$\ell_{p,q}$ -bounded sequences in function spaces

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Abstract. A sequence of measurable functions $\{f_n\}$ is called $\ell_{p,q}$ -bounded sequence, $0 < p < \infty, 0 < q \leq \infty$, if for any sequence of real numbers $a = \{\alpha_n\} \in \ell_{p,q}$ we have

$$\sup_n |\alpha_n F_n(\omega)| < \infty \quad \omega - \text{a.e.},$$

for any sequence $\{F_n\}$ such that for every n functions f_n and F_n are equimeasurable. The main result gives necessary and sufficient conditions for the sequence being $\ell_{p,q}$ -bounded.

1. Introduction

Let (Ω, μ) be a measure space with probabilistic measure μ , isomorphic, as a measure space, to $([0, 1], \mathbf{m})$. Let $d_f(\tau) := \mu(\omega : |f(\omega)| > \tau)$, $\tau \geq 0$, be a distribution function of some measurable function f . Two sequences of measurable functions $\{f_n(\omega)\}_{n=1}^\infty$, $\omega \in \Omega$ and $\{F_n(\omega')\}_{n=1}^\infty$, $\omega' \in \Omega'$ are called *similar*, if

$$d_{f_n}(\tau) = d_{F_n}(\tau), \quad \tau \geq 0, \quad n = 1, 2, \dots$$

We shall write

$$\{f_n(\omega)\}_{n=1}^\infty \sim \{F_n(\omega')\}_{n=1}^\infty$$

to denote that two sequences are similar. A sequence of measurable functions $\{f_n(t)\}$, $t \in [0, 1]$, is called $\ell_{p,q}$ -bounded sequence, if for any sequence of real numbers $a = \{\alpha_n\} \in \ell_{p,q}$ we have

$$\sup_n |\alpha_n F_n(\omega)| < \infty \quad \omega - \text{a.e.}$$

for any sequence $\{F_n(\omega)\}_{n=1}^\infty \sim \{f_n(t)\}_{n=1}^\infty$.

The Lorentz sequence space $\ell_{p,q}$ ($0 < p < \infty$, $0 < q \leq \infty$) consists of all real sequences $a = \{\alpha_n\}_{n=1}^\infty$ such that

$$\begin{aligned} \|a\|_{p,q} &= \left[\sum_{n=1}^{\infty} (\alpha_n^*)^q n^{q/p-1} \right]^{1/q} \\ &\asymp \left[\sum_{n=1}^{\infty} (\alpha_n^*)^q \left(n^{q/p} - (n-1)^{q/p} \right) \right]^{1/q} < \infty \quad \text{for } q < \infty; \end{aligned}$$

$$\|a\|_{p,\infty} = \sup_n \alpha_n^* n^{1/p} < \infty.$$

The non-increasing rearrangement of numbers $\{|\alpha_n|\}$ is denoted by $\{\alpha_n^*\}$. These spaces are Banach spaces for $1 < p < \infty$, $1 \leq q \leq \infty$, $p \neq q$ and for $1 \leq p < \infty$, $p = q$. In other cases they are only quasinormed. The functionals written above are equivalent to norms in the first case, and quasinorms in general [7]. By a quasinorm on some linear space $\mathcal{A}$ we mean the functional $\|\cdot\|$, which satisfies all norm's axioms besides a triangle inequality. This inequality is substituted by a weaker inequality

$$(1.1) \quad \|a + b\| \leq K(\|a\| + \|b\|); \quad a, b \in \mathcal{A}$$

for some $K \geq 1$. Each quasinormed space is metrizable ([5], n° 3.10), its completeness will be assumed below. Each quasinormed space has an equivalent continuous quasinorm ([19, 6. 1. 9]).

Similarly [7] the Lorentz function space $L_{p,q}$, ($0 < p < \infty$, $0 < q \leq \infty$) consists of all measurable functions f such that

$$\|f\|_{p,q} = \left(\int_0^1 \left[t^{1/p} f^*(t) \right]^q \frac{dt}{t} \right)^{1/q} < \infty \quad \text{for } q < \infty;$$

$$\|f\|_{p,\infty} = \sup_{0 < t < 1} t^{1/p} f^*(t) < \infty,$$

where $f^*(t) = \inf(\tau \geq 0 : d_f(\tau) < t)$ is a decreasing rearrangement of the function $|f|$.

$\ell_{p,q}$ -bounded sequences for some pairs (p, q) appeared naturally in various areas both in analysis and probability theory [1, 2, 12]. $\ell_{p,q}$ -bounded

sequences of independent *identically* distributed functions for any admissible pair (p, q) have been studied in [8], where the following theorem was proved.

