Research Article

Probabilistic Behavior Analysis of MIMO Fading Channels under Geometric Mean Decomposition

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Geometric mean decomposition (GMD) has been proposed as a method to realize multiple spatial links with identical gains that are intrinsic to a MIMO channel. In order to simplify system design and implementation based on knowledge regarding probability behavior of MIMO-GMD schemes, the main objective of this paper is to statistically characterize the link gains and channel capacities that can be provided via GMD. In particular, closed-form univariate and bivariate probability density functions (PDFs) for these metrics under Rayleigh fading are derived using Gamma approximations. By applying these analytical results, the fluctuations of MIMO-GMD schemes are examined by modeling both link gains and capacities using finite-state Markov chains (FSMCs).

1. Introduction

Numerous research results have shown that spatial multiplexing-based multiple-input multiple-output (MIMO) transceivers can boost the data rate of wireless communication systems without using additional power and channel bandwidth. When both transmitter and receiver are equipped with multiple antennas, multiple independent data streams can be concurrently launched into the air by the transmitter and separately decoded at the receiver, with aid of certain signal processing techniques [1]. By combining MIMO with the multicarrier schemes such as orthogonal frequency division multiplexing (the so-called MIMO-OFDM), channel capacity can be tremendously increased as the degree of freedom in frequency and spatial domain are jointly utilized.

By assuming that channel state information (CSI) is available at both ends of the link, singular value decomposition (SVD) [2] is commonly considered as the approach to extract spatial links within a MIMO channel. With SVD, the MIMO channel matrix can be transformed into a bank of parallel scalar air-pipes (widely termed as eigenmodes in the literature) by applying appropriate beam-forming matrices at

both transmitter and receiver. Since the eigenmodes realized by SVD are equivalent to singular values of the MIMO channel matrix, their link strengths could be quite different. Therefore, proper bit-allocation and power distribution (such as water-filling) algorithms must be employed to optimize the system capacity. This, however, increases the complexity in adaptive control mechanism, as bit/power allocations among eigenmodes may have to be updated frequently in rapidly time-varying channels. For the sake of convenience, one may simply apply the modulation/coding format or distribute the power uniformly on all eigenmodes. Nevertheless, this may cause degradation on error performance of the system, especially when magnitude of the minimum channel singular value is too small [3].

Geometric mean decomposition (GMD), proposed by Jiang et al. [4], has emerged as an alternative method of designing MIMO transceiver. A set of parallel spatial links with identical gains intrinsic to a MIMO channel can be realized via GMD. Remarkably, these parallel links with the identical gain are equal to the geometric mean of channel singular values (eigenmodes). The property of parallel links with identical gain makes GMD useful for adaptive MIMO-OFDM systems, as complicated joint bit/power allocation

algorithms in frequency and spatial domain can be significantly simplified. In recent years, much attention from both academia and industry has been paid to the GMD-based MIMO transceiver. For instance, GMD has been deemed as a prospective MIMO architecture in some next-generation cellular broadband networks, including two of the most important standards in this context—LTE-Advanced [5] and WiMAX (IEEE 802.16 m) [6]. Note that Jiang et al. have put forward another MIMO transceiver dubbed as uniform channel decomposition (UCD) [7], which also provides spatial links with identical gains. The scope of this paper is focussed on GMD, and the related work on UCD is, however, remained as open problems for future research.

