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Hardy operator with variable limits on monotone functions[1]

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Abstract. We characterize weighted $L^p - L^q$ inequalities with the Hardy operator of the form $\mathbb{H}f(x) = \int_{a(x)}^{b(x)} f(y)u(y)dy$ with a non-negative weight function u, restricted to the cone of monotone functions on the semiaxis. The proof is based on the Sawyer criterion and the boundedness of generalized Hardy operator with variable limits.

1. Introduction

We study the Hardy operator

$$\mathbb{H}f(x) = \int_{a(x)}^{b(x)} f(y)u(y)dy \tag{1}$$

with a non-negative weight function u, restricted to the cone of non-increasing or non-decreasing non-negative functions on the semiaxis $\mathbb{R}^+ := [0, \infty)$. The border functions $a(x)$ and $b(x)$ are supposed to be continuous and strictly increasing on $\mathbb{R}^+$ and as a model case may be taken such that

$$0 = a(0) = b(0) < a(\infty) = b(\infty) = \infty. \tag{2}$$

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The equalities in (2) are not important as well as continuity of the border functions and assumed for simplicity. Also for simplicity we assume that $a(x) \neq b(x)$ everywhere on $\mathbb{R}^+$ except a countable subset.

The operator (1) with $u = 1$ was studied in [1], but the only sufficient condition for weighted $L^p - L^q$ boundedness on monotone functions was found there. A natural way to solve this problem is to apply the Sawyer criterion [2]. However, it leads to the additional problem for weighted $L^p - L^q$ characterization of generalized Hardy operators. This was done in [3] for the case $a(x) < b(x)$, $x > 0$ (see also, [4], [5]). Using the results of [3] and the block-diagonal structure of (1) we characterize the operator $\mathbb{H}$ on monotone functions.

The paper is organized as follows. Section 2 contains the weighted $L^p - L^q$ characterization of generalized Hardy operators. In particular, for the operator $\mathbb{H}$, the "fairway" form of criteria are given. In Section 3 we remind the Sawyer criterion and formulate its dual form, more convenient for our purpose. The main result is given in Section 4.

Throughout of the paper products of the form $0 \cdot \infty$ are taken to be equal to 0. Relations $A \ll B$ mean $A \leqslant cB$ with some constants c depending only on parameters of summations and, possibly, on the constants of equivalence in the inequalities of the type (8). We write $A \approx B$ instead of $A \ll B \ll A$ or $A = cB$. $\mathbb{Z}$ denotes the set of all integers and χ_E stands for a characteristic function (indicator) of a subset E. The union of disjoint sets we denote by $\bigsqcup$ and $\sqcup$. Also we make use of marks $:\ =$ and $=:$ for introducing new quantities as well as the symbol $\square$ for the end of proof.

2. Generalized Hardy operator

Let

$$Kf(x) = \int_{a(x)}^{b(x)} k(y,x) f(y)\,dy, \qquad x > 0, \tag{3}$$

with a measurable kernel $k(x,y) \geqslant 0$ and strictly increasing functions $a(x)$ and $b(x)$, satisfying (2) such that $a(x) \neq b(x)$ for $0 < x < \infty$ except $x \in X$, where

$$X:\ = \{x_k\}, \qquad k \in \Sigma \subset \mathbb{Z} \quad \text{and} \quad a(x_k) = b(x_k). \tag{4}$$

Denote

$$(0,\infty) =: I \sqcup J,$$

$$I =: \bigsqcup_{i \in \Sigma_1} I_i, \qquad J =: \bigsqcup_{j \in \Sigma_2} J_j, \qquad \Sigma = \Sigma_1 \sqcup \Sigma_2, \tag{5}$$

$$I_i = (t_i, t_{i+1}), \qquad J_j = (s_j, s_{j+1}),$$

where $\{t_i\}_{i \in \Sigma_1} \subset \{x_k\}_{k \in \Sigma}$, $\{s_j\}_{j \in \Sigma_2} \subset \{x_k\}_{k \in \Sigma}$ such that

$$x \in I_i \quad \Leftrightarrow \quad a(x) < b(x), \tag{6}$$

$$x \in J_j \quad \Leftrightarrow \quad b(x) < a(x). \tag{7}$$

Suppose that there exists $D \geqslant 1$ such that for all $i \in \Sigma_1$ and $j \in \Sigma_2$ the condition

$$D^{-1}k(y,x) \leqslant k(y,z) + k(a(z),x) \leqslant Dk(y,x), \tag{8}$$

holds, if

$$\begin{cases} t_i \leqslant x \leqslant z \leqslant t_{i+1}, \\ a(z) \leqslant y \leqslant b(x), \end{cases}$$

and, similarly,

$$D^{-1}k(y,x) \leqslant k(y,z) + k(b(z),x) \leqslant Dk(y,x), \tag{9}$$

when

$$\begin{cases} s_j \leqslant x \leqslant z \leqslant s_{j+1}, \\ b(z) \leqslant y \leqslant a(x). \end{cases}$$

