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Estimates for convolutions in the anisotropic Nikol'skiĭ-Besov spaces

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Abstract. We obtain various estimates for convolutions in the anisotropic Nikol'skiĭ-Besov spaces of functions of several real variables possessing some common smoothness of, in general, fractional order which may be different with respect to different variables.

1. Introduction

Various problems in pure and applied mathematics, in particular some inverse problems arising in geophysics, are reduced to convolution equations with anisotropic kernels. In order to exploit to a full extent the anisotropy of such kernels one requires applying estimates for convolutions in anisotropic spaces of differentiable functions. The aim of this article is obtaining estimates for convolutions in the anisotropic Nikol'skiĭ-Besov spaces, one of the most widely used and well studied scales of anisotropic spaces of differentiable functions with arbitrary orders of smoothness.

We are mostly interested in the estimates of convolutions of the following form

$$\|f * g\|_{B^{\vec{s}_2}_{p_2,\theta_2}(\mathbb{R}^N)} \leq c_1 \|wFf\|_{L_\infty(\mathbb{R}^N)} \|g\|_{B^{\vec{s}_1}_{p_1,\theta_1}(\mathbb{R}^N)} \ , \tag{1}$$

where $c_1 > 0$ is independent of f and g, $B^{\vec{s}_1}_{p_1,\theta_1}(\mathbb{R}^N)$, $B^{\vec{s}_2}_{p_2,\theta_2}(\mathbb{R}^N)$ are the anisotropic Nikol'skiĭ-Besov spaces, Ff is the Fourier transform of f and w is a weight function. Such estimates appeared to be convenient for some applications due to the simple form of the factor $\|wFf\|_{L_\infty(\mathbb{R}^N)}$.

They are of two types. In one of them it is assumed that $p_1 \leq 2 \leq p_2$, $\theta_1 \leq 2 \leq \theta_2$, and they are based on the embedding theorems for the Nikol'skiĭ-Besov spaces.

In the other it is only assumed that $p_1 \leq p_2, \theta_1 \leq \theta_2$. However, it is assumed, in addition that the vectors $\vec{s}_1$ and $\vec{s}_2$ are proportional, and there are also additional regularity assumptions on Ff. In this case first the estimate

$$\|f * g\|_{B^{\vec{s}_2}_{p_2,\theta_2}(\mathbb{R}^N)} \leq c_2 \|f\|_{B^{\vec{s}}_{p,\theta}(\mathbb{R}^N)} \|g\|_{B^{\vec{s}_1}_{p_1,\theta_1}(\mathbb{R}^N)} \tag{2}$$

is proved, where $c_2 > 0$ is independent of f and g, with the sharp assumptions on the parameters defining the spaces (coinciding with the known necessary and sufficient conditions in the isotropic case). This estimate is of independent interest and may be applied to various problems involving convolutions.

By applying a result on equivalent norms for the anisotropic Nikol'skiĭ-Besov spaces on a cone of functions whose Fourier transforms satisfy certain regularity conditions, obtained by the authors [9], we deduce (1) from (2) with the weight function w, which for proportional $\vec{s}_1$ and $\vec{s}_2$ is the same as in the first case.

For the isotropic case the estimates described above coincide with the estimates obtained by Batyrov, Burenkov and Pankratov [2], [3] and [7].

2. Definitions and basic properties

In this section we give some basic definitions and properties that will be required to prove the main result in the next section.

2.1. Convolution.

Definition 1. *Let*

$$\tilde{S}'(\mathbb{R}^N) := \{f \in S'(\mathbb{R}^N) : f \ , \ F(f) \in L^{loc}(\mathbb{R}^N)\}$$

and let f, $g \in \tilde{S}'(\mathbb{R}^N)$, where $S'(\mathbb{R}^N)$ is the Schwartz space of tempered distributions. If the (pointwise) product $F(f) \cdot F(g) \in \tilde{S}'(\mathbb{R}^N)$, the

convolution of f and g is defined by

$$f * g := F^{-1}\left(F\left(f\right) \cdot F\left(g\right)\right) \ . \tag{3}$$

Note that under the admitted assumptions one has $F\left(\tilde{S}'\left(\mathbb{R}^N\right)\right) = \tilde{S}'\left(\mathbb{R}^N\right)$ and $f * g \in \tilde{S}'\left(\mathbb{R}^N\right)$ and $F(f * g) = Ff \cdot Fg$.

