \end{pmatrix} \end{aligned} \tag{9}$$

with Jordan canonical form $J(t_k)$ and block upper triangular matrix $\widehat{L}(t_k)$ represented as

$$\widehat{L}(t_k) = \begin{pmatrix} \xi_2(t_k) & \tau & 0 & \cdots & 0 \\ 0 & \xi_3(t_k) & \tau & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & \xi_M(t_k) \end{pmatrix} \quad (10)$$

in which $\xi_i(t_k)$, $i \in \{2,\dots,M\}$ satisfying $\mathrm{Re}(\xi_2(t_k)) \le \cdots \le \mathrm{Re}(\xi_M(t_k))$ are the nonzero eigenvalues of $L(I_M + \Lambda(t_k))$, and τ is equal to 1 or 0.

Let $m(t) = (m_1^T(t),\dots,m_M^T(t))^T = (Q^{-1}(t_k)\otimes I_2)z(t)$, $\forall t \in [t_k, t_{k+1})$ and $m_i(t) \in \mathbb{R}^2$, and then (8) gives that

$$\begin{aligned} \dot{m}(t) &= \left(Q^{-1}(t_k)\otimes I_2\right)(I_M\otimes A)\left(Q(t_k)\otimes I_2\right)m(t) \\ &\quad + \gamma\left(Q^{-1}(t_k)\otimes I_2\right)\left(L(I_M+\Lambda(t_k))\otimes B\right) \\ &\quad \cdot \left(Q(t_k)\otimes I_2\right)m(t_k) = (I_M\otimes A)\,m(t) \\ &\quad + \gamma\left(J(t_k)\otimes B\right)m(t_k), \quad t\in[t_k,t_{k+1}),\ k\in\mathbb{N}\cup\{0\} \end{aligned} \quad (11)$$

indicating

$$\dot{m}_1(t) = Am_1(t), \quad (12a)$$

$$\begin{aligned} \dot{M}_2(t) &= (I_{M-1}\otimes A)\,M_2(t) \\ &\quad + \gamma\left(\widehat{L}(t_k)\otimes B\right)M_2(t_k), \end{aligned} \quad (12b)$$

for any $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}\cup\{0\}$, with $M_2(t) = (m_2^T(t),\dots, m_M^T(t))^T$.

Next, we will verify that the asymptotic stability of subsystem (12b) is equivalent to the synchronization of (8) if the initial states satisfy $\phi^T x(0) = \phi^T v(0) = 0$. On one hand, if $\lim_{t\to\infty} m_i(t) = \mathbf{0}_2, i \in \{2,\dots,M\}$, combining with $z(t) = (Q(t_k)\otimes I_2)m(t), t \in [t_k, t_{k+1})$ and $p_1(t_k) = [1/(1+\varepsilon_1(t_k)), 1/(1+\varepsilon_1(t_k)),\dots,1/(1+\varepsilon_M(t_k))]^T$ gives $z_i(t) \to (1/(1+\varepsilon_i(t_k)))m_1(t), t \in [t_k, t_{k+1}), i \in \{1,\dots,M\}$. Moreover, by defining

$$E(t) = \begin{bmatrix} \cos\left(\sqrt{h}(t)\right) & \dfrac{1}{\sqrt{h}}\sin\left(\sqrt{h}(t)\right) \\ -\sqrt{h}\sin\left(\sqrt{h}(t)\right) & \cos\left(\sqrt{h}(t)\right) \end{bmatrix}, \quad (13)$$

the equalities $\dot{m}_1(t) = Am_1(t)$ and $\phi^T x(0) = \phi^T v(0) = 0$ tell us that

$$\begin{aligned} m_1(t) &= e^{At}m(0) = E(t)\left(\phi^T\otimes I_2\right)z(0) \\ &= \begin{bmatrix} \cos\left(\sqrt{h}t\right)\phi^T x(0) + \dfrac{1}{\sqrt{h}}\sin\left(\sqrt{h}t\right)\phi^T v(0) \\ -\sqrt{h}\sin\left(\sqrt{h}t\right)\phi^T x(0) + \cos\left(\sqrt{h}t\right)\phi^T v(0) \end{bmatrix} \\ &= \mathbf{0}_2 \end{aligned} \quad (14)$$