We call a linear operator $T\colon \mathcal{L}(U) \to \mathcal{M}(V)$ *block-diagonal*, if $U = \bigsqcup_k U_k$, $V = \bigsqcup_k V_k$ and $T = \sum_k T_k$, where $T_k\colon \mathcal{L}(U_k) \to \mathcal{M}(V_k)$ and $T_k f = \chi_{V_k} T(\chi_{U_k} f)$.

The following Lemma is frequently used. For completeness we provide the proof.

Lemma 1. *Let $U = \bigsqcup_k U_k$, $V = \bigsqcup_k V_k$ and $T = \sum_k T_k$ is a block-diagonal operator, $T_k\colon L^p(U_k) \to L^q(V_k)$. Then*

$$\|T\|_{L^p(U)\to L^q(V)} = \sup_k \|T_k\|_{L^p(U_k)\to L^q(V_k)}, \qquad 0 < p \leqslant q < \infty \tag{10}$$

and

$$\|T\|_{L^p(U)\to L^q(V)} = \left(\sum_k \|T_k\|^r_{L^p(U_k)\to L^q(V_k)}\right)^{1/r}, 0 < q < p < \infty, \frac{1}{r} = \frac{1}{q} - \frac{1}{p}. \tag{11}$$

Proof. Let $0 < p \leqslant q < \infty$. Since $\|Tf\|_{L^q(V)} \geqslant \|T_k f\|_{L^q(V_k)}$ for all k, the lower estimate in (10) is trivial. For the upper bound, applying Jensen inequality, we write

$$\begin{aligned} \|Tf\|^q_{L^q(V)} &= \sum_k \|T_k f\|^q_{L^q(V_k)} \\ &\leqslant \left(\sup_k \|T_k\|_{L^p(U_k)\to L^q(V_k)}\right)^q \sum_k \|f\chi_k\|^q_{L^p(U_k)} \\ &\leqslant \left(\sup_k \|T_k\|_{L^p(U_k)\to L^q(V_k)}\right)^q \|f\|^q_{L^p(U)}. \end{aligned} \tag{12}$$

Let $0 < q < p < \infty$. The upper bound in (11) follows similar to (12) by application of Hölder's inequality. For the lower bound let $0 < \lambda < 1$ be given and the functions $f_k \in L^p(U_k)$ are such that for all k

$$\|f_k\|_{L^p(U_k)} = \|T_k\|^{r/p}_{L^p(U_k)\to L^q(V_k)},$$
$$\lambda\|T_k\|_{L^p(U_k)\to L^q(V_k)}\|f_k\|_{L^p(U_k)} \leqslant \|T_k f_k\|_{L^p(U_k)}.$$

The inequality here follows from the definition of the norm of a linear operator and then we can change the function f_k, if necessary, by multiplying on a constant for the equality is satisfied. If we put $f = \sum_k \chi_{U_k} f_k$, then

$$\begin{aligned}
\lambda^q \sum_k \|T_k\|^r_{L^p(U_k)\to L^q(V_k)} &= \lambda^q \sum_k \left(\|T_k\|_{L^p(U_k)\to L^q(V_k)}\|f_k\|_{L^p(U_k)}\right)^q \\
&\leqslant \sum_k \|T_k f_k\|^q_{L^p(U_k)} \\
&= \|Tf\|^q_{L^q(V)} \\
&\leqslant \|T\|^q_{L^p(U)\to L^q(V)}\|f\|^q_{L^p(U)} \\
&= \|T\|^q_{L^p(U)\to L^q(V)}\left(\sum_k \|T_k\|^r_{L^p(U_k)\to L^q(V_k)}\right)^{q/p}
\end{aligned}$$

and the lower bound in (11) follows by tending $\lambda \to 1$. □

Remark. Observe, that if $0 < p < 1 \leqslant q$ and $\|T\| < \infty$, then $\|T\| = 0$ ([6], § 1.47) and, if $0 < q < \infty$, $0 < p < 1$, $\|T\| < \infty$ and T is an integral operator, then $\|T\| = 0$ ([[7], Theorem 2).

