Research Article

An Out Space Accelerating Algorithm for Generalized Affine Multiplicative Programs Problem

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This paper presents an out space branch-and-bound algorithm for solving generalized affine multiplicative programs problem. Firstly, by introducing new variables and constraints, we transform the original problem into an equivalent nonconvex programs problem. Secondly, by utilizing new linear relaxation technique, we establish the linear relaxation programs problem of the equivalent problem. Thirdly, based on the out space partition and the linear relaxation programs problem, we construct an out space branch-and-bound algorithm. Fourthly, to improve the computational efficiency of the algorithm, an out space reduction operation is employed as an accelerating device for deleting a large part of the investigated out space region. Finally, the global convergence of the algorithm is proved, and numerical results demonstrate the feasibility and effectiveness of the proposed algorithm.

1. Introduction

The generalized affine multiplicative programs problem (GAMP) arises naturally in many practical applications including management science, engineering optimization design, optimal control, Euclidean geometry, economic planning, production planning, and combinatorial mathematics [1, 2]. The problem (GAMP) usually contains many local

optimum points which are not global optimum solution, so that important theoretical difficulties and computational complexities exist. Therefore, it is very necessary to establish an effective algorithm for globally solving the problem (GAMP).

The mathematical modelling of the generalized affine multiplicative programs problem is given as follows:

$$\begin{aligned} v = \max \quad & f(y) = \sum_{i=1}^p \left(\sum_{j=1}^n c_{ij} y_j + d_i \right) \left(\sum_{j=1}^n e_{ij} y_j + f_i \right) \\ \text{s.t.} \quad & y \in Y = \{y \mid Ay \leq b\}, \end{aligned} \quad (\text{GAMP})$$

where $p \geq 2$, $c_{ij}, e_{ij}, d_i, f_i \in R$, $i = 1, \dots, p$, $j = 1, \dots, n$.

In the last decades, many algorithms have been intended for solving the generalized linear multiplicative programming problem, such as parametric simplex algorithms [3, 4],

quadratic programming algorithms [5–9], rectangle branch-and-bound algorithms [10, 11], approximate algorithms [12, 13], finite algorithm [14], outcome space algorithm [15], cutting plane algorithm [16], heuristic algorithm [17], monotonic

optimization algorithms [18, 19], and simplicial branch-and-bound algorithm [20]. Although there exist some algorithms which can be used to solve the generalized linear multiplicative programming problem, as far as we know, it is still necessary to propose a more efficient algorithm for globally solving (GAMP).

In this paper, we shall present an effective algorithm for globally solving (GAMP). First of all, we transform (GAMP) into an equivalent problem (EQ). Secondly, by utilizing the new linear relaxation technique, we systematically convert (EQ) into a series of linear relaxation programs problems, which can infinitely approximate the global optimal solution of (EQ) by a successive partition. Thirdly, based on the out space partition and the linear relaxation programs problem, we construct an out space branch-and-bound algorithm. In this algorithm, the proposed branching operation takes place in R^p , rather than R^n or R^{2p} ; this economizes the required computation. Fourthly, to improve the computational efficiency of the algorithm, an out space reduction operation is employed as an accelerating device for deleting a large part of the investigated out space region. Finally, the global convergence of the algorithm is proved, and numerical experimental results demonstrate the feasibility and effectiveness of the proposed algorithm.

This paper is organized as follows. In Section 2, by introducing new variables and constraints, we establish the equivalent problem (EQ) of the initial problem (GAMP). Next, by utilizing the linearization technique, we derive the linear relaxation programming problem of (EQ), which can provide a reliable upper bound for the global optimal value of the (EQ). In Section 3, an out space region reduction operation is introduced, and by combining this operation with bounding technique, an out space accelerating branch-and-bound algorithm is established and its global convergence is proved. Section 4 presents the numerical experimental results obtained by solving some test examples. Finally, the some concluding remarks of this paper are elaborated.