Theorem A. *Let $\{f_k(t)\}_{k=1}^\infty$ be a sequence of identically distributed functions, $0 < p < \infty$, $0 < q \leq \infty$. The following assertions are equivalent:*

(A.1) $\{f_k\}_{k=1}^\infty$ is a $\ell_{p,q}$ -bounded sequence;

(A.2) for any $a = \{\alpha_k\} \in \ell_{p,q}$

$$\lim_{k \rightarrow \infty} \alpha_k F_k = 0 \text{ a.e.}, \quad \{F_n\} \sim \{f_n\};$$

(A.3) for any $a = \{\alpha_k\} \in \ell_{p,q}$

$$\sup_{k \in \mathbb{N}} |\alpha_k F_k| \in L_{p,\infty}, \quad \{F_n\} \sim \{f_n\};$$

(A.4) $f_1 \in L_{p,s}$, where s satisfies the following relations:

$$\begin{aligned} \frac{1}{q} + \frac{1}{s} &= \frac{1}{p}, \quad \text{for } q > p; \\ s &= \infty, \quad \text{for } q \leq p. \end{aligned}$$

Theorem A was proved for the sequence of independent (in probabilistic sense) functions. Here we can drop this assumption, since there is a sequence of independent functions $\{F_n^{(i)}\}_{n=1}^\infty \sim \{f_n\}_{n=1}^\infty$ [3, 12]. Theorem A was proved by using Banach theorem about operators with values in the space $L_\circ$ of all measurable functions [10, 11], a variant of the principle of uniform boundedness. The assumption of identical distribution was essential in the application of this theorem.

The main result of this paper is the following theorem.

Theorem 1.1. *Let $0 < p < \infty$, $0 < q \leq \infty$. The sequence $\{f_k\}_{k=1}^\infty$ is a $\ell_{p,q}$ -bounded sequence iff the function*

$$\sup_k f_k^*(t) \in L_{p,s}$$

where s satisfies the following relations (cf. Theorem A):

$$\begin{aligned} \frac{1}{q} + \frac{1}{s} &= \frac{1}{p}, \quad \text{for } q > p; \\ s &= \infty, \quad \text{for } q \leq p. \end{aligned}$$

In spite of the proximity of assertions stated in Theorem A and Theorem 1.1, the proof of the last one requires other methods, which will be described below.

2. Auxiliary propositions

Lemma 2.1. *Let X be a quasi-Banach space. There exists the sequence of positive numbers (ε_n) such that for any sequence $(x_n) \in X^{\mathbb{N}}$ with $\|x_n\| \leq \varepsilon_n$, $n \in \mathbb{N}$, the series $\sum x_n$ converges in X .*

Proof. By induction we can prove the inequality

$$\left\| \sum_{i=1}^l x_i \right\| \leq \sum_{i=1}^{l-1} K^i \|x_i\| + K^{l-1} \|x_l\|, \quad l = 2, 3, \dots$$

Let $\varepsilon_i = (2K)^{-i}$, $i = 1, 2, \dots$. If $\|x_i\| \leq \varepsilon_i$, then

$$\begin{aligned} \left\| \sum_{i=k+1}^{k+l} x_i \right\| &\leq \sum_{i=1}^{l-1} K^i \|x_{k+i}\| + K^{l-1} \|x_{k+l}\| \\ &\leq \sum_{i=1}^{l-1} K^i \varepsilon_{k+i} + K^{l-1} \varepsilon_{k+l} \\ &= \sum_{i=1}^{l-1} \frac{1}{2^{i+k} K^k} + \frac{1}{2^{k+l} K^{k+1}} \leq \frac{1}{(2K)^k}. \end{aligned}$$

So Cauchy criterion is fulfilled for the series $\sum x_i$. $\square$

For any $\ell_{p,q}$ -bounded sequence $\{f_n\}$ we can define the family of sublinear operators

$$T_{\{F_n\}} : a \in \ell_{p,q} \rightarrow \sup_n |a_n F_n(\omega)|$$

from $\ell_{p,q}$ to $L_\circ(\Omega)$, where $\{F_n\} \sim \{f_n\}$.

The following theorem plays the same role as Banach theorem does in the proof of Theorem A.

Theorem 2.2. *Let $\{f_n\}$ be a $\ell_{p,q}$ -bounded sequence. Then the family of operators $T_{\{F_n\}}$, $\{F_n\} \sim \{f_n\}$ is uniformly bounded, i.e. the set*

$$\mathcal{T} = \{ (T_{\{F_n\}}(a))(\omega) : \|a\|_{p,q} \leq 1, \{F_n\} \sim \{f_n\} \},$$

is bounded in measure or (equivalently) in the space $L_\circ(\mu)$.