In order to calculate performance metrics such as bit error rate and outage probability of a GMD-based MIMO transceiver operating in mobile fading channels, analytical results on statistical properties of MIMO eigenmode geometric mean are required. To the best of our knowledge, however, only a handful of papers in the existing literature have offered the relevant analysis. In [8], asymptotic behavior of geometric mean of MIMO channel eigenvalues has been examined. Also, the probability density function (PDF) for the link gain of MIMO-GMD in special cases with two antennas at both transmitter and receiver can be found in [9], but the results for more general cases are not available. This motivated us to derive closed-form expressions of PDFs (either exact or approximated) for systems with arbitrary number of antennas in Rayleigh fading channels. Specifically, apart from the univariate PDFs, we are also interested in the bivariate PDFs for MIMO eigenmode geometric mean and channel capacity, since the statistical behavior relating to channel variation is crucial for systems incorporating with adaptive transmission techniques. For instance, knowledge on how channel quality evolves with time allows engineers to determine the adaptation/feedback rate for MIMO-GMD in a more judicious manner. Further, for successive interference cancellation (SIC) detection algorithm in conventional MIMO-GMD transceivers [4], the reliability of a data stream is dependent on previously decoded data streams. That is, if the receiver fails to detect a data stream correctly, the data on the remaining streams may not be decoded successfully due to error propagation. Thus, it is possible to gauge how frequently that an error propagation event would occur if time-varying characteristics are known. In correspondence, this paper applies first-order finite-state Markov chains (FSMC) [10, 11] to model the fluctuations of both link gain and channel capacity under MIMO-GMD in baseline cases of Rayleigh fading environments. Note that FSMC has been used to emulate the capacity process of general MIMO wireless channels [12], but channels with MIMO-GMD transceiver have not been considered. Also, some researchers have designed the feedback mechanisms for adaptive modulation systems with channels modeled by FSMC [13]. Based on the univariate and bivariate PDFs developed in this paper, the transition probabilities among the discrete states of an FSMC are analytically computed.

In a nutshell, this paper encompasses the following contributions.

- (1) Derivation of the exact PDFs for link gains and channel capacities of MIMO-GMD systems with two antennas at either transmitter or receiver and an arbitrary number of antennas on the other side.
- (2) Demonstration of Gamma approximations for link gains of MIMO-GMD systems with arbitrary numbers of antenna at either transmitter or receiver.
- (3) Derivation of PDFs for channel capacities of MIMO-GMD systems with arbitrary numbers of antenna at either transmitter or receiver, based on Gamma approximation of link gains.
- (4) Extending the Gamma approximation to provide bivariate PDFs for link gains and channel capacities of MIMO-GMD systems with numbers of antenna at either transmitter or receiver.
- (5) Constructing first-order FSMCs for both link gains and channel capacities of MIMO-GMD systems. The transition probabilities among multiple quantized Markov states are computed based on the derived PDFs.

The rest of the paper is organized as follows. In Section 2, assumptions on channel model and background of the MIMO-GMD architecture are briefly reviewed. Then, in Section 3, the closed-form expressions of both univariate and bivariate PDFs for eigenmode geometric mean and channel capacity of MIMO-GMD schemes are derived. Based on the derived analytical results, FSMCs are constructed in Section 4 to study the time variation of MIMO channel under GMD. Finally, this paper ends with a concise conclusion drawn in Section 5.

2. Background and Assumptions

In this section, we first describe the considered system model. Then, the concept of GMD and the fundamental structure of an FSMC are reviewed subsequently.

2.1. Channel Model. In general, this paper mainly studies a (N_t, N_r) single-user, point-to-point MIMO system that has N_t transmit antennas and N_r receive antennas. The MIMO channel, H , is therefore an $N_r \times N_t$ matrix. Here, we denote $m = \min(N_t, N_r)$ and $n = \max(N_t, N_r)$. We presume this MIMO system is coupled with OFDM setting, so channel response per subcarrier is basically flat in frequency domain. Therefore, the signal model for each subcarrier can be written as:

$$y = Hx + z, \quad (1)$$

where y is an $N_r \times 1$ vector of the received signal, x is an $N_t \times 1$ vector of the transmitted data symbols, and z is an $N_r \times 1$ vector of additive complex zero mean Gaussian noise. Since Rayleigh fading is considered, all entries of H are modeled as complex Gaussian random variables with zero mean and unit variance, $\mathcal{CN}(0, 1)$. With SVD, the MIMO channel H can be written as

$$H = U\Sigma V^\dagger, \quad (2)$$

where both U and V are unitary matrices, $(\cdot)^\dagger$ denotes Hermitian transpose, and Σ is a diagonal matrix with m non-zero elements representing the singular values of H : $s_1, s_2, \dots, s_m$. The unordered joint density for the singular values of H is [14]