2. Equivalent Problem and Its Linear Relaxation

In this section, we will convert the problem (GAMP) into an equivalent nonconvex problem (EQ). Since $\sum_{j=1}^n c_{ij}y_j + d_i$ and $\sum_{j=1}^n e_{ij}y_j + f_i$ are all finite affine functions defined in Y , there exist positive constant numbers M_i and N_i such that $\sum_{j=1}^n c_{ij}y_j + d_i + M_i > 0$ and $\sum_{j=1}^n e_{ij}y_j + f_i + N_i > 0$ for any $y \in Y$. Therefore, in the following, without loss of generality, we can assume that $\sum_{j=1}^n c_{ij}y_j + d_i \geq 0$ and $\sum_{j=1}^n e_{ij}y_j + f_i \geq 0$ for all $y \in Y$.

Without loss of generality, we let

$$\begin{aligned} L_i^0 &= \min_{y \in Y} \left(\sum_{j=1}^n e_{ij}y_j + f_i \right), \\ U_i^0 &= \max_{y \in Y} \left(\sum_{j=1}^n e_{ij}y_j + f_i \right), \\ i &= 1, \dots, p. \end{aligned} \quad (1)$$

Define the set $Z^0 = \{z \in R^p \mid L_i^0 \leq z_i \leq U_i^0, i = 1, \dots, p\}$. For any $y \in Y$ and $z \in Z^0$, define the function $g(y, z) = \sum_{i=1}^p [z_i (\sum_{j=1}^n c_{ij}y_j + d_i)]$, and consider the following equivalent problem:

$$\begin{aligned} \max \quad & g(y, z) = \sum_{i=1}^p \left[z_i \left(\sum_{j=1}^n c_{ij}y_j + d_i \right) \right] \\ \text{s.t.} \quad & \sum_{j=1}^n e_{ij}y_j + f_i - z_i \geq 0, \quad i = 1, \dots, p, \\ & y \in Y, \\ & z \in Z^0. \end{aligned} \quad (\text{EQ})$$

Obviously, if (y^*, z^*) is a global optimal solution for (EQ), then y^* is a global optimal solution for (GAMP), and $z_i^* = \sum_{j=1}^n e_{ij}y_j^* + f_i, i = 1, \dots, p$. If y^* is a global optimal solution for (GAMP), then (y^*, z^*) is a global optimal solution for (EQ), where $z_i^* = \sum_{j=1}^n e_{ij}y_j^* + f_i, i = 1, \dots, p$.

In the following, the main computation is to globally solve (EQ). Next, we will construct the linear relaxation programs problem of (EQ), which can provide the upper bounds for (EQ) in the proposed branch-and-bound algorithm. Assume that $Z = \{z \in R^p \mid L_i \leq z_i \leq U_i, i = 1, \dots, p\} \subseteq Z^0$. And without loss of generality, let

$$\begin{aligned} T_i^+ &= \{j \mid c_{ij} \geq 0, j = 1, \dots, n\}, \\ T_i^- &= \{j \mid c_{ij} < 0, j = 1, \dots, n\}, \\ D_i^+ &= \{i \mid d_i \geq 0, i = 1, \dots, p\}, \\ D_i^- &= \{i \mid d_i < 0, i = 1, \dots, p\}. \end{aligned} \quad (2)$$

Then the linear relaxation programs (U_Z) of (EQ) in Z can be given as follows:

$$\begin{aligned} u_Z = \max \quad & \sum_{i=1}^p \left(\sum_{T_i^+} U_i c_{ij} y_j + \sum_{T_i^-} L_i c_{ij} y_j \right) + \sum_{D_i^+} U_i d_i \\ & + \sum_{D_i^-} L_i d_i \end{aligned} \quad (U_Z)$$

$$\begin{aligned} \text{s.t.} \quad & L_i - \left(\sum_{j=1}^n e_{ij}y_j + f_i \right) \leq 0, \quad i = 1, \dots, p, \\ & y \in Y, \end{aligned}$$

where (EQ) in Z is defined by

$$\begin{aligned} v_Z = \max \quad & g(y, z) = \sum_{i=1}^p z_i \left(\sum_{j=1}^n c_{ij}y_j + d_i \right) \\ \text{s.t.} \quad & \left(\sum_{j=1}^n e_{ij}y_j + f_i \right) - z_i \geq 0, \\ & i = 1, \dots, p, \\ & y \in Y, \\ & z \in Z \subseteq Z^0. \end{aligned} \quad (\text{EQ}_Z)$$

Theorem 1. (i) For any $Z \subseteq Z^0$, one has $u_Z \geq v_Z$. (ii) If $\bar{Z} = \{z \in R^p \mid \bar{L} \leq z \leq \bar{U}\} \subseteq Z$ and $\bar{Z} \neq \emptyset$, then $u_{\bar{Z}} \leq u_Z$.