This definition of the convolution was introduced and used in [2], [3] and [7]. It differs from the standard definition of the convolution in the theory of distributions (see for example, Vladimirov [21]). It is not wider than the standard definition, but may be applied to some pairs of f and g, which are of interest from the point of view of applications, for which the standard definition may not be applied.

2.2. Nikol'skiĭ-Besov spaces. Suppose[1] that $0 < p,\ \theta \leq \infty$, $\vec{s} = (s_1, ..., s_N)$, where $-\infty < \vec{s} < \infty$, and all s_j have the same sign. Thus either $\vec{s} > 0$ or $\vec{s} < 0$ or $\vec{s} = 0$. If $\vec{s} > 0$ or $\vec{s} < 0$, let $\dfrac{1}{s} = \dfrac{1}{N}\sum_{j=1}^{N}\dfrac{1}{s_j}$. Furthermore, let $\vec{a} = \dfrac{s}{\vec{s}}$, i.e. $\vec{a} = (a_1, ..., a_N)$, where $a_j = \dfrac{s}{s_j}$, $j = 1, ..., N$. Note that all $a_j > 0$ and $\sum_{j=1}^{N} a_j = N$. Thus, s is a certain mean smoothness and $\vec{a}$ measures the anisotropy. If $\vec{s} = 0$, then $s = 0$ and $\vec{a}$ is an arbitrary vector with positive components satisfying $\sum_{j=1}^{N} a_j = N$. (For different $\vec{a}$ one gets different spaces.) In the sequel $\vec{a}$ will be called the *anisotropy vector*. Define the anisotropic distance of $t = (t_1, ..., t_N) \in \mathbb{R}^N$ from the origin as $|t|_{\vec{a}} = \left(\sum_{j=1}^{N} |t_j|^{\frac{2}{a_j}}\right)^{\frac{1}{2}}$ and the anisotropic ball $B_r = \left\{t \in \mathbb{R}^N : |t|_{\vec{a}} < r\right\}$. Next, for $k \in \mathbb{N}_0$, let $\varphi_k \in C_0^\infty\left(\mathbb{R}^N\right)$, $\varphi_k \geq 0$, and $\operatorname{supp}\varphi_0 \subset B_2$; $\operatorname{supp}\varphi_k \subset B_{2^{k+1}} \setminus B_{2^{k-1}}$ for all $k \in \mathbb{N}$. Moreover, let for every multi-index α

$$\sup_{k\in\mathbb{N}_0}\ \sup_{t\in\mathbb{R}^N} 2^{k\langle\vec{a},\alpha\rangle}\left|\left(D^\alpha\varphi_k\right)(t)\right| < \infty \ , \tag{4}$$

where $\langle\vec{a},\alpha\rangle = \sum\limits_{k=1}^{N} a_k\alpha_k$, and for all $t \in \mathbb{R}^N$ $\sum\limits_{k=0}^{\infty}\varphi_k\left(t\right) = 1$. Finally, let $\Phi^{\vec{a}}\left(\mathbb{R}^N\right)$ be the set of all the collections $\{\varphi_k\}_{k=0}^{\infty}$ with the properties listed above.