Proof. Assume that the set $\mathcal{T}$ is unbounded in $L_\circ(\mu)$. Then for some $\delta > 0$ and for any $k \in \mathbb{N}$ there exist $0 \leq a^{(k)} \in \ell_{p,q}$, $\|a^{(k)}\|_{p,q} \leq 1$ and a sequence of functions $\{F_n^{(k)}\}_{n=1}^\infty \sim \{f_n\}_{n=1}^\infty$ such that

$$\begin{aligned} \mu(E_k) &:= \mu \left(\omega : \left(T_{\{F_n^{(k)}\}}(a^{(k)}) \right)(\omega) > \frac{k}{\varepsilon_k} \right) \\ &= \mu \left(\omega : \sup_n \left| \varepsilon_k a_n^{(k)} F_n^{(k)}(\omega) \right| > k \right) \\ &\geq \delta, \end{aligned}$$

where ε_k is defined by Lemma 2.1. Let $n \rightarrow (k, m)$ be a bijection from $\mathbb{N}$ on $\mathbb{N} \times \mathbb{N}$. If we set $\alpha_n := \varepsilon_k \alpha_m^{(k)}$, $F_n := F_m^{(k)}$, $n = 1, 2, \dots$, then $a := \{\alpha_n\} \in \ell_{p,q}$ by virtue of Lemma 2.1.

Now we see, that if $\omega \in M := \limsup E_k = \bigcap_{l=1}^{\infty} \bigcup_{k=l}^{\infty} E_k$, then the sequence $\{\alpha_n F_n(\omega)\}_{n=1}^{\infty}$ has an infinite subsequence (may be rearranged) $\{\varepsilon_k \alpha_m^{(k)} F_m^{(k)}(\omega)\}_{k=1, m=1}^{\infty, \infty}$, such that

$$\sup_m \left| \varepsilon_k \alpha_m^{(k)} F_m^{(k)}(\omega) \right| > k, \quad k \in \mathbf{K} \subset \mathbb{N}, \quad \text{card } \mathbf{K} = \infty.$$

Using the rearrangement invariance of spaces $\ell_{p,q}$ and ℓ_{∞} we get that

$$\sup_n |\alpha_n F_n(\omega)| = +\infty, \quad \omega \in M, \quad \mu(M) \geq \delta.$$

We arrived at a contradiction with definition of $\ell_{p,q}$ -bounded sequence. $\square$

Corollary 2.3. *Let $\{f_n\}$ be a $\ell_{p,q}$ -bounded sequence. There exists function*

$$\chi : \mathbb{R}_+ \rightarrow [0, 1], \quad \lim_{\tau \rightarrow \infty} \chi(\tau) = 0$$

such that

$$\sup_{\|a\|_{p,q} \leq 1} \mu \left(\omega : \sup_n |a_n F_n(\omega)| > \tau \right) \leq \chi(\tau), \quad \tau \geq 0,$$

where $\{F_n\} \sim \{f_n\}$.

Theorem 2.4. *A sequence of measurable functions $\{f_n(t)\}$, $t \in [0, 1]$ is $\ell_{p,q}$ -bounded sequence iff there exists $\tau_o > 0$ such that the series $\sum_{n=1}^{\infty} d_{a_n f_n}(\tau_o)$ converges for any $a \in \ell_{p,q} : \|a\|_{p,q} \leq 1$.*

Proof. Among sequences $\{F_n\} \sim \{f_n\}$ there exists a sequence $\{F_n^{(i)}\}$ of independent functions. By applying 2.3 we get $\tau_o > 0$ such that for any a with $\|a\|_{p,q} \leq 1$,

$$\mu \left(\omega : \sup_n |a_n F_n^{(i)}(\omega)| > \tau_o \right) < 1.$$

Using independence we get

$$\begin{aligned} \mu \left(\omega : \sup_n |a_n F_n^{(i)}(\omega)| > \tau_o \right) &= 1 - \mu \left(\omega : \sup_n |a_n F_n^{(i)}(\omega)| \leq \tau_o \right) \\ &= 1 - \prod_n \mu \left(|a_n F_n^{(i)}(\omega)| \leq \tau_o \right) \\ &= 1 - \prod_n \left(1 - \mu \left(|a_n F_n^{(i)}| > \tau_o \right) \right). \end{aligned}$$

So $\prod_n \left(1 - \mu \left(\left| \alpha_n F_n^{(i)} \right| > \tau_o \right) \right) > 0$, it means the convergence of the series $\sum_n \mu \left(\left| \alpha_n F_n^{(i)} \right| > \tau_o \right) = \sum_n d_{\alpha_n f_n}(\tau_o)$.