Proof. (i) If $v_Z = -\infty$, then we can easily obtain $u_Z = -\infty$; it is obvious that $u_Z \geq v_Z$, if v_Z is finite and achieved. Denote by $(\bar{y}, \bar{z})$ a global optimal solution for (EQ_Z) . Then $\bar{y} \in Y, \bar{z} \in Z$. For each $i \in \{1, \dots, p\}$, we have $L_i \leq \bar{z}_i \leq U_i$. Therefore, we have

$$\sum_{i=1}^p \bar{z}_i \left(\sum_{j=1}^n c_{ij} \bar{y}_j + d_i \right) \leq \sum_{i=1}^p \left(\sum_{T_i^+} U_i c_{ij} \bar{y}_j + \sum_{T_i^-} L_i c_{ij} \bar{y}_j \right) + \sum_{D_i^+} U_i d_i + \sum_{D_i^-} L_i d_i. \quad (3)$$

Since $(\bar{y}, \bar{z})$ is a global optimal solution for (EQ_Z) , it follows that

$$v_Z = \sum_{i=1}^p \bar{z}_i \left(\sum_{j=1}^n c_{ij} \bar{y}_j + d_i \right). \quad (4)$$

Since $\bar{y}$ is a feasible solution for (U_Z) ,

$$u_Z \geq \sum_{i=1}^p \left(\sum_{T_i^+} U_i c_{ij} \bar{y}_j + \sum_{T_i^-} L_i c_{ij} \bar{y}_j \right) + \sum_{D_i^+} U_i d_i + \sum_{D_i^-} L_i d_i. \quad (5)$$

By the above discussions, we have $u_Z \geq v_Z$.

(ii) Let $\bar{Z} = \{z \in R^p \mid \bar{L} \leq z \leq \bar{U}\} \subseteq Z$ and $\bar{Z} \neq \emptyset$; we have the following conclusions.

If (U_Z) is infeasible, then we can easily prove that $(U_{\bar{Z}})$ is also infeasible; the conclusion is obvious.

If (U_Z) is feasible, since Y is compact set, then u_Z is finite. If $(U_{\bar{Z}})$ is infeasible, then $u_{\bar{Z}} = -\infty < u_Z$. Thus, in the following, we assume that $(U_{\bar{Z}})$ is feasible, then $u_{\bar{Z}}$ is finite, and there exists a point $\bar{y} \in Y$ such that $\bar{L}_i \leq \sum_{j=1}^n e_{ij} \bar{y}_j + f_i, i = 1, \dots, p$. And

$$u_{\bar{Z}} = \sum_{i=1}^p \left(\sum_{T_i^+} \bar{U}_i c_{ij} \bar{y}_j + \sum_{T_i^-} \bar{L}_i c_{ij} \bar{y}_j \right) + \sum_{D_i^+} \bar{U}_i d_i + \sum_{D_i^-} \bar{L}_i d_i. \quad (6)$$

By $\bar{Z} \subseteq Z$, we have $0 < L_i \leq \bar{L}_i \leq \bar{U}_i \leq U_i, i = 1, \dots, p$. It follows that $\bar{y}$ is a feasible solution of problem (U_Z) and

$$u_{\bar{Z}} \leq \sum_{i=1}^p \left(\sum_{T_i^+} U_i c_{ij} \bar{y}_j + \sum_{T_i^-} L_i c_{ij} \bar{y}_j \right) + \sum_{D_i^+} U_i d_i + \sum_{D_i^-} L_i d_i. \quad (7)$$