We consider the inequality

$$\left(\int_0^\infty |Kf(x)|^q w^q(x)dx\right)^{1/q} \leqslant C\left(\int_0^\infty |f(x)|^p v^p(x)dx\right)^{1/p}, \tag{13}$$

where $w(x) \geqslant 0$ and $v(x) \geqslant 0$ are locally integrable functions (weights) and K is given by (3) with the border functions $a(x)$ and $b(x)$ satisfying (2), (4-7) and the kernel $k(y,x) \geqslant 0$ satisfying (8) and (9).

Denote

$$L^p_v\colon = \left\{f\colon \|fv\|_p\colon = \left(\int_0^\infty |f(x)|^p v^p(x)dx\right)^{1/p} < \infty\right\}$$

and let $\|K\|_{X\to Y}$ stand for the norm of $K\colon X \to Y$.

Obviously, under the above assumptions on $a(x)$ and $b(x)$ K has a block-diagonal structure and Lemma 1 gives

$$\|K\|_{L^p_v\to L^q_w} = \sup_{k\in\Sigma} \mathbb{A}_k, \qquad 1 < p \leqslant q < \infty, \tag{14}$$

$$\|K\|_{L^p_v\to L^q_w} = \left(\sum_{k\in\Sigma}\mathbb{A}_k^r\right)^{1/r}, \qquad 1<q<p<\infty, \tag{15}$$

where

$$\mathbb{A}_k = \|K\|_{L^p_v(a(U_k))\to L^q_w(U_k)}, \tag{16}$$

U_k runs the intervals I_i and J_j, and $a(U) = (a(u_1), a(u_2))$ if $U = (u_1, u_2) \subset \mathbb{R}^+$.

The norms (16) we estimate with the help of [3] by the following scheme.

We have two types of the intervals U_k, namely I_i (see (6)) and J_j (see (7)). Since on each of I_i the operator K with the kernel $k(y,x) \in$ (8) satisfies the conditions of Theorem 2 [3], then

(17)
$$\mathbb{A}_i \approx \begin{cases} A_i = A_0^{(i)} + A_1^{(i)}, & 1<p\leqslant q<\infty, \\ B_i = \left[\sum_{k\in\mathbb{Z}_i}\left(B_{k,1}^{(i)} + B_{k,2}^{(i)} + B_{k,3}^{(i)} + B_{k,4}^{(i)}\right)^r\right]^{1/r}, & 1<q<p<\infty, \end{cases}$$

where

$$A_0^{(i)} = \sup_{s\in I_i}\sup_{s\leqslant t\leqslant a^{-1}(b(s))}\left(\int_s^t k^q(a(t),x)w^q(x)dx\right)^{1/q}\left(\int_{a(t)}^{b(s)} v^{-p'}(x)dx\right)^{1/p'},$$

$$A_1^{(i)} = \sup_{s\in I_i}\sup_{s\leqslant t\leqslant a^{-1}(b(s))}\left(\int_s^t w^q(x)dx\right)^{1/q}\left(\int_{a(t)}^{b(s)} k^{p'}(x,t)v^{-p'}(x)dx\right)^{1/p'},$$

$$B_{k,1}^{(i)} = \left(\int_{a(\xi_k^{(i)})}^{a(\xi_{k+1}^{(i)})}\left(\int_{\xi_k^{(i)}}^{a^{-1}(t)} k^q(t,x)w^q(x)dx\right)^{r/q} \times\left(\int_t^{a(\xi_{k+1}^{(i)})} v^{-p'}(x)dx\right)^{r/q'} v^{-p'}(t)dt\right)^{1/r},$$

$$B_{k,2}^{(i)} = \left(\int_{\xi_k^{(i)}}^{\xi_{k+1}^{(i)}}\left(\int_{\xi_k^{(i)}}^{t} w^q(x)dx\right)^{r/p} \times\left(\int_{a(t)}^{a(\xi_{k+1}^{(i)})} k^{p'}(x,t)v^{-p'}(x)dx\right)^{r/p'} w^q(t)dt\right)^{1/r},$$

$$B_{k,3}^{(i)} = \left(\int_{b(\xi_k^{(i)})}^{b(\xi_{k+1}^{(i)})} \left(\int_{b^{-1}(t)}^{\xi_{k+1}^{(i)}} k^q(t,x)w^q(x)dx\right)^{r/q} \times \left(\int_{b(\xi_k^{(i)})}^{t} v^{-p'}(x)dx\right)^{r/q'} v^{-p'}(t)dt\right)^{1/r},$$