Definition 2. One says that $f \in B_{p,\theta}^{\vec{s}}\left(\mathbb{R}^N\right)$ if $f \in S'\left(\mathbb{R}^N\right)$ and

$$\|f\|_{B_{p,\theta}^{\vec{s}}(\mathbb{R}^N)} = \left(\sum_{k=0}^{\infty}\left(2^{ks}\left\|F^{-1}\left(\varphi_k F\left(f\right)\right)\right\|_{L_p(\mathbb{R}^N)}\right)^\theta\right)^{\frac{1}{\theta}} < \infty \tag{5}$$

[1]We assume here and in the sequel that for $\vec{a} = (a_1, ..., a_N)$ and $\vec{b} = (b_1, ..., b_N)$ the inequalities $\vec{a} < \vec{b}$, $\vec{a} \leq \vec{b}$ mean that $a_j < b_j$, $a_j \leq b_j$ respectively, for all $j \in \{1, ..., N\}$. If $b \in \mathbb{R}$, then $\vec{a} < b$, $\vec{a} \leq b$, $\vec{a} = b$ mean that $a_j < b$, $a_j \leq b$, $a_j = b$ respectively, for all $j \in \{1, ..., N\}$.

for $\theta < \infty$ or

$$\|f\|_{B^{\vec{s}}_{p,\infty}(\mathbb{R}^N)} = \sup_{k\in\mathbb{N}_0}\left(2^{ks}\left\|F^{-1}\left(\varphi_k F\left(f\right)\right)\right\|_{L_p(\mathbb{R}^N)}\right) < \infty \tag{6}$$

for $\theta = \infty$, where $\{\varphi_k\}_{k=0}^{\infty} \in \Phi^{\vec{a}}\left(\mathbb{R}^N\right)$.

If $1 \le p, \theta \le \infty$, then (5) and (6) are norms, in general case they are quasinorms. For different collections $\{\varphi_k\}_{k=0}^{\infty} \in \Phi^{\vec{a}}\left(\mathbb{R}^N\right)$ quasinorms (5) (or (6)) are equivalent[2]. The norm $\|f\|_{B^{\vec{s}}_2(\mathbb{R}^N)} := \|f\|_{B^{\vec{s}}_{2,2}(\mathbb{R}^N)}$ is equivalent to

$$\|f\|^{(1)}_{B^{\vec{s}}_2(\mathbb{R}^N)} = \left\|\left(1+|\omega|^2_{\vec{a}}\right)^{\frac{s}{2}}\left(Ff\right)(\omega)\right\|_{L_2(\mathbb{R}^N)} .$$

If $\vec{s} > 0$ and $1 \le p$, $\theta \le \infty$, then for $\theta < \infty$ norm (5) is equivalent to

$$\|f\|^{(2)}_{B^{\vec{s}}_{p,\theta}(\mathbb{R}^N)} = \|f\|_{L_p(\mathbb{R}^N)} + \sum_{j=1}^{N}\left(\int_0^{\infty}\left(h^{-\gamma_j}\left\|\Delta^2_{h,j}\left(\frac{\partial^{r_j} f}{\partial x_j^{r_j}}\right)\right\|_{L_p(\mathbb{R}^N)}\right)^{\theta}\frac{dh}{h}\right)^{\frac{1}{\theta}}$$

and norm (6) is equivalent to

$$\|f\|^{(2)}_{B^{\vec{s}}_{p,\infty}(\mathbb{R}^N)} = \|f\|_{L_p(\mathbb{R}^N)} + \sum_{j=1}^{N}\sup_{h>0}\left(h^{-\gamma_j}\left\|\Delta^2_{h,j}\left(\frac{\partial^{r_j} f}{\partial x_j^{r_j}}\right)\right\|_{L_p(\mathbb{R}^N)}\right) .$$

Here r_j is the greatest integer which is less than s_j, $\gamma_j = s_j - r_j$; $\Delta^2_{h,j} = f\left(x+2he_j\right) - 2f\left(x+he_j\right) + f\left(x\right)$, where $h \in \mathbb{R}$ and e_j is the jth unit vector in $\mathbb{R}^N$, the one with 1 in the jth place and 0 in the other places; and $\dfrac{\partial^{r_j} f}{\partial x_j^{r_j}}$ are the weak derivatives.