Since $\bar{y}$ is a feasible solution of (U_Z) , we have

$$u_Z \geq \sum_{i=1}^p \left(\sum_{T_i^+} U_i c_{ij} \bar{y}_j + \sum_{T_i^-} L_i c_{ij} \bar{y}_j \right) + \sum_{D_i^+} U_i d_i + \sum_{D_i^-} L_i d_i. \quad (8)$$

By the above discussions, we have $u_Z \geq u_{\bar{Z}}$, and the proof is completed. $\square$

By Theorem 1, (U_Z) can provide the valid upper bound for (EQ_Z) .

3. Algorithm and Its Convergence

In this section, we shall present a branch-and-bound algorithm for solving (EQ). The critical operation in guaranteeing the global convergence of the proposed algorithm is the selection of a suitable branching technique. In this paper, we choose a simple bisection method. For any selected rectangle $\widehat{Z} = \{z \in R^p \mid \widehat{L}_i \leq z_i \leq \widehat{U}_i, i = 1, \dots, p\} \subseteq Z^0$, this branching rule is given as follows:

$$(1) \text{ Let } \widehat{U}_k - \widehat{L}_k = \max\{\widehat{U}_i - \widehat{L}_i \mid i = 1, \dots, p\}.$$

$$(2) \text{ Let}$$

$$\begin{aligned} \widehat{Z}_1 &= \left\{ z \in R^p \mid \widehat{L}_i \leq z_i \leq \widehat{U}_i, i = 1, \dots, p, i \neq k; \widehat{L}_k \right. \\ &\quad \left. \leq z_k \leq \frac{1}{2} (\widehat{L}_k + \widehat{U}_k) \right\}, \end{aligned} \quad (9)$$

$$\begin{aligned} \widehat{Z}_2 &= \left\{ z \in R^p \mid \widehat{L}_i \leq z_i \leq \widehat{U}_i, i = 1, \dots, p, i \right. \\ &\quad \left. \neq k; \frac{1}{2} (\widehat{L}_k + \widehat{U}_k) \leq z_k \leq \widehat{U}_k \right\}. \end{aligned}$$

3.1. New Region Reduction Operation. For any rectangle $Z^k \subseteq Z^0$, during the process of iteration, we want to recognize whether or not Z^k contains a global optimal solution of (EQ). The proposed new reduction operation aims at replacing the rectangle $Z^k = [L^k, U^k]$ with a smaller rectangle $Z' = [L', U']$ without deleting any global optimal solution of (EQ).

Without loss of generality, we can assume that LB^k is a known lower bound of the optimal value for $EQ(Z^0)$ and that $v(Z)$ is the maximum value of $g(y, z)$ over Z and Y , and let

$$\begin{aligned} l_i^0 &= \min_{y \in Y} \left(\sum_{j=1}^n c_{ij} y_j + d_i \right), \quad i = 1, 2, \dots, p, \\ u_i^0 &= \max_{y \in Y} \left(\sum_{j=1}^n c_{ij} y_j + d_i \right), \quad i = 1, 2, \dots, p, \end{aligned} \quad (10)$$

$$RUB^k = \sum_{i=1}^p u_i^0 U_i^k,$$

$$\rho_i^k = \frac{LB^k - RUB^k + u_i^0 U_i^k}{u_i^0}, \quad i = 1, 2, \dots, p.$$

Theorem 2. For any subrectangle $Z^k = (Z_i^k)_{p \times 1} = [L_i^k, U_i^k]_{p \times 1} \subseteq Z^0$, the following conclusions hold:

(i) If $RUB^k < LB^k$, then there is no global optimal solution of $EQ(Z^0)$ over Z^k .