$$f(s_1, \dots, s_m) = \frac{2^m}{m! \prod_{i=1}^m (n-i)! (m-i)!} \times \exp\left(-\sum_{i=1}^m s_i^2\right) \times \prod_{i=1}^m s_i^{2l+1} \prod_{i < j} (s_i^2 - s_j^2)^2, \quad (3)$$

where $l = n - m$. Note that the eigenvalues of the channel correlation matrix, $\lambda_i = s_i^2$, are equivalent to power gains of the spatial links intrinsic to the MIMO channel, and their joint statistics are governed by the central Wishart distribution as given in [15]

$$f(\lambda_1, \dots, \lambda_m) = \frac{1}{m! \prod_{i=1}^m (n-i)! (m-i)!} \times \exp\left(-\sum_{i=1}^m \lambda_i\right) \times \prod_{i=1}^m \lambda_i^l \prod_{i < j} (\lambda_i - \lambda_j)^2. \quad (4)$$

It is further assumed that the MIMO channel evolves over time in accordance to

$$H(t + \tau) = J_0(2\pi f_D \tau) H(t) + \sqrt{1 - J_0(2\pi f_D \tau)^2} \Psi, \quad (5)$$

where $J_0(\cdot)$ is the zeroth-order Bessel function, f_D and τ are the Doppler frequency and time displacement, respectively, τ is a $N_r \times N_t$ random matrix with zero-mean, unit variance complex Gaussian elements. For the sake of simplicity, we denote the autocorrelation function as $\zeta(\tau) = J_0(2\pi f_D \tau)$ in the remainder of this paper.

2.2. Review of Geometric Mean Decomposition. The authors of [4] suggested designing MIMO transceivers via geometric mean decomposition (GMD), in which the MIMO channel is decomposed as:

$$H = QRP^\dagger, \quad (6)$$

where P and Q are semiunitary matrices and R represents an $m \times m$ upper triangular matrix with identical diagonal elements. If the data signal x is precoded by P at the transmitter and y is filtered by $Q^\dagger$ at the receiver, the signaling model (1) is tantamount to multiplication between x and the upper triangular matrix, R . Hence, the transmitted signal could be extracted and decoded using *nulling and cancellation* procedure as elucidated in [4]. Assuming ideal operations without error propagation, the data signals are sent on m parallel spatial links with a common value of amplitude:

$$g_s = \left(\prod_{i=1}^m s_i \right)^{1/m}. \quad (7)$$

In other words, the amplitudes of these links are equivalent to the geometric mean of MIMO channel singular values, s_i . Since these m spatial links have an identical gain, the “worst subchannel” problem, in which the overall error performance is dominated by the weakest spatial link, is no longer a concern. Thus, one merit of using GMD is the exemption of complicated bit/power allocations over spatial links. By uniformly allocating the transmission power among all parallel spatial links realized by GMD, the channel capacity can be expressed as:

$$C = m \log_2(1 + \gamma g_s^2) = m \log_2(1 + \gamma g), \quad (8)$$

where γ denotes the signal-to-noise ratio (SNR) on each of the parallel links. Also, it is easy to see that the link power gain of these spatial links is the geometric mean of λ s:

$$g = g_s^2 = \left(\prod_{i=1}^m \lambda_i \right)^{1/m}. \quad (9)$$

Conventionally, the channel capacity of a general MIMO system can be written as the sum of multiple random variables; hence, central-limited theorem (CLT) could be used to approximate the MIMO channel capacity as a Gaussian random variable [16]. For MIMO-GMD schemes, however, Gaussian approximation may not be suitable as (8) is simply a scalar integer multiple of a single random variable.