When $s_1 = \cdots = s_N = s$ we have the isotropic case, and we write $B^s_{p,\theta}\left(\mathbb{R}^N\right)$ for $B^{(s,\ldots,s)}_{p,\theta}\left(\mathbb{R}^N\right)$.

2.3. Embedding theorems for the anisotropic Nikol'skiĭ-Besov spaces. Let $0 < p$, $\theta \le \infty$, $-\infty < \vec{s} < \infty$ and all the components of $\vec{s}$ have the same sign. Then the following embeddings are valid:

$$B^{\vec{s}}_{p,\theta}\left(\mathbb{R}^N\right) \hookrightarrow B^{\vec{s}}_{p,\theta_1}\left(\mathbb{R}^N\right) , \tag{7}$$

where $\theta < \theta_1 \le \infty$;

$$B^{\vec{s}+\vec{\varepsilon}}_{p,\theta_1}\left(\mathbb{R}^N\right) \hookrightarrow B^{\vec{s}}_{p,\theta}\left(\mathbb{R}^N\right) \hookrightarrow B^{\vec{s}-\vec{\varepsilon}}_{p,\theta_2}\left(\mathbb{R}^N\right) , \tag{8}$$

[2]Let X be a quasinormed space with the quasinorm $\|\cdot\|_1$, and let the quasinorm $\|\cdot\|_2$ be defined on a subset $Y \subset X$. One says that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent on Y (briefly $\|\cdot\|_1 \sim \|\cdot\|_2$ on Y) if there exist two constants $a, b > 0$ such that $a\|x\|_1 \le \|x\|_2 \le b\|x\|_1$ for all $x \in Y$. If $Y = X$, one says that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent briefly ($\|\cdot\|_1 \sim \|\cdot\|_2$).

where $\vec{\varepsilon} > 0$ is proportional to $\vec{s}$, $0 < \theta_1,\ \theta_2 \leq \infty$;

$$B^{\vec{s}}_{p,\theta}\left(\mathbb{R}^N\right) \hookrightarrow B^{\vec{\rho}}_{q,\theta}\left(\mathbb{R}^N\right) \ , \tag{9}$$

where $p < q \leq \infty$, $\vec{\rho} = \varkappa\vec{s}$, $\varkappa = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)\frac{N}{s}$. (If $\vec{s} = 0$, then $\vec{\rho} = -\left(\frac{1}{p} - \frac{1}{q}\right)\frac{N}{\vec{a}}$, where $\vec{a}$ is the anisotropy vector defining $B^{\vec{0}}_{p,\theta}\left(\mathbb{R}^N\right)$. If $\vec{s} \neq 0$ and $\varkappa = 0$, hence $\vec{\rho} = 0$, then the anisotropy vector defining $B^{\vec{0}}_{p,\theta}\left(\mathbb{R}^N\right)$ is equal to $\frac{s}{\vec{s}}$.)

Here continuous embedding $\hookrightarrow$ is defined as follows:

Definition 3. Let X, Y be two quasinormed spaces. If $X \subset Y$ and in addition there exists $c_3 > 0$ such that

$$\|x\|_Y \leq c_3 \|x\|_X \ ,$$

by all $x \in X$, where $\|\cdot\|_Y$, $\|\cdot\|_X$ are the quasinorms in Y, X respectively, then it is said that the continuous embedding

$$X \hookrightarrow Y$$

holds.

For more details about Nikol'skiĭ-Besov spaces see Nikol'skiĭ [13], Besov, Il'in and Nikol'skiĭ [1], Triebel [16] - [20], Schmeisser and Triebel [14], Burenkov [4].