(ii) If $RUB^k \geq LB^k$, then, for each $s \in \{1, 2, \dots, p\}$, there is no global optimal solution of the $EQ(Z^0)$ over $\underline{Z}^k$, where

$$\underline{Z}^k = (\underline{Z}_i^k)_{p \times 1} \subseteq Z^0$$

$$\text{with } \underline{Z}_i^k = \begin{cases} Z_i^k, & i \neq s, i = 1, 2, \dots, p, \\ [L_i^k, \rho_i^k] \cap Z_i^k & i = s \in \{1, 2, \dots, p\}. \end{cases} \quad (11)$$

Proof. (i) If $RUB^k < LB^k$, then we have

$$\begin{aligned} v(Z^k) &= \max_{z \in Z^k, y \in Y} \sum_{i=1}^p \left(\sum_{j=1}^n c_{ij} y_j + d_i \right) z_i \\ &\leq \sum_{i=1}^p \max_{z \in Z^k, y \in Y} \left[\left(\sum_{j=1}^n c_{ij} y_j + d_i \right) z_i \right] = \sum_{i=1}^p u_i^0 U_i^k \quad (12) \\ &= RUB^k < LB^k. \end{aligned}$$

Therefore, there exists no global optimal solution of the problem $EQ(Z^0)$ over Z^k .

(ii) If $RUB^k \geq LB^k$, then we have the following conclusions.

For each $s \in \{1, 2, \dots, p\}$, for any $y \in Y$ and $z \in \underline{Z}^k$, and for each $i = 1, 2, \dots, p$, since

$$\begin{aligned} 0 &\leq l_i^0 \leq \sum_{j=1}^n c_{ij} y_j + d_i \leq u_i^0, \\ 0 &\leq L_i^k \leq z_i \leq U_i^k, \end{aligned} \quad (13)$$

$$i \neq s,$$

$$0 \leq L_s^k \leq z_s \leq \rho_s^k, \quad i = s,$$

we have

$$\begin{aligned} v(\underline{Z}^k) &= \max_{z \in \underline{Z}^k, y \in Y} \left[\sum_{i=1}^p \left(\sum_{j=1}^n c_{ij} y_j + d_i \right) z_i \right] \\ &\leq \max_{z \in \underline{Z}^k, y \in Y} \left[\sum_{i=1, i \neq s}^p \left(\sum_{j=1}^n c_{ij} y_j + d_i \right) z_i \right] \\ &\quad + \max_{z \in \underline{Z}^k, y \in Y} \left[\left(\sum_{j=1}^n c_{sj} y_j + d_s \right) z_s \right] \quad (14) \\ &< \sum_{i=1, i \neq s}^p u_i^0 U_i^k + u_s^0 \rho_s^k \\ &= RUB^k - u_s^0 U_s^k + LB^k - (RUB^k - u_s^0 U_s^k) \\ &= LB^k. \end{aligned}$$

Therefore, there exists no global optimization solution of $EQ(Z^0)$ over $\underline{Z}^k$. $\square$

By Theorem 2, we can construct a new reduction operation to cut away a part of region in which the global optimal solution of (EQ) does not exist. Assuming that a subrectangle

$$Z^k = (Z_i^k)_{p \times 1} \subseteq Z^0 \quad \text{with } Z_i^k = [L_i^k, U_i^k] \quad (15)$$

will be compressed, then, according to Theorem 2, the original rectangle Z^k should be replaced with a new subrectangle:

$$\begin{aligned} Z' &= (Z'_i)_{p \times 1} \\ &\text{with } Z'_i = [\rho_i^k, U_i^k] \cap Z_i^k, \quad i \in \{1, 2, \dots, p\}. \end{aligned} \quad (16)$$

Let $UB(Z^k)$ be the optimal value of (U_Z) in Z^k ; the basic steps of the proposed algorithm are summarized as follows.

3.2. Algorithm Steps

Step 0. Initialize the iteration counter $k = 0$, the active node set $\Omega_0 = \{Z^0\}$, the lower bound $LB_0 = -\infty$, and the feasible points set $F = \emptyset$.

Solve (U_{Z^0}) ; obtain u_{Z^0} and $y(Z^0)$. Set $UB_0 = UB(Z^0) = u_{Z^0}$, $y^c = y(Z^0)$, and set $z_i^c = \sum_{j=1}^n e_{ij} y_j^c + f_i$ ($i = 1, 2, \dots, p$), $LB_0 = g(y^c, z^c)$. Update $F = F \cup \{(y^c, z^c)\}$.