Conversely let $a \in \ell_{p,q}$, $\|a\| \leq 1$, let $\{F_n\}$ be a sequence similar to $\{f_n\}$. We have

$$\begin{aligned} \mu \left(\omega : \sup_n |\alpha_n F_n(\omega)| > \tau \right) &= \mu \left(\bigcup_n (\omega : |\alpha_n F_n(\omega)| > \tau) \right) \\ &\leq \sum_n \mu (\omega : |\alpha_n F_n(\omega)| > \tau) \\ &= \sum_n d_{\alpha_n f_n}(\tau) < \infty \end{aligned}$$

for $\tau \geq \tau_o$. It turns out that the sum

$$\sum_n d_{\alpha_n f_n}(\tau) = \sum_n \mathbf{m}(t : |\alpha_n f_n(t)| > \tau)$$

is a distribution function for the some function on the half-line, namely, for the function $\sum \oplus \alpha_n f_n$, defined by a transplantation of $\alpha_n f_n$ to the interval $[n-1, n)$ and by pasting together all such pieces. The function $\sum \oplus \alpha_n f_n \in L_o([0, \infty))$ and $d_{\sum \oplus \alpha_n f_n}(\tau) < \infty$ for $\tau \geq \tau_o$, from this we deduce that $d_{\sum \oplus \alpha_n f_n}(\tau) \rightarrow 0$ for $\tau \rightarrow \infty$.

So $\mu(\omega : \sup_n |\alpha_n F_n(\omega)| = \infty) = 0$. $\square$

3. Proof of Theorem 1.1

Lemma 3.1. *Let $0 < q \leq p < \infty$. A sequence of measurable functions $\{f_n(t)\}$, $t \in [0, 1]$ is $\ell_{p/q,1}$ -bounded sequence iff $\sup_n f_n^* \in L_{p/q, \infty}$.*

Proof. For $s > 0$ we put

$$\rho(s) := \sup_{0 \leq t \leq s} \sup_n d_{t f_n}(\tau_o),$$

where the number τ_o is defined by Th.2.4, according to it the series $\sum \rho(|\alpha_n|)$ converges for a with $\|a\|_{p/q,1} \leq 1$.

Let $\tilde{\rho}(t) := \sup\{s \geq 0 : \rho(s) \leq t\}$. The function $\tilde{\rho}(t)$ increases. We show first, that

$$\inf_{n \in \mathbb{N}} \tilde{\rho} \left(\frac{1}{4n} \right) (2n)^{q/p} > 0.$$

Let's suppose, that

$$\inf_{n \in \mathbb{N}} \tilde{\rho} \left(\frac{1}{4n} \right) (2n)^{q/p} = 0.$$

Then we inductively construct the sequence of integers $n_1 < n_2 < \dots$ in such a way:

$$1) \quad n_j \geq N_{j-1}, \text{ where } N_{j-1} = \sum_{i=1}^{j-1} n_i, j \geq 2; N_0 := 0.$$

(We can deduce from that: $N_j \leq 2n_j$.)

$$2) \quad \tilde{\rho} \left(\frac{1}{4n_j} \right) (2n_j)^{q/p} \leq \frac{1}{2^{j+1}}, j = 1, 2, \dots$$

Now we define the sequence

$$\alpha_n := 2\tilde{\rho} \left(\frac{1}{4n_j} \right) \text{ for } N_{j-1} < n \leq N_j.$$

The sequence $a = \{\alpha_n\}$ satisfies the conditions:

$$\begin{aligned} 1) \quad \alpha_n &\geq 0, \alpha_n \downarrow; \\ 2) \quad \|a\|_{p/q,1} &= \sum_n \alpha_n \left((n)^{q/p} - (n-1)^{q/p} \right) \\ &= 2 \sum_{j=1}^{\infty} \tilde{\rho} \left(\frac{1}{4n_j} \right) \left((N_j)^{q/p} - (N_{j-1})^{q/p} \right) \\ &\leq 2 \sum_{j=1}^{\infty} \tilde{\rho} \left(\frac{1}{4n_j} \right) (2n_j)^{q/p} \\ &\leq 2 \sum \frac{1}{2^{j+1}} = 1. \end{aligned}$$

On the other hand we have that $\sum \rho(\alpha_n) = \infty$, since

$$\sum_{n=N_{j-1}+1}^{N_j} \rho(\alpha_n) = n_j \rho \left(2\tilde{\rho} \left(\frac{1}{4n_j} \right) \right) > n_j \frac{1}{4n_j} = \frac{1}{4},$$

contrary to the convergence of the series $\sum \rho(\alpha_n)$. So, we have shown that

$$\varkappa := \inf_n \tilde{\rho} \left(\frac{1}{4n} \right) (2n)^{q/p} > 0.$$

For $n := \left\lceil \frac{t}{2} \right\rceil$, we obtain the inequality

$$\tilde{\rho} \left(\frac{1}{t} \right) t^{q/p} \geq \tilde{\rho} \left(\frac{1}{4n} \right) (2n)^{q/p} \geq \varkappa,$$

then

$$\frac{1}{t} \geq \rho \left(\frac{\varkappa}{\varphi(t)} \right), \quad t \geq 2.$$