$$B_{k,4}^{(i)} = \left(\int_{\xi_k^{(i)}}^{\xi_{k+1}^{(i)}} \left(\int_{t}^{\xi_{k+1}^{(i)}} w^q(x)dx\right)^{r/p} \times \left(\int_{b(\xi_k^{(i)})}^{b(t)} k^{p'}(x,t)v^{-p'}(x)dx\right)^{r/p'} w^q(t)dt\right)^{1/r},$$

and the sequence $\{\xi_k^{(i)}\}_{k\in\mathbb{Z}_i}, \mathbb{Z}_i \subset \mathbb{Z}$ is such that $\xi_0^{(i)} \in I_i$ and $\xi_k^{(i)} = (a^{-1} \circ b)^k(\xi_0^{(i)})$. Analogously, we can estimate the norm of the operator $Kf(x)$ for $x \in J_j$. Since the kernel $k(y,x)$ satisfies the condition (9) and $b(x) < a(x)$, then

(18)
$$\mathbb{A}_j \approx \begin{cases} A_j = A_0^{(j)} + A_1^{(j)}, & 1 < p \leqslant q < \infty, \\ B_j = \left[\sum_{k\in\mathbb{Z}_j}\left(B_{k,1}^{(j)} + B_{k,2}^{(j)} + B_{k,3}^{(j)} + B_{k,4}^{(j)}\right)^r\right]^{1/r}, & 1 < q < p < \infty, \end{cases}$$

where

$$A_0^{(j)} = \sup_{s\in J_j}\sup_{s\leqslant t\leqslant b^{-1}(a(s))}\left(\int_s^t k^q(b(t),x)w^q(x)dx\right)^{1/q}\left(\int_{b(t)}^{a(s)} v^{-p'}(x)dx\right)^{1/p'},$$

$$A_1^{(j)} = \sup_{s\in J_j}\sup_{s\leqslant t\leqslant b^{-1}(a(s))}\left(\int_s^t w^q(x)dx\right)^{1/q}\left(\int_{b(t)}^{a(s)} k^{p'}(x,t)v^{-p'}(x)dx\right)^{1/p'},$$

$$B_{k,1}^{(j)} = \left(\int_{b(\xi_k^{(j)})}^{b(\xi_{k+1}^{(j)})} \left(\int_{\xi_k^{(j)}}^{b^{-1}(t)} k^q(t,x)w^q(x)dx\right)^{r/q} \times \left(\int_{t}^{b(\xi_{k+1}^{(j)})} v^{-p'}(x)dx\right)^{r/q'} v^{-p'}(t)dt\right)^{1/r},$$

$$B_{k,2}^{(j)} = \left(\int_{\xi_k^{(j)}}^{\xi_{k+1}^{(j)}} \left(\int_{\xi_k^{(j)}}^{t} w^q(x)dx\right)^{r/p} \times \left(\int_{b(t)}^{b(\xi_{k+1}^{(j)})} k^{p'}(x,t)v^{-p'}(x)dx\right)^{r/p'} w^q(t)dt\right)^{1/r},$$

$$B_{k,3}^{(j)} = \left(\int_{a(\xi_k^{(j)})}^{a(\xi_{k+1}^{(j)})} \left(\int_{a^{-1}(t)}^{\xi_{k+1}^{(j)}} k^q(t,x)w^q(x)dx\right)^{r/q} \times \left(\int_{a(\xi_k^{(j)})}^{t} v^{-p'}(x)dx\right)^{r/q'} v^{-p'}(t)dt\right)^{1/r},$$

$$B_{k,4}^{(j)} = \left(\int_{\xi_k^{(j)}}^{\xi_{k+1}^{(j)}} \left(\int_{t}^{\xi_{k+1}^{(j)}} w^q(x)dx\right)^{r/p} \times \left(\int_{a(\xi_k^{(j)})}^{a(t)} k^{p'}(x,t)v^{-p'}(x)dx\right)^{r/p'} w^q(t)dt\right)^{1/r},$$

and the sequence $\{\xi_k^{(j)}\}_{k\in\mathbb{Z}_j}, \mathbb{Z}_j \subset \mathbb{Z}$ is such as $\xi_0^{(j)} \in J_j$ and $\xi_k^{(j)} = (b^{-1} \circ a)^k(\xi_0^{(j)})$.