for any $t \in \mathbb{R}^+\cup\{0\}$. Thus we get $\lim_{t\to\infty} z_i(t) = \mathbf{0}_2, i \in \{1,\dots,M\}$ which means that the synchronization of (8) is guaranteed with synchronization state $\mathbf{0}_2$. On the other hand, if the synchronization of (8) is ensured, by letting the synchronization state as $z^*(t) = (x^*(t), v^*(t))^T \in \mathbb{R}^2$, we will show $(x^*(t), v^*(t))^T = \mathbf{0}_2$. In fact, it holds that $m_1(t) = (\phi^T\otimes I_2)z(t) = \sum_{i=1}^M \phi_i z_i(t)$ according to $m(t) = (Q^{-1}\otimes I_2)z(t)$. Based on $\phi^T 1_M = 1$, we get $z^*(t) = m_1(t)$. Combined with (14), it gives $x^*(t) = v^*(t) = 0$, and thus $\lim_{t\to\infty} z(t) = \mathbf{0}_{2M}$. Taking $m(t) = (Q^{-1}(t_k)\otimes I_2)z(t), \forall t \in [t_k, t_{k+1})$, into consideration, we get $\lim_{t\to\infty} m_i(t) = \mathbf{0}_2, i \in \{1,\dots,M\}$, which ensures the asymptotic stability of subsystem (12b).

By the claim above and (10), we know that the asymptotic stability of (12b) is equal to that of the following subsystems:

$$\dot{m}_i(t) = Am_i(t) + \gamma\xi_i(t_k)\,Bm_i(t_k), \quad i\in\{2,\dots,M\}, \quad (15)$$

the solution of which is as

$$\begin{aligned} m_i(t) &= e^{A(t-t_k)}m_i(t_k) + \gamma\xi_i(t_k)\int_{t_k}^{t} e^{A(t-s)}ds\,Bm_i(t_k) \\ &= e^{A(t-t_k)}m_i(t_k) + \gamma\xi_i(t_k)\int_{0}^{t-t_k} e^{As}ds\,Bm_i(t_k) \\ &= E_i(t-t_k)\,m_i(t_k) + F_i\left(\xi_i(t_k), t-t_k\right)m_i(t_k) \end{aligned} \quad (16)$$

for any $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}\cup\{0\}$, $i \in \{1,\dots,M\}$, where

$$\begin{aligned} &E_i(t-t_k) \\ &= \begin{pmatrix} \cos\left(\sqrt{h}(t-t_k)\right) & \dfrac{1}{\sqrt{h}}\sin\left(\sqrt{h}(t-t_k)\right) \\ -\sqrt{h}\sin\left(\sqrt{h}(t-t_k)\right) & \cos\left(\sqrt{h}(t-t_k)\right) \end{pmatrix} \\ &F_i\left(\xi_i(t_k), t-t_k\right) \\ &= \begin{pmatrix} \dfrac{\gamma\xi_i(t_k)\left(1-\cos\left(\sqrt{h}(t-t_k)\right)\right)}{h} & 0 \\ \dfrac{\gamma\xi_i(t_k)\sin\left(\sqrt{h}(t-t_k)\right)}{\sqrt{h}} & 0 \end{pmatrix}. \end{aligned} \quad (17)$$

Let $S_i(\xi_i(t_k), t-t_k) = E_i(t-t_k) + F_i(\xi_i(t_k), t-t_k)$; then equality (16) gives

$$\begin{aligned} m_i(t) &= S_i\left(\xi_i(t_k), t-t_k\right)S_i\left(\xi_i(t_{k-1}), T\right) \\ &\quad \cdot S_i\left(\xi_i(t_{k-2}), T\right),\dots,S_i\left(\xi_i(0), T\right)m_i(0), \\ &\qquad i\in\{2,\dots,M\} \end{aligned} \quad (18)$$

for any $t \in [t_k, t_{k+1}), k \in \mathbb{N}\cup\{0\}$.

The boundedness of $S_i(\xi_i(t_k), t-t_k)$ results in that $m_i(t)$ tends to $\mathbf{0}_2$ if $\rho(S_i(\xi_i(t_k), T)) < 1, k \in \mathbb{N}\cup\{0\}, i \in \{2,\dots,M\}$. Denoting the possible values for $\xi_i(t_k), k \in \mathbb{N}\cup\{0\}, i \in \{2,\dots,M\}$ by $\xi_i^l, l \in \mathbb{N}$, then $\rho(S_i(\xi_i(t_k), T)) < 1$ is ensured by

$\rho(S_i(\xi_i^l, T)) < 1$ which is equal to the fact that the following equation's solutions satisfy $|\lambda| < 1$:

$$\begin{aligned} f_{il}(\lambda, T) &= \det\left(\lambda I_2 - S_i\left(\xi_i^l, T\right)\right) \\ &= \lambda^2 \\ &\quad - \left(2\cos\left(\sqrt{h}T\right) + \frac{1-\cos\left(\sqrt{h}T\right)}{h}\gamma\xi_i^l\right)\lambda \\ &\quad + \left(1 + \frac{\cos\left(\sqrt{h}T\right)-1}{h}\gamma\xi_i^l\right) = 0. \end{aligned} \tag{19}$$