3. Estimates for convolutions in the anisotropic Nikol'skiĭ-Besov spaces

3.1. Estimates for convolutions based on the embedding theorems.

Theorem 1. *Assume that* $-\infty < \vec{s}_1,\ \vec{s}_2 < \infty$, *for each* $i = 1, 2$ *all the components of* $\vec{s}_i$ *have the same sign,* $0 < p_1 \leq 2 \leq p_2 \leq \infty$, $0 < \theta_1 \leq 2 \leq \theta_2 \leq \infty$. *Moreover, let*

$$\varkappa_1 = 1 - \left(\frac{1}{p_1} - \frac{1}{2}\right)\frac{N}{s_1} \ , \ \ \varkappa_2 = 1 - \left(\frac{1}{p_2} - \frac{1}{2}\right)\frac{N}{s_2} \ , \tag{10}$$

where s_1 *and* s_2 *are mean smoothnesses, and* $\vec{\rho}_1 = \varkappa_1\vec{s}_1$, $\vec{\rho}_2 = \varkappa_2\vec{s}_2$, $\vec{a}_1 = \frac{s_1}{\vec{s}_1}$, $\vec{a}_2 = \frac{s_2}{\vec{s}_2}$ *if* $\vec{s}_1 \neq 0$, $\vec{s}_2 \neq 0$. *(If* $\vec{s}_i = 0$, *then* $\vec{a}_i$ *is an arbitrary vector with positive components* $a_{i,j}$ *satisfying* $a_{i,1} + \cdots + a_{i,N} = N$ *and* $\vec{\rho}_1 = -\left(\frac{1}{p_1} - \frac{1}{2}\right)\frac{N}{\vec{a}_i}$.)

If $f \in \tilde{S}'\left(\mathbb{R}^N\right)$, $g \in B^{\vec{s}_1}_{p_1,\theta_1}\left(\mathbb{R}^N\right) \cap \tilde{S}'\left(\mathbb{R}^N\right)$ *and*

$$\left(1 + |\omega|^2_{\vec{a}_2}\right)^{\frac{\rho_2}{2}} \left(1 + |\omega|^2_{\vec{a}_1}\right)^{-\frac{\rho_1}{2}} (Ff)(\omega) \in L_\infty\left(\mathbb{R}^N\right),$$

then the convolution $f * g$ exists and

$$\begin{aligned}&\|f*g\|_{B^{\vec{s}_2}_{p_2,\theta_2}(\mathbb{R}^N)}\\ &\le c_4\left\|\left(1+|\omega|^2_{\vec{a}_2}\right)^{\frac{\rho_2}{2}}\left(1+|\omega|^2_{\vec{a}_1}\right)^{-\frac{\rho_1}{2}}(Ff)(\omega)\right\|_{L_\infty(\mathbb{R}^N)}\|g\|_{B^{\vec{s}_1}_{p_1,\theta_1}(\mathbb{R}^N)}\,,\end{aligned}\tag{11}$$

where $c_4>0$ is independent of f and g.

Remark 1. If [3] $\vec{a}_1=\vec{a}_2=\vec{a}$ (11) takes the form

$$\|f*g\|_{B^{\vec{s}_2}_{p_2,\theta_2}(\mathbb{R}^N)}\le c_4\left\|\left(1+|\omega|^2_{\vec{a}}\right)^{\frac{1}{2}(s_2-\tau_1)}(Ff)(\omega)\right\|_{L_\infty(\mathbb{R}^N)}\|g\|_{B^{\vec{s}_1}_{p_1,\theta_1}(\mathbb{R}^N)}\,,$$

where $\tau_1=s_1-N\left(\dfrac{1}{p_1}-\dfrac{1}{p_2}\right)$. Note that τ_1 is the mean smoothness of the smoothness vector $\vec{\tau}_1$ in the embedding theorem

$$B^{\vec{s}_1}_{p_1,\theta_1}\left(\mathbb{R}^N\right)\hookrightarrow B^{\vec{\tau}_1}_{p_2,\theta_1}\left(\mathbb{R}^N\right)$$

In particular, in the isotropic case in which $\vec{s}_1=(s_1,...,s_1)$, $\vec{s}_2=(s_2,...,s_2)$ this inequality reduces to

$$\begin{aligned}&\|f*g\|_{B^{s_2}_{p_2,\theta_2}(\mathbb{R}^N)}\\ &\le c_4\left\|\left(1+|\omega|^2\right)^{\frac{1}{2}(s_2-\tau_1)}(F(f))(\omega)\right\|_{L_\infty(\mathbb{R}^N)}\|g\|_{B^{s_1}_{p_1,\theta_1}(\mathbb{R}^N)}\,.\end{aligned}\tag{12}$$

This case was considered in Burenkov & Pankratov [7].