If $s := \frac{\varkappa}{t^{q/p}}$, $t \geq 2$, then

$$\rho(s) \leq \left(\frac{s}{\varkappa}\right)^{p/q},$$

and

$$d_{f_n} \left(\frac{t}{\tau_o} \right) \leq \left(\frac{\varkappa}{t} \right)^{p/q}, \quad n = 1, 2, \dots$$

it means, that $\sup_n f_n^* \in L_{p/q, \infty}$.

Conversely, if

$$\sup_n f_n^*(t) \in L_{p/q, \infty},$$

then there exist $k > 0, t_o > 0$ such that for each $s \geq 0$, $d_{sf_n}(t_o) \leq (ks)^{p/q}$ for any $n \in \mathbb{N}$. Let $\delta := 1/k$ and $a \in \ell_{p,q}$ with $\|a\| \leq \delta$. Let $h_j := \alpha_j^* - \alpha_{j+1}^*$,

then we have $\alpha_n^* = \sum_{j=n}^{\infty} h_j$, $n \in \mathbb{N}$. It is easily checked that

$$\begin{aligned} \sum_{j=1}^{\infty} j^{q/p} h_j &= \sum_{j=1}^{\infty} \sum_{n=1}^j h_j \left(n^{q/p} - (n-1)^{q/p} \right) \\ &= \sum_{n=1}^{\infty} \left(\sum_{j=n}^{\infty} h_j \right) \left(n^{q/p} - (n-1)^{q/p} \right) \\ &= \sum_{n=1}^{\infty} \alpha_n^* \left(n^{q/p} - (n-1)^{q/p} \right) \\ &= \|a\|_{p/q, 1} \leq \delta. \end{aligned}$$

Now we consider for fixed $n \in \mathbb{N}$

$$\begin{aligned} (k\alpha_n^*)^{p/q} &= \left(k \sum_{j=n}^{\infty} h_j \right)^{p/q} \\ &= \left(\sum_{j=n}^{\infty} k j^{q/p} h_j \frac{1}{j^{q/p}} \right)^{p/q} \\ &\leq \sum_{j=n}^{\infty} k j^{q/p} h_j \left(\frac{1}{j^{q/p}} \right)^{p/q} \\ &= k \sum_{j=n}^{\infty} j^{q/p-1} h_j, \end{aligned}$$

since $p \geq q$ and

$$\sum_{j=n}^{\infty} k j^{q/p} h_j \leq 1.$$

Finally, we receive that

$$\begin{aligned} \sum_{n=1}^{\infty} d_{\alpha_n f_n}(t_o) &\leq \sum_{n=1}^{\infty} (k |\alpha_n|)^{p/q} \\ &= \sum_{n=1}^{\infty} (k \alpha_n^*)^{p/q} \\ &\leq k \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} j^{q/p-1} h_j \\ &= k \sum_{j=1}^{\infty} j^{q/p} h_j \leq k \delta = 1, \end{aligned}$$

according to Theorem 2.4 it means that the sequence $\{f_n\}$ is $\ell_{p/q,1}$ -bounded. $\square$

Corollary 3.2. *Let $0 < q \leq p < \infty$. A sequence of measurable functions $\{f_n(t)\}$, $t \in [0, 1]$ is $\ell_{p,q}$ -bounded iff $\sup_n f_n^* \in L_{p,\infty}$.*

Proof. It's clear directly from definition that the sequence $\{f_n\}$ is $\ell_{p,q}$ -bounded iff the sequence $\{f_n^q\}$ is $\ell_{p/q,1}$ -bounded. So according to Lemma 3.1 we receive that

$$\sup_n f_n^{q*} \in L_{p/q,\infty},$$

or

$$\sup_n f_n^* \in L_{p,\infty}. \quad \square$$

Lemma 3.3. *Let $p > 0$. A sequence of measurable functions $\{f_n(t)\}$, $t \in [0, 1]$ is $\ell_{p,\infty}$ -bounded iff $\sup_n f_n^* \in L_p$.*

Proof. By applying Lemma 2.2, Th. 2.4 and definition of the space $\ell_{p,\infty}$ we obtain that a sequence of measurable functions $\{f_n(t)\}$, $t \in [0, 1]$ is $\ell_{p,\infty}$ -bounded sequence iff the series

$$\sum_{k=1}^{\infty} d_{\sup_n f_n^*} \left(k^{1/p} \tau_o \right)$$

for some $\tau_o > 0$. This leads naturally to the convergence of the integrals $\int_0^{\infty} d_{\sup_n f_n^*} (s^{1/p} \tau_o) ds$ and $\int_0^1 (\sup_n f_n^*(t))^p dt$. $\square$