If $UB_0 - LB_0 \leq \epsilon$, then the algorithm stops with (y^c, z^c) and y^c as global optimal solutions of (EQ) and (GAMP), respectively. Otherwise, proceed to next step.

Step 1. Use the proposed reduction operation to compress the investigated rectangle Z^k , and still represent remaining part of the rectangle as Z^k .

Step 2. Subdivide Z^k into two subrectangles; denote by Θ_k its partitioning set.

For each $Z \in \Theta_k$, solve (U_Z) to obtain $UB(Z) = u_Z$ and $y(Z)$. If $UB(Z) < LB_k$, then remove the subrectangle Z from Θ_k , that is, $\Theta_k := \Theta_k \setminus Z$, and skip to next element of Θ_k .

If $\Theta_k = \emptyset$, set $LB_{k+1} = LB_k$, and go to Step 3. Otherwise, set $z_i^c = z_i(Z) = \sum_{j=1}^n e_{ij} y_j(Z) + f_i$, $i = 1, 2, \dots, p$, update $F = F \cup \{(y(Z), z(Z))\}$, set $LB_{k+1} = \max_{(y,z) \in F} g(y, z)$, and denote by $(y^c, z^c) = \arg\max_{(y,z) \in F} g(y, z)$ the best known feasible point.

Step 3. Denote by $\Omega_{k+1} = (\Omega_k \setminus Z^k) \cup \Theta_k$ the remaining partition set. We can obtain a new upper bound $UB_k := \inf_{Z \in \Omega_{k+1}} UB(Z)$.

Step 4. Fathom any nonimproving nodes by setting $\Omega_{k+1} = \Omega_k \setminus \{Z : UB(Z) - LB \leq \epsilon, Z \in \Omega_k\}$. If $\Omega_{k+1} = \emptyset$, then stop: LB is the optimal value of (EQ), and y^c is a global ϵ -optimal solutions for (GAMP). Otherwise, set $k = k + 1$ and return to Step 1.

3.3. Convergence of the Algorithm. In this subsection, the convergence of the algorithm is given.

Theorem 3. If the algorithm is finite, upon termination, we can obtain the ϵ -global optimal solutions for the problems (EQ) and

(GAMP), respectively. If the algorithm is infinite, for each $k \geq 0$, denote by y^k the incumbent solution of (GAMP), which is found at the end of iteration k . Then, $\lim_{k \rightarrow \infty} y^k = y^*$ will be a global optimal solution for (GAMP).

Proof. If the algorithm is finite, then the algorithm terminates at iteration k , $k \geq 0$. By the termination condition of the algorithm and the updating method of the upper bound, it can follow easily that the conclusion is obvious.

If the algorithm is infinite, then, it generates a sequence of incumbent solutions for problem (EQ), which may be denoted by $\{(y^k, z^k)\}$. For each $k \geq 1$, (y^k, z^k) is found by solving problem (U_{Z^k}) , for some rectangle $Z^k \subseteq Z^0$, for an optimal solution $y^k \in Y$, and setting $z_i^k = \sum_{j=1}^n e_{ij} y_j^k + f_i$, $i = 1, \dots, p$. Therefore, the sequence $\{y^k\}$ consists of feasible solutions for problem (GAMP). Let y^* be an accumulation point of $\{y^k\}$, and without loss of generality we assume that $\lim_{k \rightarrow \infty} y^k = y^*$. Then, since Y is a compact set, $y^* \in Y$. Furthermore, since $\{y^k\}$ is infinite, we may assume without loss of generality that, for each k , $Z^{k+1} \subseteq Z^k$. Since the rectangles Z^k , $k \geq 1$, are formed by rectangular bisection, this implies that, for some point $z^* \in R^p$, we have