2.3. Finite-State Markov Chain. As one of the objectives in this paper is to model the time-varying behavior of MIMO-GMD systems based on FSMC, the fundamental structure of an FSMC is outlined here for pedagogical reasons. To model the fluctuation of a random process using an FSMC, the process is initially quantized into a number of discrete states. Then, the dynamic behavior is described by the transition probabilities among these states. For simplicity, only first-order FSMC is considered in this paper, which means the transition probability is solely dependent on the most recent observation. In our case, both processes of eigenmode geometric mean and capacity are quantized based on their magnitudes. In particular, the random process of interest is simply partitioned into S states by setting $S - 1$ threshold levels. In the first state (denoted as $\mathcal{S}_0$), the process has a value smaller than the lowest threshold level T_0 . In the final state ($\mathcal{S}_{S-1}$), the process is larger than the highest threshold level T_{S-2} . Otherwise, the process is said to be in the i th state (denoted as $\mathcal{S}_i$) if its value falls in the range between threshold levels T_{i-1} and T_i . For an illustrative example, Figure 1 shows an eigenmode geometric mean process that is partitioned into four states using three threshold levels: T_0 , T_1 , and T_2 . As shown in the later sections of this paper, the transition probabilities can be computed based on both univariate and bivariate PDFs of the process.

3. Derivations of Statistical Distributions

This section targets to derive univariate and bivariate PDFs for both spatial link power gain, g , and channel capacity, C ,

FIGURE 1: An example of Markov state quantization. The time-varying eigenmodes geometric mean process in a (2,2) MIMO channel is partitioned into $S = 4$ states using three threshold levels: T_0 , T_1 , and T_2 .

for MIMO-GMD schemes. For univariate distributions, the exact analytical results are first presented for cases with two spatial links ($m = 2$), and then the cases with more spatial links are dealt with by Gamma approximations.

3.1. Univariate Distributions for Cases with $m = 2$. When either the transmitter or receiver is equipped with only two antennas, only $m = 2$ spatial links are realizable as the rank of a MIMO channel is $\min(N_t, N_r)$. From (3), the joint PDF for MIMO channel singular values with $m = 2$ is

$$f(s_1, s_2) = \frac{2 \exp(-s_1^2 - s_2^2) s_1^{2n-3} s_2^{2n-3} (s_1^2 - s_2^2)^2}{(n-1)!(n-2)!}. \quad (10)$$

Since the geometric mean, g_s , is a function of singular values, Jacobian transform can be applied to obtain the joint PDF for s_2 and g_s . We have

$$s_1 = \frac{g_s^2}{s_2} \Rightarrow \frac{\partial s_1}{\partial g_s} = \frac{2g_s}{s_2}, \quad (11)$$

and hence

$$\begin{aligned} f(s_2, g_s) &= \left| \frac{\partial s_1}{\partial g_s} \right| \times f(s_1, s_2)|_{s_1 \rightarrow s_1(g_s)} \\ &= \frac{4g_s \exp\left[-(g_s^2/s_2)^2 - s_2^2\right] (g_s^2/s_2)^{(2n-3)} s_2^{2n-3} \left[(g_s^2/s_2)^2 - s_2^2\right]^2}{s_2 (n-1)!(n-2)!}. \end{aligned} \quad (12)$$

The PDF for g_s can be acquired by integrating (12) with respect to s_2 . After tedious calculation and little algebra, we find the following result:

$$\begin{aligned} f(g_s) &= \int_0^\infty f(s_2, g_s) ds_2 \\ &= \frac{8g_s^{(4n-3)}}{(n-1)!(n-2)!} K_1(2g_s^2), \end{aligned} \quad (13)$$

where $K_1(\cdot)$ represents modified Bessel function of the second kind of order 1. The derivation from (12) to (13) could be carried out by certain symbolic manipulation software packages. Note that (13) is a generalized version for cases with $m = 2$; by setting $n = 2$, (13) can be reduced to the special case result for a (2,2) MIMO-GMD system given in [9]. As g_s is the square root of g , we may apply simple transformation on (13) to get the PDF for g :

$$f(g) = \frac{4g^{2(n-1)}}{(n-1)!(n-2)!} K_1(2g). \quad (14)$$

Similarly, Jacobian transform can be employed to cope with the channel capacity of MIMO-GMD schemes, as the PDF for g is now readily available. Based on (8), we have

$$g = \frac{2^{C/m} - 1}{\gamma} \Rightarrow \frac{\partial g}{\partial C} = \frac{\ln(2) 2^{C/m}}{m\gamma}. \quad (15)$$