Lemma 3.4. *Let $0 < p < q < \infty$. For arbitrary increasing function $\rho(t) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ the following asserions are equivalent:*

- 1) *the series $\sum \rho(\alpha_n)$ converges for any $a \in \ell_{p/q,1}$;*
- 2) *the function $\varkappa(t) := \rho(t)/t$, $t > 0$ belongs to the space $L_{\frac{q}{q-p}}$.*

Proof. 1) $\Rightarrow$ 2) Using convexity of function $t \rightarrow t^{q/p}$ we get inductively the inequality

$$(t_1 + \dots + t_m)^{q/p} \leq \sum_{i=1}^m \left(2^{q/p-1}\right)^{m-i+1} (t_i)^{q/p}, \quad m = 1, 2, \dots; t_1, \dots, t_m \geq 1. \quad (*)$$

For $\gamma \in (0, 2^{1-q/p})$ we introduce the sequence $a = \{\alpha_n\}$ in the following way:

$$\underbrace{1, \dots, 1}_{n_1}, \underbrace{\gamma, \dots, \gamma}_{n_2}, \dots, \underbrace{\gamma^{j-1}, \dots, \gamma^{j-1}}_{n_j}, \dots,$$

numbers $n_1, n_2, \dots$ will be defined more exactly later. Using Abel identity and (*) we obtain

$$\begin{aligned} \|a\|_{p/q,1} &= \sum_{n=1}^{\infty} n^{q/p} (\alpha_n - \alpha_{n+1}) \\ &= (1-\gamma) \sum_{j=1}^{\infty} (n_1 + \dots + n_j)^{q/p} \gamma^{j-1} \\ &\leq (1-\gamma) \sum_{j=1}^{\infty} \gamma^{j-1} \sum_{i=1}^j \left(2^{q/p-1}\right)^{j-i+1} (n_i)^{q/p} \\ &= (1-\gamma) \sum_{i=1}^{\infty} (n_i)^{q/p} \sum_{j=1}^{\infty} \left(2^{q/p-1}\right)^{j-i+1} \gamma^{j-1} \\ &= \frac{1-\gamma}{2^{2-q/p}-\gamma} \sum_{i=1}^{\infty} (n_i)^{q/p} \gamma^{i-1} \end{aligned} \quad (**)$$

We can reproduce the series $\sum \rho(\alpha_n)$ in the following function mode using characteristic function $\mathbf{1}_e$ of a measurable set e :

$$\begin{aligned} \sum_{n=1}^{\infty} \rho(\alpha_n) &= \sum_{j=1}^{\infty} n_j \rho(\gamma^{j-1}) \\ &= \frac{\gamma}{1-\gamma} \int_0^1 \sum_{j=1}^{\infty} n_j \mathbf{1}_{(\gamma^j, \gamma^{j-1}]}(t) \cdot \sum_{j=1}^{\infty} \frac{\rho(\gamma^{j-1})}{\gamma^j} \mathbf{1}_{(\gamma^j, \gamma^{j-1}]}(t) dt. \end{aligned}$$

Our aim now is to show, that

$$\sum_{j=1}^{\infty} \frac{\rho(\gamma^{j-1})}{\gamma^j} \mathbf{1}_{(\gamma^j, \gamma^{j-1}]}(t) \in L_{\frac{q}{q-p}}([0, 1]).$$

If we assume the opposite, then there exists a sequence of positive numbers $\{s_j\}$ such that

$$\left\| \sum_{j=1}^{\infty} s_j \mathbf{1}_{(\gamma^j, \gamma^{j-1}]} \right\|_{L_{q/p}([0,1])} \leq 1,$$

and

$$\int_0^1 \sum_{j=1}^{\infty} s_j \mathbf{1}_{(\gamma^j, \gamma^{j-1}]}(t) \cdot \sum_{j=1}^{\infty} \frac{\rho(\gamma^{j-1})}{\gamma^j} \mathbf{1}_{(\gamma^j, \gamma^{j-1}]}(t) dt = \infty.$$

Now we can define more exactly numbers n_j , namely we put $n_j := [s_j] + 1$, $j \in \mathbb{N}$, then

$$\int_0^1 \sum_{j=1}^{\infty} n_j \mathbf{1}_{(\gamma^j, \gamma^{j-1}]}(t) \cdot \sum_{j=1}^{\infty} \frac{\rho(\gamma^{j-1})}{\gamma^j} \mathbf{1}_{(\gamma^j, \gamma^{j-1}]}(t) dt = \infty,$$

hence, $\sum \rho(\alpha_n) = \infty$.