On the base of the above estimates and Lemma 1 we obtain

Theorem 1. *Let $1 < p \leqslant q < \infty$, then the least constant C of inequality* (13) *is equivalent to $\mathbb{A}$, where*

$$\mathbb{A} = \sup_{i\in\Sigma_1} A_i + \sup_{j\in\Sigma_2} A_j.$$

If $1 < q < p < \infty$, then $C \approx \mathbb{B}$, where

$$\mathbb{B} = \left(\sum_{i\in\Sigma_1} B_i^r\right)^{1/r} + \left(\sum_{j\in\Sigma_2} B_j^r\right)^{1/r}.$$

Remark 2. Theorem 1 has a counterpart for the operator

$$\bar{K}f(x) = \int_{a(x)}^{b(x)} k(x,y)f(y)dy,$$

where the kernel $k(x,y) \geqslant 0$ satisfies the conditions

$$D^{-1}k(x,y) \leqslant k(x,b(z)) + k(z,y) \leqslant Dk(x,y)$$

and

$$D^{-1}k(x,y) \leqslant k(x,a(z)) + k(z,y) \leqslant Dk(x,y)$$

instead of (8) and (9), respectively. We omit details which can be restored by using ([3], Theorem 3).

Theorem 1 has a particularly transparent form, when $k(y,x) = 1$, with the help of the following

Definition 1. Given boundary functions $a(x)$ and $b(x)$, satisfying the conditions (2), a number $p \in (1,\infty)$ and a weight function $v(x)$ such that $0 < v(x) < \infty$ almost everywhere for $x \in \mathbb{R}^+$, $v^p(x)$ and $v^{-p'}(x)$, $p' = p/(p-1)$, are locally integrable on $\mathbb{R}^+$, we define *fairway* - the function $\sigma(x)$ such that $\min(a(x),b(x)) < \sigma(x) < \max(a(x),b(x))$, $x \notin \{x_k\}$ and

$$\int_{\min(a(x),b(x))}^{\sigma(x)} v^{-p'}(y)dy = \int_{\sigma(x)}^{\max(a(x),b(x))} v^{-p'}(y)dy \qquad \text{for all } x \in \mathbb{R}^+ \setminus X.$$

Put

$$\Delta(x) = [\min(a(x),b(x)), \max(a(x),b(x))],$$

$$\delta(x) = \begin{cases} \left[b^{-1}(\sigma(x)), a^{-1}(\sigma(x))\right], & x \in I, \\ \left[a^{-1}(\sigma(x)), b^{-1}(\sigma(x))\right], & x \in J, \end{cases}$$

where $a^{-1}(y)$ and $b^{-1}(y)$ are the functions converse to $y = a(x)$ and $y = b(x)$, respectively. Applying Theorem 1 of [3] we obtain the following

Theorem 2. *Let the operator $\mathbb{H}$ of the form (1) be given with $u = 1$ and the boundary functions $a(x)$ and $b(x)$ satisfying the conditions (2), (4-7). Then for the norm of $\mathbb{H}$ and $1 < p \leqslant q < \infty$ the estimate*

$$\alpha_0(p,q)\mathcal{A} \leqslant \|\mathbb{H}\|_{L^p_v \to L^q_w} \leqslant \alpha_1(p,q)\mathcal{A},$$

holds, where

$$\mathcal{A} = \sup_{t>0} \left(\int_{\delta(t)} w^q(y)dy\right)^{1/q} \left(\int_{\Delta(t)} v^{-p'}(y)dy\right)^{1/p'}.$$

If $0 < q < p < \infty$, $p > 1$, $1/r = 1/q - 1/p$, then

$$\beta_0(p,q)\mathcal{B} \leqslant \|\mathbb{H}\|_{L^p_v \to L^q_w} \leqslant \beta_1(p,q)\mathcal{B},$$

where

$$\mathcal{B} = \left(\int_0^\infty \left(\int_{\delta(t)} w^q(y)dy\right)^{r/p} \left(\int_{\Delta(t)} v^{-p'}(y)dy\right)^{r/p'} w^q(t)dt\right)^{1/r}.$$

3. Sawyer criterion

Let

$$Tf(x) = \int_0^\infty t(x,y)f(y)dy$$

be an integral operator with a non-negative kernel and

$$T^*g(y) = \int_0^\infty t(x,y)g(x)dx$$

is formally adjoint to T. If, for instance,

$$Hf(x) = \int_0^x f(y)dy,$$

then

$$H^*g(y) = \int_y^\infty g(x)dx.$$

By the Sawyer criterion [2] the weighted inequality

(19)
$$\left(\int_0^\infty (Tf)^q(x)w^q(x)dx\right)^{1/q} \leqslant C\left(\int_0^\infty f^p(x)v^p(x)dx\right)^{1/p}, \quad 0 \leqslant f \downarrow,$$

for all non-increasing function f is equivalent, when $1 < p, q < \infty$, to the following two inequalities