Proof. Step 1: We note that

$$\begin{aligned}&\left\|\left(1+|\omega|^2_{\vec{a}_2}\right)^{\frac{\rho_2}{2}}(Ff)(\omega)(Fg)(\omega)\right\|_{L_2(\mathbb{R}^N)}\\ &\le\left\|\left(1+|\omega|^2_{\vec{a}_2}\right)^{\frac{\rho_2}{2}}\left(1+|\omega|^2_{\vec{a}_1}\right)^{-\frac{\rho_1}{2}}(Ff)(\omega)\right\|_{L_\infty(\mathbb{R}^N)}\\ &\quad\times\left\|\left(1+|\omega|^2_{\vec{a}_1}\right)^{\frac{\rho_1}{2}}(Fg)(\omega)\right\|_{L_2(\mathbb{R}^N)}\\ &=\left\|\left(1+|\omega|^2_{\vec{a}_2}\right)^{\frac{\rho_2}{2}}\left(1+|\omega|^2_{\vec{a}_1}\right)^{-\frac{\rho_1}{2}}(Ff)(\omega)\right\|_{L_\infty(\mathbb{R}^N)}\|g\|^{(1)}_{B^{\vec{\rho}_1}_2(\mathbb{R}^N)}\,.\end{aligned}$$

Since $0<p_1,\,\theta_1\le 2$,

$$B^{\vec{s}_1}_{p_1,\theta_1}\left(\mathbb{R}^N\right)\hookrightarrow B^{\vec{\rho}_1}_{2,\theta_1}\left(\mathbb{R}^N\right)\hookrightarrow B^{\vec{\rho}_1}_{2}\left(\mathbb{R}^N\right)\,.$$

[3] If $\vec{s}_1,\,\vec{s}_2\neq 0$, this means that $\vec{s}_1$ and $\vec{s}_2$ are proportional and $\vec{a}=\dfrac{s_1}{\vec{s}_1}=\dfrac{s_2}{\vec{s}_2}$.

(See (9) and (7).) Consequently

$$
\begin{aligned}
&\left\| \left(1+|\omega|^2_{\vec{a}_2}\right)^{\frac{\rho_2}{2}} (Ff)(\omega)(Fg)(\omega) \right\|_{L_2(\mathbb{R}^N)} \\
&\le c_5 \left\| \left(1+|\omega|^2_{\vec{a}_2}\right)^{\frac{\rho_2}{2}} \left(1+|\omega|^2_{\vec{a}_1}\right)^{-\frac{\rho_1}{2}} (Ff)(\omega) \right\|_{L_\infty(\mathbb{R}^N)} \|g\|_{B^{\vec{s}_1}_{p_1,\theta_1}(\mathbb{R}^N)} ,
\end{aligned} \tag{13}
$$

where $c_5 > 0$ is independent of f and g.

Step 2: Let us verify that the convolution $f * g$ exists (in the sense of (3)). First of all (13) implies that $Ff \cdot Fg \in L^{loc}\left(\mathbb{R}^N\right)$.
Next let $\varphi \in S\left(\mathbb{R}^N\right)$. By the Cauchy-Schwartz-Bunyakovskiĭ inequality

$$
\left| \int_{\mathbb{R}^N} (Ff)(\omega)(Fg)(\omega)\varphi(\omega)\,d\omega \right| \le c_6 \left\| \left(1+|\omega|^2_{\vec{a}_2}\right)^{-\frac{\rho_2}{2}} \varphi(\omega) \right\|_{L_2(\mathbb{R}^N)} < \infty,
$$

where

$$
c_6 = \left\| \left(1+|\omega|^2_{\vec{a}_2}\right)^{\frac{\rho_2}{2}} (Ff)(\omega)(Fg)(\omega) \right\|_{L_2(\mathbb{R}^N)} < \infty .
$$