If we set $\lambda = (s+1)/(s-1)$, characteristic equation (19) is rewritten as

$$\begin{aligned} &2\left(1-\cos\left(\sqrt{h}T\right)\right)\left(1-\frac{\gamma\xi_i^l}{h}\right)s^2 \\ &\quad + 2\left(1-\cos\left(\sqrt{h}T\right)\right)\frac{\gamma\xi_i^l}{h}s + 2\left(1+\cos\left(\sqrt{h}T\right)\right) \\ &= 0 \end{aligned} \tag{20}$$

for any $i \in \{2,\dots,M\}$, $l \in \mathbb{N}$. According to the property of bilinear transformation, $|\lambda| < 1$ achieves if and only if $\operatorname{Re}(s) < 0$. Moreover, if we define $\overline{\Lambda} = \delta I_M$ and denote the nonzero eigenvalues of $L(I_M + \overline{\Lambda})$ and $L(I_M - \overline{\Lambda})$ as $\tilde{\xi}_i$ and $\bar{\xi}_i, i \in \{2,\dots,M\}$, respectively, according to the relationship between the eigenvalues and the parameters $\varepsilon_i, i \in \{1,\dots,M\}$, we get

$$\min_{i\in\{2,\dots,M\}} \operatorname{Re}\left(\bar{\xi}_i\right) \le \operatorname{Re}\left(\xi_i^l\right) \le \max_{i\in\{2,\dots,M\}} \operatorname{Re}\left(\tilde{\xi}_i\right) \tag{21}$$

which ensures that $h \neq \gamma\xi_i^l, i \in \{2,\dots,M\}, l \in \mathbb{N}$, if (5a) is satisfied. Combining with (5b) gives that equality (20) can be transferred into, $\forall i \in \{2,\dots,M\}, l \in \mathbb{N}$,

$$s^2 + \frac{\gamma\xi_i^l}{h-\gamma\xi_i^l}s + \frac{1+\cos\left(\sqrt{h}T\right)}{1-\cos\left(\sqrt{h}T\right)}\frac{h}{h-\gamma\xi_i^l} = 0. \tag{22}$$

Considering $(1+\cos(\sqrt{h}T))/(1-\cos(\sqrt{h}T)) > 0$, Lemma 2 tells us that the roots of (22) located in the left-half plane are equal to

$$\frac{a_{il}}{\left|h-\gamma\xi_i^l\right|^2} > 0, \tag{23}$$

$$\frac{a_{il}b_{il}^2 + a_{il}^2 c_{il}}{\left|h-\gamma\xi_i^l\right|^2} - \frac{1+\cos\left(\sqrt{h}T\right)}{1-\cos\left(\sqrt{h}T\right)}b_{il}^2 > 0, \tag{24}$$

where

$$\begin{aligned} a_{il} &= h\gamma\operatorname{Re}\left(\xi_i^l\right) - \gamma^2\left|\xi_i^l\right|^2, \\ b_{il} &= h\gamma\operatorname{Im}\left(\xi_i^l\right), \\ c_{il} &= h^2 - h\gamma\operatorname{Re}\left(\xi_i^l\right) \end{aligned} \tag{25}$$

for all $i \in \{2,\dots,M\}$, $l \in \mathbb{N}$. Recalling (5a), we have that condition (23) is equivalent to

$$\gamma < \frac{h\operatorname{Re}\left(\xi_i^l\right)}{\left|\xi_i^l\right|^2}, \quad i \in \{2,\dots,M\},\ l \in \mathbb{N}, \tag{26}$$

which is guaranteed by

$$\gamma < \min_{i\in\{2,\dots,M\}, l\in\mathbb{N}} \left\{\frac{h\operatorname{Re}\left(\xi_i^l\right)}{\left|\xi_i^l\right|^2}\right\}. \tag{27}$$

Based on the relationship between the eigenvalues and the parameters $\varepsilon_i, i \in \{1,\dots,M\}$ and inequality (21), we know that

$$\min_{i\in\{2,\dots,M\}, l\in\mathbb{N}} \left\{\frac{h\operatorname{Re}\left(\xi_i^l\right)}{\left|\xi_i^l\right|^2}\right\} = \frac{h\operatorname{Re}\left(\tilde{\xi}_{i_{\max}}\right)}{\left|\tilde{\xi}_{i_{\max}}\right|^2} \tag{28}$$

with $i_{\max} = \arg\max_{i\in\{2,\dots,M\}}|\tilde{\xi}_i|$.