(20)
$$\left(\int_0^\infty (HT^*g)^{p'}(x)V^{-p'}(x)v^p(x)dx\right)^{1/p'} \leqslant C_1\left(\int_0^\infty g^{q'}(x)w^{-q'}(x)dx\right)^{1/q'},$$

and

(21)
$$\int_0^\infty (T^*g)(x)dx \leqslant C_2\left(\int_0^\infty v^p(x)dx\right)^{1/p}\left(\int_0^\infty g^{q'}(x)w^{-q'}(x)dx\right)^{1/q'},$$

where $g \geqslant 0$, $p' = p/(p-1)$, $q' = q/(q-1)$ and $V(x) = \int_0^x v^p(y)dy$. We assume the constants C, C_1 and C_2 as the least possible. The second inequality (21) is easily characterized by the duality in Lebesgue's space and

(22)
$$C_2 = A_0 := \left(\int_0^\infty \left(\int_0^\infty t(x,y)dy\right)^q w^q(x)dx\right)^{1/q}\left(\int_0^\infty v^p(x)dx\right)^{-1/p}.$$

As for the first inequality (20) it is more convenient for our purpose to use the dual form of (20)

(23)
$$\left(\int_0^\infty (TH^*g)^q(x)w^q(x)dx\right)^{1/q} \leqslant C_1\left(\int_0^\infty g^p(x)V^p(x)v^{-p^2/p'}(x)dx\right)^{1/p},$$
for all $g \geqslant 0$.

Thus, if we characterize (23) by a finiteness of, say, a constant A_1, then for (19) we have

$$C \approx A_0 + A_1. \tag{24}$$

The similar criterion is valid for the inequality (19) restricted to the cone of non-decreasing functions. We omit details (see [8]).

4. Main result

Let $a(x)$ and $b(x)$ satisfy the assumptions of § 1 and 2. Taking into account (6) and (7) we observe, that the inequality

(25)
$$\left(\int_0^\infty |\mathbb{H}f(x)|^q w^q(x)dx\right)^{1/q} \leqslant C\left(\int_0^\infty f^p(x)v^p(x)dx\right)^{1/p}, \qquad 0 \leqslant f \downarrow,$$
is equivalent to

(26)
$$\left(\int_0^\infty \left[\overline{\mathbb{H}}f(x)\right]^q w^q(x)dx\right)^{1/q} \leqslant C\left(\int_0^\infty f^p(x)v^p(x)dx\right)^{1/p}, \qquad 0 \leqslant f \downarrow,$$
where

$$\overline{\mathbb{H}}f(x) = \begin{cases} \mathbb{H}f(x), & x \in I_i, \\ -\mathbb{H}f(x), & x \in J_j. \end{cases} \tag{27}$$

By the Sawyer criterion for the least possible constant C in (26) we have
$$C \approx A_0 + A_1,$$
where

$$A_0 = \left(\int_0^\infty \left|\int_{a(x)}^{b(x)} u(y)dy\right|^q w^q(x)dx\right)^{1/q}\left(\int_0^\infty v^p(x)dx\right)^{-1/p} \tag{28}$$

and A_1 is a characterization constant for the inequality

(29)
$$\left(\int_0^\infty \left(\overline{\mathbb{H}}H^*g\right)^q(x)w^q(x)dx\right)^{1/q} \leqslant C_1\left(\int_0^\infty g^p(x)V^p(x)v^{-p^2/p'}(x)dx\right)^{1/p},$$

for all $g \geqslant 0$. Using (27) and notations of § 2 we write for the left hand side of (29)

$$\begin{aligned}
\Phi: \quad &= \int_0^\infty \left(\overline{\mathbb{H}} H^* g\right)^q (x) w^q(x) dx \\
&= \sum_{i\in\Sigma_1} \int_{t_i}^{t_{i+1}} \left(\int_{a(x)}^{b(x)} u(s) ds \int_s^\infty g(y) dy \right)^q w^q(x) dx \\
&\quad + \sum_{j\in\Sigma_2} \int_{s_j}^{s_{j+1}} \left(\int_{b(x)}^{a(x)} u(s) ds \int_s^\infty g(y) dy \right)^q w^q(x) dx \\
&=: \sum_{i\in\Sigma_1} F_i + \sum_{j\in\Sigma_2} G_j.
\end{aligned}$$