$$\lim_{k \rightarrow \infty} Z^k = \bigcap_k Z^k = \{z^*\}. \quad (17)$$

Let $Z^* = \{z^*\}$ and, for each k , let Z^k be given by

$$Z^k = \{z \in R^p \mid L^k \leq z \leq U^k\}. \quad (18)$$

For each k , from Step 2 of the algorithm,

$$UB(Z^k) = u_{Z^k}. \quad (19)$$

Since $Z^{k+1} \subset Z^k \subseteq Z^0$, for each k , by Theorem 1 and Step 4, this implies that $\{UB(Z^k)\}$ is a nonincreasing sequence bounded by v . Therefore, $\lim_{k \rightarrow \infty} UB(Z^k)$ is a finite number and satisfies

$$\lim_{k \rightarrow \infty} UB(Z^k) \geq v. \quad (20)$$

For each k , from Step 1, $UB(Z^k)$ is equal to the optimal value of problem (U_{Z^k}) given by

$$\begin{aligned} \max \quad & \sum_{i=1}^p \left(\sum_{T_i^+} U_i^k c_{ij} y_j + \sum_{T_i^-} L_i^k c_{ij} y_j \right) + \sum_{D_i^+} U_i^k d_i \\ & + \sum_{D_i^-} L_i^k d_i \end{aligned} \quad (U_{Z^k})$$

$$\text{s.t.} \quad L_i^k - \left(\sum_{j=1}^n e_{ij} y_j + f_i \right) \leq 0, \quad i = 1, \dots, p,$$

$$y \in Y,$$

and y^k is an optimal solution for this problem. By the branching technique, we have

$$\lim_{k \rightarrow \infty} L^k = \lim_{k \rightarrow \infty} U^k = \{z^*\} = Z^*. \quad (21)$$

Since $\lim_{k \rightarrow \infty} y^k = y^*$, $L_i^k \leq \sum_{j=1}^n e_{ij} y_j^k + f_i \leq U_i^k$ and the continuity of $(\sum_{j=1}^n e_{ij} y_j + f_i)$, for each $i = 1, \dots, p$, $\sum_{j=1}^n e_{ij} y_j^* + f_i = z_i^*$. This implies that (y^*, z^*) is a feasible solution for (EQ). Therefore, $g(y^*, z^*) \leq v$. Combining (20), we obtain that

$$g(y^*, z^*) \leq v \leq \lim_{k \rightarrow \infty} UB(Z^k), \quad (22)$$

since

$$\begin{aligned} \lim_{k \rightarrow \infty} UB(Z^k) &= \lim_{k \rightarrow \infty} \left(\sum_{i=1}^p \left(\sum_{T_i^+} U_i^k c_{ij} y_j^k + \sum_{T_i^-} L_i^k c_{ij} y_j^k \right) \right. \\ &\quad \left. + \sum_{D_i^+} U_i^k d_i + \sum_{D_i^-} L_i^k d_i \right) = \sum_{i=1}^p z_i^* \left(\sum_{j=1}^n c_{ij} y_j^* + d_i \right) \\ &= g(y^*, z^*). \end{aligned} \quad (23)$$

From (22) and (23), since $UB(Z^k) = UB_k$, we have $\lim_{k \rightarrow \infty} UB_k = v = g(y^*, z^*)$. Therefore, (y^*, z^*) is a global optimal solution for (EQ). By Theorem 1, this implies that y^* is a global optimal solution for (GAMP). The proof is complete. $\square$

4. Numerical Experiments

To verify the performance of the proposed algorithm, some test problems are implemented on an Intel(R) Core(TM)2 Duo CPU (1.58 GHZ) microcomputer. The algorithm is coded in C++, and each linear programs problem is solved by simplex method in our experiments. These test problems are given in the following, and their numerical results are listed in Tables 1 and 2.

Example 1 (see [14]). One has

$$\begin{aligned} \max \quad & (3y_1 + 5y_2 + 3y_3 + 50) \\ & \times (3y_1 + 4y_2 + 5y_3 + 50) + (3y_1 + 4y_2 + 50) \\ & \times (4y_1 + 3y_2 + 2y_3 + 50) \\ & + (4y_1 + 2y_2 + 4y_3 + 50) \\ & \times (5y_1 + 4y_2 + 3y_3 + 50) \\ \text{s.t.} \quad & 6y_1 + 3y_2 + 3y_3 \leq 10, \\ & 10y_1 + 3y_2 + 8y_3 \leq 10, \\ & y_1, y_2, y_3 \geq 0. \end{aligned} \quad (24)$$

TABLE 1: Computational results of Examples 1 and 2.