Moreover, (26) gives $\gamma < h/\operatorname{Re}(\xi_i^l)$ indicating $a_{il} > 0$ and $c_{il} > 0$. By defining

$$e_{il} = \frac{\left|h-\gamma\xi_i^l\right|^2 b_{il}^2}{a_{il}b_{il}^2 + a_{il}^2 c_{il}}, \tag{29}$$

we get that (24) holds if

$$T \in \Gamma_T\left(\max_{i\in\{2,\dots,M\}, l\in\mathbb{N}} \arctan\sqrt{e_{il}}\right) \tag{30}$$

with $\Gamma_T(\varphi), \varphi \in [0, \pi/2)$ defined by $\Gamma_T(\varphi) = \{T \mid \tan^2(\sqrt{h}T/2) > \tan^2\varphi\}$. Therefore, if (5a) and (5b) are satisfied, we obtain that the roots of (20) hold negative real part if (5c) and (5d) hold which means that $|\lambda| < 1$, where λ denotes the eigenvalues of $S_i(\xi_i^l, T)$. Thus $\rho(S_i(\xi_i^l, T)) < 1$ is guaranteed which gives $\lim_{t\to\infty} m_i(t) = \mathbf{0}_2$ indicating $\lim_{t\to\infty} z(t) = \mathbf{0}_{2M}$. Up to now, it shows that the synchronization of network (1) under protocol (2) is obtained when (5a), (5b), (5c), and (5d) are satisfied. This completes the proof. □

Remark 4. In this paper, the constraint on the initial state, $\phi^T x(0) = \phi^T v(0) = 0$, is a necessary condition to ensure the synchronization of (8) according to $z_i(t) \to (1/(1+\varepsilon_i(t_k)))m_1(t), t \in [t_k, t_{k+1}), i \in \{1,\dots,M\}$, in which ε_i and ε_j, $(i \neq j)$ are different from each other. How to relax the constraint on the initial state is one of our further research directions, which will need completely different proof line.

Remark 5. When we design parameters based on the above theorem, it is obvious that the difficulty lies in ascertaining the value of $\varphi = \max_{i\in\{2,\dots,M\}, l\in\mathbb{N}} \arctan\sqrt{e_{il}}$. In fact, it can always increase or decrease the value of $\varepsilon_i, i \in \{1,\dots,M\}$ to study the proportional relationship between ε_i and e_{il} and thus determine the value of φ and the feasible interval for T.

Remark 6. It is worth mentioning that the synchronization expression of CHO (1) under protocol (2) is the same as the one under protocol [9]

$$u_i(t) = \gamma \sum_{l \in \mathscr{M}_i} a_{il} \left(x_i(t_k) - x_j(t_k) \right), \quad t \in [t_k, t_{k+1}), \ i \in \{1, \ldots, M\}, \ k \in \mathbb{N}, \tag{31}$$

where the current relative sampled position data has not been quantized. This is due to the fact that $E(t)$ defined in (13), which determines the synchronization state, is not influenced by quantization.

4. Simulation

A simulation example is given in this section to show the usefulness of the result. The CHO network composed of four nodes and the adjacency matrix $\mathscr{A}$ is given as

$$\mathscr{A} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \tag{32}$$

Select parameters as $h = 0.8$, $\delta = 0.1$, and γ satisfying

$$\begin{aligned} \gamma &= 0.2 \\ &\notin \left[\frac{h}{\max_{i \in \{2,\ldots,M\}} \operatorname{Re}(\tilde{\xi}_i)}, \frac{h}{\min_{i \in \{2,\ldots,M\}} \operatorname{Re}(\tilde{\xi}_i)} \right] \\ &= [0.2424, 0.3556], \\ \gamma &= 0.2 < \frac{h \operatorname{Re}(\tilde{\xi}_{i_{\max}})}{|\tilde{\xi}_{i_{\max}}|^2} = 0.2424. \end{aligned} \tag{33}$$