Thus, the PDF for channel capacity with $m = 2$ is

$$\begin{aligned} f(C) &= \left| \frac{\partial g}{\partial C} \right| \times f(g)|_{g \rightarrow g(C)} \\ &= \frac{2 \ln(2) \sqrt{2}^C (2^{C/2} - 1)^{2(n-1)}}{\gamma^{(n-0.75)} (n-1)!(n-2)!} K_1 \left[\frac{2}{\gamma} (2^{C/2} - 1) \right]. \end{aligned} \quad (16)$$

The validity of (14) and (16) can be verified through Monte Carlo simulations. In Figure 2, the empirical distributions of eigenvalue geometric mean for (2,2), (2,4), and (2,8) MIMO systems are compared with their respective analytical PDFs that we have derived (14). On the other hand, the comparison between theoretical channel capacity PDFs (16) and simulation results are shown in Figure 3. In all cases, excellent agreements between analytical and simulation results can be observed.

3.2. Univariate Distributions for Cases with $m > 2$. For systems with more than two spatial links, that is, $m > 2$, the similar approach invoked in Section 3.1 could be used. Nonetheless, it is difficult to get closed-form results like (14) and (16) because multiple integrals are required in the step of obtaining $f(g_s)$ from $f(s_2, \dots, s_m, g_s)$ as in (13). Fortunately, some previous studies have concluded that MIMO eigenvalues ($\lambda_1, \lambda_2, \dots, \lambda_m$) can be very accurately approximated by Gamma random variables [14]. Moreover, [17] claims that the geometric mean of multiple independent Gamma random variables could be either a Gamma or a mixture of Gamma distributions. Although MIMO eigenvalues are not independent processes, they are weakly correlated. Thus, we may make a hypothesis stating that

$$f(g) \approx \text{Gamma}(k, \theta) = \frac{g^{k-1} \exp(-g/\theta)}{\Gamma(k) \theta^k}, \quad (17)$$

where the parameters $k = E(g)/\theta$ and $\theta = \text{var}(g)/E(g)$ are shape factor and scale factor of Gamma distribution respectively, and $\Gamma(\cdot)$ represents Gamma function. Since $\text{var}(g) = E(g^2) - E(g)^2$, the computation for $\text{var}(g)$ requires the first

FIGURE 2: Comparison between simulation and calculation results for distributions of eigenvalue geometric means (link gains of GMD-based transceivers) in (2,2), (2,4), and (2,8) MIMO Rayleigh fading channels.

FIGURE 3: Comparison between simulation and calculation results for distributions of capacities of GMD-based transceivers in (2,2), (2,4), and (2,8) MIMO Rayleigh fading channels. Assuming $\gamma = 5$ dB.

two moments of g . Note that the t -th moment of g can be calculated as

$$E(g^t) = \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^m \lambda_i \right)^{t/m} f(\lambda_1, \dots, \lambda_m) d\lambda_1, \dots, d\lambda_m, \quad (18)$$

where $f(\lambda_1, \dots, \lambda_m)$ is the joint PDF of all m eigenvalues of the channel correlation matrix $H^\dagger H$, as given in (4). To

TABLE 1: Parameters for Gamma approximation for some selected MIMO configurations.

	(4,4)	(4,8)	(8,8)
$E(g)$	1.8075	6.0279	3.2150
$\text{var}(g)$	0.5280	1.5306	0.5507
k	6.1880	23.7393	18.7687
θ	0.2921	0.2539	0.1713

make this paper more self-contained, we have tabulated the numerical values of $E(g)$, $\text{var}(g)$, k , and θ for (4,4), (4,8), and (8,8) cases in Table 1. Hence, one may construct the PDFs for g in these MIMO cases accordingly.