On the other hand we have

$$\begin{aligned} (1 - \gamma) \sum_{j=1}^{\infty} (n_j)^{q/p} \gamma^{j-1} &\leq (1 - \gamma) \sum_{j=1}^{\infty} (s_j + 1)^{q/p} \gamma^{j-1} \\ &\leq (1 - \gamma) \sum_{j=1}^{\infty} \left[2^{q/p} (s_j)^{q/p} + 2^{q/p} \right] \gamma^{j-1} \\ &= 2^{q/p} + (1 - \gamma) 2^{q/p} \sum_{j=1}^{\infty} (s_j)^{q/p} \gamma^{j-1} \leq 2^{q/p+1}, \end{aligned}$$

and together with (**) we receive that

$$\|a\|_{p/q,1} \leq \frac{2^{q/p+1}}{2^{1-q/p} - \gamma}.$$

The received contradiction means that

$$\sum_{j=1}^{\infty} \frac{\rho(\gamma^{j-1})}{\gamma^j} \mathbf{1}_{(\gamma^j, \gamma^{j-1}]}(t) \in L_{\frac{q}{q-p}}([0, 1]).$$

Since

$$\varkappa(t) \leq \sum_{j=1}^{\infty} \frac{\rho(\gamma^{j-1})}{\gamma^j} \mathbf{1}_{(\gamma^j, \gamma^{j-1}]}(t), \quad 0 < t \leq 1,$$

we obtain that $\varkappa(t) \in L_{\frac{q}{q-p}}([0, 1])$.

2) $\Rightarrow$ 1). We suppose the sequence of nonnegative numbers $a = \{\alpha_n\}$ decreasing and $\|a\|_{p/q,1} \leq 1/4$. Let's introduce the sequence of integers (N_j) in the following way:

$$N_0 := 0, \quad N_j := \sup\{n \in \mathbb{N} : \alpha_n \geq \frac{\alpha_1}{2^j}\}, \quad j \in \mathbb{N}.$$

Then we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \rho(\alpha_n) &\leq \sum_{j=1}^{\infty} (N_j - N_{j-1}) \rho\left(\frac{\alpha_1}{2^{j-1}}\right) \\
&\leq \sum_{j=1}^{\infty} N_j \rho\left(\frac{\alpha_1}{2^{j-1}}\right) \\
&= 2 \int_0^{2\alpha_1} \sum_{j=1}^{\infty} N_j \mathbf{1}_{\left(\frac{\alpha_1}{2^{j-1}}, \frac{\alpha_1}{2^{j-2}}\right]}(t) \cdot \sum_{j=1}^{\infty} \frac{\rho\left(\frac{\alpha_1}{2^{j-1}}\right)}{\frac{\alpha_1}{2^{j-2}}} \mathbf{1}_{\left(\frac{\alpha_1}{2^{j-1}}, \frac{\alpha_1}{2^{j-2}}\right]}(t) dt \\
&\leq 2 \int_0^{2\alpha_1} \sum_{j=1}^{\infty} N_j \mathbf{1}_{\left(\frac{\alpha_1}{2^{j-1}}, \frac{\alpha_1}{2^{j-2}}\right]}(t) \cdot \frac{\rho(t)}{t} dt \\
&\leq 4 \left\| \sum_{j=1}^{\infty} N_j \mathbf{1}_{\left(\frac{\alpha_1}{2^{j-1}}, \frac{\alpha_1}{2^{j-2}}\right]} \right\|_{L_{q/p}([0, 2\alpha_1])} \cdot \|\mathcal{K}\|_{L_{q/(q-p)}([0, 2\alpha_1])}.
\end{aligned}$$

Now we notice that

$$\begin{aligned}
&\left\| \sum_{j=1}^{\infty} N_j \mathbf{1}_{\left(\frac{\alpha_1}{2^{j-1}}, \sum_{j=1}^{\infty} \varphi(N_j) \frac{\alpha_1}{2^{j-1}} = \frac{\alpha_1}{2^{j-2}}\right]} \right\|_{L_{q/p}([0, 2\alpha_1])} \\
&= \sum_{j=1}^{\infty} (N_j)^{q/p} \frac{\alpha_1}{2^{j+1}} = 4 \sum_{j=1}^{\infty} (N_j)^{q/p} \frac{\alpha_1}{2^{j+1}} \\
&= 4 \left\| \underbrace{\frac{\alpha_1}{2}, \dots, \frac{\alpha_1}{2}}_{N_1}, \underbrace{\frac{\alpha_1}{2^2}, \dots, \frac{\alpha_1}{2^2}}_{N_2 - N_1}, \dots, \underbrace{\frac{\alpha_1}{2^j}, \dots, \frac{\alpha_1}{2^j}}_{N_j - N_{j-1}}, \dots \right\|_{p/q, 1} \leq 4 \|(\alpha_n)\|_{p/q, 1} \leq 1,
\end{aligned}$$