Plainly

$$\begin{aligned}
F_i \quad &\approx \int_{t_i}^{t_{i+1}} \left(\int_{a(x)}^{b(x)} u(s) ds \right)^q \left(\int_{b(x)}^\infty g(y) dy \right)^q w^q(x) dx \\
&\quad + \int_{t_i}^{t_{i+1}} \left(\int_{a(x)}^{b(x)} \left(\int_{a(x)}^y u(s) ds \right) g(y) dy \right)^q w^q(x) dx,
\end{aligned}$$

$$\begin{aligned}
G_j \quad &\approx \int_{s_j}^{s_{j+1}} \left(\int_{b(x)}^{a(x)} u(s) ds \right)^q \left(\int_{a(x)}^\infty g(y) dy \right)^q w^q(x) dx \\
&\quad + \int_{s_j}^{s_{j+1}} \left(\int_{b(x)}^{a(x)} \left(\int_{b(x)}^y u(s) ds \right) g(y) dy \right)^q w^q(x) dx,
\end{aligned}$$

Hence,

$$\begin{aligned}
\Phi \quad &\approx \int_0^\infty \left| \int_{a(x)}^{b(x)} u(s) ds \right|^q \left(\int_{\max(a(x),\, b(x))}^\infty g(y) dy \right)^q w^q(x) dx \\
&\quad + \int_0^\infty \left| \int_{a(x)}^{b(x)} \mathbf{k}(y,x) g(y) dy \right|^q w^q(x) dx,
\end{aligned}$$

where

$$\mathbf{k}(y,x) := \begin{cases} \displaystyle\int_{a(x)}^y u(s) ds, & a(x) \leqslant y \leqslant b(x),\ x \in I_i, \\ \displaystyle\int_{b(x)}^y u(s) ds, & b(x) \leqslant y \leqslant a(x),\ x \in J_j. \end{cases} \tag{30}$$

Clearly, the kernel $\mathbf{k}(y,x) \geqslant 0$ satisfies the condition (8) and (9) with $D = 1$. Thus, the inequality (29) is equivalent to the following two inequalities

$$(31)\quad \left(\int_0^\infty \left(\int_{\max(a(x),\,b(x))}^\infty g(y)dy\right)^q \left|\int_{a(x)}^{b(x)} u(s)ds\right|^q w^q(x)dx\right)^{1/q}$$
$$\leqslant C_{1,0}\left(\int_0^\infty g^p(x)V^p(x)v^{-p^2/p'}(x)dx\right)^{1/p}, \qquad g\geqslant 0,$$

and

$$(32)\quad \left(\int_0^\infty \left|\int_{a(x)}^{b(x)} \mathbf{k}(y,x)g(y)dy\right|^q w^q(x)dx\right)^{1/q}$$
$$\leqslant C_{1,1}\left(\int_0^\infty g^p(x)V^p(x)v^{-p^2/p'}(x)dx\right)^{1/p}, \qquad g\geqslant 0,$$

with $C_1\approx C_{1,0}+C_{1,1}$. Characterization of (31) is known, because $\max(a(x),b(x))$ is an increasing function, $C_{1,0}\approx A_{1,0}$, where for $1<p\leqslant q<\infty$

$$(33)\quad A_{1,0} := \sup_{t>0}\left(\int_0^t \left|\int_{a(x)}^{b(x)} u(s)ds\right|^q w^q(x)dx\right)^{1/q}$$
$$\times\left(\int_{\max(a(t),\,b(t))}^\infty V^{-p'}(x)v^p(x)dx\right)^{1/p'},$$

and for $1<q<p<\infty$ with $1/r=1/q-1/p$

$$(34)\quad A_{1,0} := \left(\int_0^\infty\left(\int_0^t \left|\int_{a(x)}^{b(x)} u(s)ds\right|^q w^q(x)dx\right)^{r/p}\right.$$
$$\left.\times\left(\int_{\max(a(t),\,b(t)}^\infty V^{-p'}(x)v^p(x)dx\right)^{r/p'}\left|\int_{a(t)}^{b(t)} u(s)ds\right|^q w^q(t)dt\right)^{1/r}$$

Applying Theorem 1 we obtain the main result of the paper.