The approximations based on Gamma distributions (17) can be further leveraged to find the PDFs for channel capacity in MIMO-GMD schemes. Using (15), we have

$$\begin{aligned} f(C) &= \left| \frac{\partial g}{\partial C} \right| \times f(g)|_{g=g(C)} \\ &= \frac{\ln(2) 2^{C/m} (2^{C/m} - 1)^{k-1}}{m \gamma^k \Gamma(k) \theta^k} \exp\left(\frac{1 - 2^{C/m}}{\gamma \theta}\right). \end{aligned} \quad (19)$$

To justify the suitability of Gamma approximations for eigenvalue geometric means, similar comparisons as in Section 3.1 are carried out for (4,4), (4,8), and (8,8) cases. In Figure 4, we can see that Gamma distributions provide excellent approximation for eigenvalue geometric means. Although we did not show it here, it has been found that Gamma distributions also fit the simulation data in $m = 2$ cases. Hence, we can claim that, for practical MIMO systems ($m \leq 8$) at least, the distributions of link gains in MIMO-GMD can be accurately approximated by Gamma random variables. For channel capacity, we compare the distributions of simulation samples with the capacity PDF (19), which was derived based on Gamma approximation of g . The comparisons are shown in Figure 5, and it is clear that the simulation data and computed results are almost aligned with each other. An important observation is that, in contrast to conventional MIMO systems, Gaussian approximation for capacity distribution [16] does not seem to be appropriate in GMD-based MIMO transceiver schemes.

3.3. Bivariate Distributions. Since we have observed that the MIMO-GMD link gains can be accurately approximated by a Gamma random variable, it can be conjectured that a bivariate Gamma PDF is well suited for the joint density of their magnitudes at two correlated time instants, $g(t)$ and $g(t + \tau)$. Thus, by modifying the bivariate Gamma PDF given in [18], we have

$$\begin{aligned} f(g, \hat{g}) &\approx \frac{(g\hat{g})^{(k-1)/2}}{\theta^{k+1} \Gamma(k) (1-\rho) \rho^{(k-1)/2}} \exp\left\{ \frac{-(g + \hat{g})}{\theta(1-\rho)} \right\} \\ &\times I_{k-1}\left(\frac{2}{\theta(1-\rho)} \sqrt{\rho g \hat{g}} \right), \end{aligned} \quad (20)$$

FIGURE 4: Comparison between simulation and Gamma approximation results for distributions of eigenvalue geometric means (link gains of GMD-based transceivers) in (4,4), (4,8), and (8,8) MIMO Rayleigh fading channels.

FIGURE 5: Comparison between simulation and calculation results for distributions of capacities of GMD-based transceivers in (4,4), (4,8), and (8,8) MIMO Rayleigh fading channels. Assuming $\gamma = 5$ dB.

where $\hat{g} = g(t + \tau)$ and ρ represents the correlation coefficient between g and $\hat{g}$, which is defined as

$$\rho = \frac{E[g(t)g(t + \tau)] - E[g(t)]E[g(t + \tau)]}{\sqrt{\text{var}[g(t)]\text{var}[g(t + \tau)]}} \quad (21)$$

$$= \frac{E(g\hat{g}) - E(g)^2}{\text{var}(g)}.$$

In (21), both $E(g)$ and $\text{var}(g)$ can be calculated by utilizing (4) and (18). Since g and $\hat{g}$ are functions of $(\lambda_1(t), \dots, \lambda_m(t))$ and $(\lambda_1(t + \tau), \dots, \lambda_m(t + \tau))$, respectively, the computation for the joint moment, $E(g\hat{g})$, requires the joint density for MIMO eigenvalues at two time instants, which has been provided in [19]:

$$f(\lambda_1, \dots, \lambda_m, \hat{\lambda}_1, \dots, \hat{\lambda}_m) = \frac{\prod_{i < j} [(1/\alpha^2)(\hat{\lambda}_i - \hat{\lambda}_j)] \prod_{i < j} (\lambda_i - \lambda_j) \prod_{i=1}^m \lambda_i^l}{\alpha^{2m} \prod_{i=1}^m [(n-i)!(m-i)!]} \times \exp\left(-\sum_{i=1}^m \lambda_i\right) \times G(\lambda), \quad (22)$$