so

$$\left\| \sum_{j=1}^{\infty} N_j \mathbf{1}_{\left(\frac{\alpha_1}{2^{j-1}}, \frac{\alpha_1}{2^{j-2}}\right]} \right\|_{L_{q/p}([0, 2\alpha_1])} \leq 1,$$

hence,

$$\sum_{n=1}^{\infty} \rho(\alpha_n) \leq 4 \|\mathcal{K}\|_{L_{q/(q-p)}([0, 2\alpha_1])}$$

□

Corollary 3.5. *Let $0 < p < q < \infty$. For arbitrary increasing function $\rho(t) : \mathbb{R}_+ \rightarrow [0, 1]$ the following assertions are equivalent:*

1) *the series $\sum \rho(\alpha_n)$ converges for any $a \in \ell_{p,q}$;*

2) function $1/\lambda$, where $\lambda(s) := \sup\{t \geq 0 : \rho(t) \leq s\}$, $s \in [0, 1]$ belongs to the space $L_{p,r}([0, 1], \mathbf{m})$, number r is defined by the equation $1/r = 1/p - 1/q$.

Proof. Evidently, assertion 1) means that the series $\sum \rho(\beta_n^{1/q})$ converges for any $b = \{\beta_n\} \in \ell_{p/q,1}$. According to Lemma 3.4 we receive from this the convergence of the integral

$$\int_0^1 \left[\frac{\rho(t^{1/q})}{t} \right]^{q/(q-p)} dt.$$

By changing variables we obtain firstly the convergence of the integral $\int_0^1 [\lambda(\tau^{p/r})]^{-r} d\tau$, and then (put $s := \tau^{p/r}$) the convergence of the integral

$$\int_0^1 \left[\frac{1}{\lambda(s)} \right]^r d(s^{r/p}),$$

which means that function $\frac{1}{\lambda}$ belongs to the space $L_{p,r}$. □

Proof of Theorem 1.1. The cases $0 < q \leq p < \infty$ and $q = \infty$ were considered in corollary 3.2 and Lemma 3.3 respectively. The case $0 < p < q < \infty$ may be examined by applying corollary 3.5 to the function

$$\rho(t) := d_{\sup_n} f_n^* \left(\frac{1}{t} \right).$$

So we receive that $\sup_n f_n^* \in L_{p,s}([0, 1])$, where $1/s = 1/p - 1/q$. □

Applications of the proved theorem to the operator theory were demonstrated in [13].

References

- [1] E.M. Nikishin, *Resonance Theorems and Superlinear Operators*. Uspekhi mat. nauk, 1970, V. 25, No. 6, 129-191. (Russian)
- [2] B. S. Kashin and A. A. Saakyan, *Orthogonal Series*, M.: AFC, 1999. (Russian)
- [3] K. R. Parthasarathy, *Introduction to Probability and Measure*, New Delhi, 1980.
- [4] J. Bergh and J. Löfström, *Interpolation Spaces*, Springer, B.-H.-NY., 1976.
- [5] S. G. Krein, Yu. I. Petunin and E. M. Semenov, *Interpolation of Linear Operators*, M.: Nauka, 1978.

- [6] J. Lindenstrauss and L. Tzafriri L., *Classical Banach Spaces, I*, Springer, Berlin, 1977.
- [7] R. A. Hunt, *On $L(p, q)$ Spaces*, L'Enseignement Math., 1966, 12, 4, 249-276.
- [8] S. Ya. Novikov and A. M. Steinberg, *Lorentz Spaces and Boundedness Almost Everywhere*, Sib. Mat. Zhurnal, 1989, V. No. 2, 138-144.
- [9] A. Pietsch, *Operator Ideals*, VEB, Berlin, 1978.
- [10] S. Banach, *Sur la Convergence Presque Partout*, Bull. Sci. Math. Annalen, 1926, V. 50. 27-43.
- [11] N. Dunford and J. T. Schwartz, *Linear Operators, Part I*, London, I.P. N.Y., 1958.
- [12] N. N. Vakhaniya, V. I. Tarieladze and Chobanyan, *Probability Distributions in Banach Spaces*, Nauka, Moscow, 1985. (Russian)
- [13] S. Ya. Novikov, *Sequence Spaces $\ell_{p,q}$ in Probabilistic Characterizations of the Weak Type Operators*, Zapiski nauchnykh seminarov POMI, 2001, 282, 160-191. (Russian)

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