Example	Ref.	Optimal solution	Optimal value	Iteration	ϵ	Time(s)
Example 1	This paper	(0, 3.33333, 0)	11611.1	2	10^{-6}	0.0065
	[14]	(0, 3.33333, 0)	11611.1	2	10^{-6}	0.0074
Example 2	This paper	(0, 0.625, 1.875)	14214.8	1	10^{-6}	0.0198
	[14]	(0, 0.625, 1.875)	14214.8	2	10^{-6}	0.0228

TABLE 2: Numerical results for Example 3.

n	p	Time(s)
10	10	0.0285
20	10	0.0192
30	10	0.0279
30	30	0.0590
30	40	0.1102
30	50	0.0945
35	50	0.1283
40	50	0.2423
50	50	0.2899

Example 2 (see [14]). One has

$$\begin{aligned}
\max \quad & (4y_1 + 3y_2 + 3y_3 + 50) \times (3y_2 + 3y_3 + 50) \\
& + (3y_1 + 4y_3 + 50) \times (4y_1 + 4y_2 + 5y_3 + 50) \\
& + (y_1 + 2y_2 + 5y_3 + 50) \\
& \times (y_1 + 5y_2 + 5y_3 + 50) \\
& + (y_1 + 2y_2 + 4y_3 + 50) \times (5y_2 + 4y_3 + 50) \\
\text{s.t.} \quad & 2y_1 + y_2 + 5y_3 \leq 10, \\
& y_1 + 6y_2 + 3y_3 \leq 10, \\
& 5y_1 + 9y_2 + 2y_3 \leq 10, \\
& 9y_1 + 7y_2 + 3y_3 \leq 10, \\
& y_1, y_2, y_3 \geq 0.
\end{aligned} \tag{25}$$

In Table 1, the notations have been used for column headers: iteration, number of iterations; time: execution time(s) of the algorithm in seconds.

Example 3. One has

$$\begin{aligned}
\max \quad & \sum_{i=1}^p \left(\sum_{j=1}^n c_{ij} y_j + d_i \right) \left(\sum_{j=1}^n e_{ij} y_j + f_i \right) \\
\text{s.t.} \quad & 0 \leq y_j \leq \gamma_j, \quad j = 1, \dots, n,
\end{aligned} \tag{26}$$

where c_{ij}, e_{ij}, d_i, f_i , $i = 1, \dots, p$, $j = 0, 1, \dots, n$ are randomly generated between 0 and 1; γ_j ($j = 1, \dots, n$) were randomly generated between 0 and 16.

Numerical results for Example 3 are given in Table 2, where the convergence tolerance is set to $\epsilon = 10^{-6}$.

From Tables 1 and 2, it is seen that the proposed algorithm in this paper can be used to globally solve the problem (GAMP) with the effectiveness and robustness.

5. Concluding Remarks

In this article, an out space accelerating branch-and-bound algorithm is presented for globally solving the generalized affine multiplicative programs problem (GAMP). First of all, we transform the (GAMP) into an equivalent problem (EQ). By utilizing the new linear relaxation technique, we systematically convert (EQ) into a series of linear relaxation programs problems, which can infinitely approximate the global optimal solution of (EQ) by a successive partition. To improve the computational efficiency of the algorithm, we introduce an out space reduction operation, which offers a theoretical possibility of reducing a large part of the investigated out space region and which can be seen as an accelerating device for improving the convergent speed of the proposed algorithm. By combining the established linear relaxed programs problem with the new reduction operation in a branch-and-bound framework, we design an out space accelerating algorithm for effectively solving (GAMP). By subsequently dividing the initial rectangle and subsequently solving a series of linear relaxed programs problems, the presented algorithm is convergent to the global minimum of (GAMP). Compared with the known algorithms, numerical results demonstrate that the proposed algorithm has higher computational efficiency.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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