where

$$G(\lambda) = \det \begin{bmatrix} f(\hat{\lambda}_1 | \lambda_1) & f(\hat{\lambda}_2 | \lambda_1) & \cdots & f(\hat{\lambda}_m | \lambda_1) \\ f(\hat{\lambda}_1 | \lambda_2) & f(\hat{\lambda}_2 | \lambda_2) & & \vdots \\ \vdots & & \ddots & \vdots \\ f(\hat{\lambda}_1 | \lambda_m) & \cdots & \cdots & f(\hat{\lambda}_m | \lambda_m) \end{bmatrix},$$

$$f(\hat{\lambda} | \lambda) = \frac{\alpha^2}{\beta^2} \left(\frac{\hat{\lambda}}{\alpha^2 \lambda} \right)^{\nu/2} \exp\left[\frac{-\alpha^2 \lambda - \hat{\lambda}}{\beta^2} \right] I_l \left(2 \frac{\alpha}{\beta^2} \sqrt{\lambda \hat{\lambda}} \right) \quad (23)$$

with $\hat{\lambda}_i = \lambda_i(t + \tau)$, $\alpha = \zeta$, $\beta = \sqrt{1 - \zeta^2}$, and $\zeta = J_0(2\pi f_D \tau)$ as before. The joint moment, $E(g\hat{g})$, can thereby be computed with (22) as

$$E(g\hat{g}) = \int_0^\infty \cdots \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^m \lambda_i \right)^{1/m} \left(\prod_{i=1}^m \hat{\lambda}_i \right)^{1/m} \times f(\lambda_1, \dots, \lambda_m, \hat{\lambda}_1, \dots, \hat{\lambda}_m) d\lambda_1, \dots, d\lambda_m, \hat{\lambda}_1, \dots, \hat{\lambda}_m. \quad (24)$$

Then, ρ can be calculated by substituting (24) into (21). Apparently, the closed-form expression for (24) is difficult, if not impossible, to obtain. However, it still can be solved via numerical integrations.

Once the value of ρ is acquired, (20) can be extended to obtain the bivariate PDF for MIMO-GMD channel capacity

process. Using Jacobian transform, $f(C, \hat{C})$ can be written as

$$\begin{aligned}
 f(C, \hat{C}) &= \det \begin{bmatrix} \frac{\partial g}{\partial C} & \frac{\partial g}{\partial \hat{C}} \\ \frac{\partial \hat{g}}{\partial C} & \frac{\partial \hat{g}}{\partial \hat{C}} \end{bmatrix} \times f(g, \hat{g})|_{g=g(C), \hat{g}=\hat{g}(\hat{C})} \\
 &= 0.4805 \\
 &\times \frac{2^{(C+\hat{C})/m} [(2^{C/m} - 1)(2^{\hat{C}/m} - 1)/\gamma^2 \rho]^{(k-1)/2}}{m^2 \gamma^2 \Gamma(k) \theta^{k+1} (1 + \rho)} \\
 &\times \exp \left[\frac{2 - 2^{C/m} - 2^{\hat{C}/m}}{\gamma \theta (1 - \rho)} \right] \\
 &\times I_{k-1} \left[\frac{2 \sqrt{\rho (2^{C/m} - 1)(2^{\hat{C}/m} - 1)}}{\gamma \theta (1 - \rho)} \right].
 \end{aligned} \quad (25)$$

With PDFs derived in this section, one may construct an FSMC to model the fluctuation of MIMO-GMD link gain and capacity, as explained in Section 4.