Theorem 3. *For the least possible constant in the inequality* (25) *the two-sided estimate*

$$C\approx A_0+A_{1,0}+A_{1,1}$$

holds,
where A_0 and $A_{1,0}$ are given by (28) *and* (33) - (34)*, respectively, and*

$$(35)\quad A_{1,1} = \sup_{i\in\Sigma_1}\left[\sup_{s\in I_i}\ \sup_{s\leqslant t\leqslant a^{-1}(b(s))}(A_0(s,t)+A_1(s,t))\right]$$
$$+\sup_{j\in\Sigma_2}\left[\sup_{s\in J_j}\ \sup_{s\leqslant t\leqslant b^{-1}(a(s))}(A_0(s,t)+A_1(s,t))\right], 1<p\leqslant q<\infty,$$

$$(36)\quad A_{1,1} = \left(\sum_{i\in\Sigma_1}\sum_{k\in\mathbb{Z}_i} B_{k,1}^r + B_{k,2}^r + B_{k,3}^r + B_{k,4}^r\right)^{1/r} + \left(\sum_{j\in\Sigma_2}\sum_{k\in\mathbb{Z}_j} B_{k,1}^r + B_{k,2}^r + B_{k,3}^r + B_{k,4}^r\right)^{1/r}, 1<q<p<\infty,$$

$$A_0(s,t) = \left(\int_s^t \mathbf{k}^q(\min(a(t),\, b(t)),x)w^q(x)dx\right)^{1/q} \times \left(\int_{\min(a(t),\, b(t))}^{\max(a(s),b(s))} V^{-p'}(x)v^p(x)dx\right)^{1/p'},$$

$$A_1(s,t) = \left(\int_s^t w^q(x)dx\right)^{1/q} \times \left(\int_{\min(a(t),\, b(t))}^{\max(a(s),b(s))} \mathbf{k}^{p'}(x,t)V^{-p'}(x)v^p(x)dx\right)^{1/p'},$$

$$B_{k,1} = \left(\int_{\min(a(\xi_k),\, b(\xi_k))}^{\min(a(\xi_{k+1}),\, b(\xi_{k+1}))} \left(\int_{\xi_k}^{\max(a^{-1}(t),\, b^{-1}(t))} \mathbf{k}^q(t,x)w^q(x)dx\right)^{r/q} \times \left(\int_t^{\min(a(\xi_{k+1}),\, b(\xi_{k+1}))} V^{-p'}(x)v^p(x)dx\right)^{r/q'} V^{-p'}(t)v^p(t)dt\right)^{1/r},$$

$$B_{k,2} = \left(\int_{\xi_k}^{\xi_{k+1}} \left(\int_{\xi_k}^t w^q(x)dx\right)^{r/p} \times \left(\int_{\min(a(t),\, b(t))}^{\min(a(\xi_{k+1}),\, b(\xi_{k+1}))} \mathbf{k}^{p'}(x,t)V^{-p'}(x)v^p(x)dx\right)^{r/p'} w^q(t)dt\right)^{1/r},$$

$$B_{k,3} = \left(\int_{\max(a(\xi_k),\, b(\xi_k))}^{\max(a(\xi_{k+1}),\, b(\xi_{k+1}))} \left(\int_{\min(a^{-1}(t),\, b^{-1}(t))}^{\xi_{k+1}} \mathbf{k}^q(t,x)w^q(x)dx\right)^{r/q} \times \left(\int_{\max(a(\xi_k),\, b(\xi_k))}^t V^{-p'}(x)v^p(x)dx\right)^{r/q'} V^{-p'}(t)v^p(t)dt\right)^{1/r},$$

$$B_{k,4} = \left(\int_{\xi_k}^{\xi_{k+1}} \left(\int_t^{\xi_{k+1}} w^q(x)dx \right)^{r/p} \right.$$

$$\left. \times \left(\int_{\max(a(\xi_k),\, b(\xi_k))}^{\max(a(t),\, b(t))} \mathbf{k}^{p'}(x,t) V^{-p'}(x) v^p(x) dx \right)^{r/p'} w^q(t)dt \right)^{1/r},$$

and

$$\xi_k = \begin{cases} (a^{-1} \circ b)^k(\xi_0), & \xi_0 \in I_i, \quad \textit{if } a(x) < b(x), \\ (b^{-1} \circ a)^k(\xi_0), & \xi_0 \in J_j, \quad \textit{if } b(x) < a(x). \end{cases}$$

Remark 3. The similar result as Theorem 3 is also true for non-decreasing functions. We omit details.

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