Based on $\varphi = \max_{i \in \{2,\ldots,M\}, l \in \mathbb{N}} \arctan \sqrt{e_{il}} = 0.6255$ and [9]

$$\begin{aligned} \Gamma_T(\varphi) &= \left\{ T \mid \tan^2\left(\frac{\sqrt{h}T}{2} \right) > \tan^2 \varphi \right\} \\ &= \left(\bigcup_{l=0}^{\infty} \left(\frac{2(\varsigma\pi + \varphi)}{\sqrt{h}}, \frac{(2\varsigma + 1)\pi}{\sqrt{h}} \right) \right) \\ &\quad \cup \left(\bigcup_{l=1}^{\infty} \left(\frac{(2\varsigma - 1)\pi}{\sqrt{h}}, \frac{2(\varsigma\pi - \varphi)}{\sqrt{h}} \right) \right) \end{aligned} \tag{34}$$

we know $T \in (1.3987, 3.5124)$ by setting $\varsigma = 0$. Selecting $T = 3 \in \Gamma_T(\varphi)$ and the initial states as $z(0) = [-1, 0, 2, 0, -1, 2, -1, -1]^T$ satisfying $\phi^T x(0) = \phi^T v(0) = 0$ with $\phi = [5/21, 7/21, 3/21, 6/21]^T$, the synchronization states can be shown in Figure 1, from which we see that the CHO network reaches synchronization. If the initial state is set as $z(0) = [1, 0, 2, 0, -1, 2, -1, 1]^T$ which violates $\phi^T x(0) = \phi^T v(0) = 0$, Figure 2 shows that the synchronization errors do not tend to zeros, which means that the synchronization for CHO network is not reached.

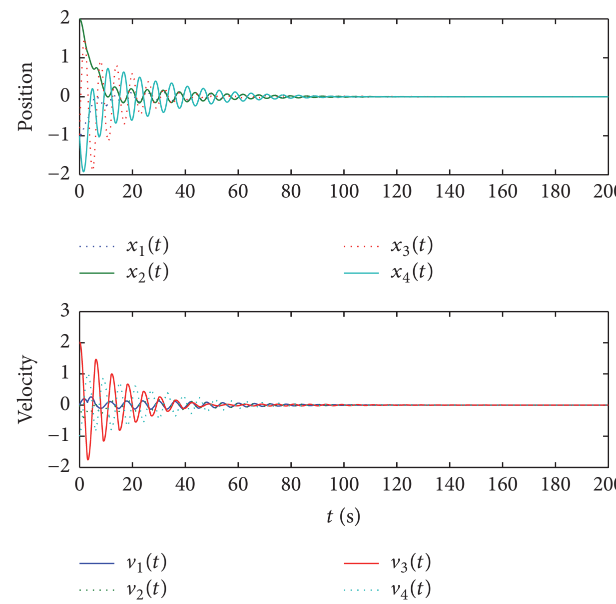

Figure 1: Synchronization with initial states $z(0) = [-1, 0, 2, 0, -1, 2, -1, -1]^T$.

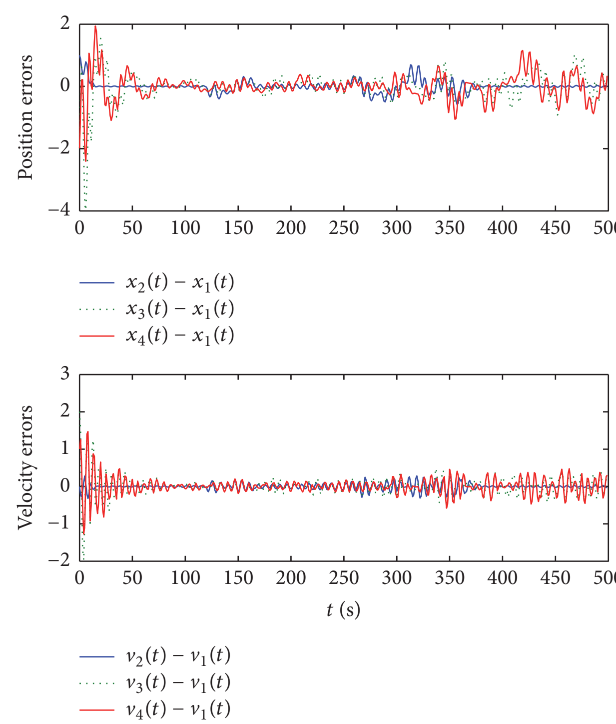

Figure 2: Synchronization errors with initial states $z(0) = [1, 0, 2, 0, -1, 2, -1, 1]^T$.

5. Conclusion

By using the quantized current sampled position data, the sufficient conditions for synchronization of CHO are given according to coupling strength, sampling period, and quantizer parameter. It is worth mentioning that the results here need some restrictions on initial states. How to obtain the synchronization for CHO network without these restrictions holds some challenges, and it is one of our further research directions.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The work was supported by the National Natural Science Foundation of China (41501562, 61403125, and 61603108) and the State Key Laboratory of Coal Resources and Safe Mining (China University of Mining and Technology) (SKLCRSM16KFD02).

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