4. FSMC Construction

In order to investigate the time-varying behavior of MIMO-GMD systems, FSMC is adopted to model both link gain and channel capacity. Based on the FSMC structure elaborated in Section 2.3, we delineate how transition probabilities can be analytically calculated using the PDFs derived in Section 3. The transition probabilities from $\mathcal{S}_i$ to $\mathcal{S}_j$, denoted as $P_{i,j}$, for the link gain process $g(t)$ can be computed as

$$P_{i,j} = \frac{\text{Prob}(g(t) \in \mathcal{S}_i, g(t+\tau) \in \mathcal{S}_j)}{\text{Prob}(g(t) \in \mathcal{S}_i)}, \quad (26)$$

where

$$\text{Prob}(g(t) \in \mathcal{S}_i, g(t+\tau) \in \mathcal{S}_j) = \int_{\mathcal{S}_j} \int_{\mathcal{S}_i} f(g, \hat{g}) dg d\hat{g}, \quad (27)$$

$$\text{Prob}(g(t) \in \mathcal{S}_i) = \int_{\mathcal{S}_i} f(g) dg. \quad (28)$$

Note that $f(g)$ has been derived as (14) or (17), depending on the value of m . Analytical calculations of (27) can be quite difficult, so numerical integrations are used in this work instead. For the sake of simplicity, this paper assumes that the channel variation is slow enough so the process would only transit to one of the adjacent states (from $\mathcal{S}_i$ to $\mathcal{S}_{i-1}$ or $\mathcal{S}_{i+1}$) or stay in the same state (from $\mathcal{S}_i$ to $\mathcal{S}_i$). Therefore, we have

$$P_{i,i} = 1 - P_{i,i+1} - P_{i,i-1}. \quad (29)$$

For channel capacity in MIMO-GMD scheme, the transition probabilities can be computed in a similar fashion.

FIGURE 6: Comparison of approximated and simulated transition probabilities for link gains (with $f_D = 30$ Hz, $\tau = 0.001$ sec) in a (2,4) MIMO system.

FIGURE 7: Comparison of approximated and simulated transition probabilities for MIMO-GMD channel capacity (with $f_D = 30$ Hz, $\tau = 0.001$ sec and $\gamma = 10$ dB) in a (2,2) system.

In order to verify that FSMC is an appropriate tool to model the fluctuation of MIMO-GMD channel, several Monte Carlo simulations have been carried out. In particular, simulation results on transition probabilities for both MIMO eigenvalue geometric mean and channel capacity are compared with our calculations by (26) and (29). Note that the threshold levels for state quantization are set arbitrarily

in this work. In practical adaptive modulation schemes, for example, the threshold levels for state quantization could be set based on minimum required channel gain for a target error performance. In all simulations, we have $f_D = 30$ Hz, $\tau = 0.001$ sec and $\gamma = 10$ dB. In Figure 6, the link gain of a (2,4) MIMO-GMD scheme is modeled as an FSMC consisting of four states with $\{T_0, T_1, T_2\} = \{1.5, 2.5, 3.5\}$, and transition probabilities from both simulation and calculations are plotted. Similarly, we approximate the MIMO-GMD capacity process in a (2,2) system using a four-state FSMC with $\{T_0, T_1, T_2\} = \{4, 7, 9\}$ bps/Hz in Figure 7. In both cases, it is apparent that our calculations can provide very accurate approximations.

5. Conclusion

By applying GMD to a MIMO-OFDM system, multiple spatial links with identical gains can be realized within each subcarrier. This assuages the complexity at the transmitter side as the spatial domain bit/power allocation can be simplified. In order to seek for potential opportunities to further decrease the system complexity, this paper has investigated the statistical properties of MIMO systems using GMD. In particular, we have derived PDFs for link gains (MIMO eigenvalue geometric mean) and capacities. Although exact results are only available for cases with two parallel links, Gamma approximations can be used to model eigenvalue geometric means for general MIMO cases with arbitrary antenna configurations. Moreover, these results are extended to derive the bivariate PDFs, which is important for the analysis of time-varying behavior. These results are employed to model the channel variation in MIMO-GMD schemes by constructing FSMCs. To be specific, the transition probabilities among states could be computed using the PDFs derived in this paper. This paper proposed a potential approach to predict the fluctuation of MIMO channels under GMD, which may allow the engineers to relent the feedback rate and hence reduce the system burden. The analytical PDF results presented in this paper are for baseline cases of Rayleigh flat fading. For scenarios involve spatial correlation and Ricean fading, statistical properties of MIMO-GMD schemes are still open problems and should be addressed in the future.

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