Hence the functional

$$
(Ff \cdot Fg, \varphi) = \int_{\mathbb{R}^N} (Ff)(\omega)(Fg)(\omega)\varphi(\omega)\,d\omega
$$

is defined on $S\left(\mathbb{R}^N\right)$ and by the properties of $S\left(\mathbb{R}^N\right)$ and $S'\left(\mathbb{R}^N\right)$ is continuous on $S\left(\mathbb{R}^N\right)$, thus $Ff \cdot Fg \in S'\left(\mathbb{R}^N\right) \cap L^{loc}\left(\mathbb{R}^N\right) = \tilde{S}'\left(\mathbb{R}^N\right)$. By Definition 1 $f * g \in \tilde{S}'\left(\mathbb{R}^N\right)$ exists and $F(f*g) = Ff \cdot Fg$.

Step 3: First assume that $\vec{s}_2 \ne 0$, $\varkappa_2 \ne 0$. Since $2 \le p_2$, $\theta_2 \le \infty$,

$$
\begin{aligned}
\varkappa &= 1 - \left(\frac{1}{2} - \frac{1}{p_2}\right) \sum_{j=1}^{N} \frac{1}{\rho_{2,j}} = 1 - \left(\frac{1}{2} - \frac{1}{p_2}\right) \frac{N}{\rho_2} \\
&= 1 - \left(\frac{1}{2} - \frac{1}{p_2}\right) \frac{N}{\varkappa_2 s_2} = \frac{1}{\varkappa_2} \left(\varkappa_2 - \left(\frac{1}{2} - \frac{1}{p_2}\right) \frac{N}{s_2}\right) \\
&= \frac{1}{\varkappa_2} ,
\end{aligned}
$$

and $\varkappa \vec{\rho}_2 = \vec{s}_2$, by (7) and (9) we have

$$
B^{\vec{\rho}_2}_2\left(\mathbb{R}^N\right) \hookrightarrow B^{\vec{\rho}_2}_{2,\theta_2}\left(\mathbb{R}^N\right) \hookrightarrow B^{\vec{s}_2}_{p_2,\theta_2}\left(\mathbb{R}^N\right) . \tag{14}
$$

If $\vec{s}_2 = 0$, then $\vec{\rho}_2 = -\left(\frac{1}{p_2} - \frac{1}{2}\right) \frac{N}{\vec{a}_2}$, where $\vec{a}_2$ is an arbitrary vector with positive components $a_{2,j}$ satisfying $a_{2,1} + \cdots + a_{2,N} = N$, $\rho_2 = -\left(\frac{1}{p_2} - \frac{1}{2}\right) N$, $\varkappa = 1 - \left(\frac{1}{2} - \frac{1}{p_2}\right) \frac{N}{\rho_2} = 0$ and (14) follows by (7) and (9).

If $\vec{s}_2 \neq 0$ and $\varkappa_2 = 1 - \left(\frac{1}{p_2} - \frac{1}{2}\right)\frac{N}{s_2} = 0$, then $\vec{\rho}_2 = 0$, $\vec{a}_2 = \frac{s_2}{\vec{s}_2}$, $-\left(\frac{1}{2} - \frac{1}{p_2}\right)\frac{N}{\vec{a}_2} = \frac{s_2}{\vec{a}_2} = \vec{s}_2$ and again (14) follows by (7) and (9).

Therefore

$$\begin{aligned} \|f * g\|_{B^{\vec{s}_2}_{p_2,\theta_2}(\mathbb{R}^N)} &\leq c_7 \|f * g\|^{(1)}_{B^{\vec{\rho}_2}_2(\mathbb{R}^N)} \\ &= c_7 \left\| \left(1 + |\omega|^2_{\vec{a}_2}\right)^{\frac{\rho_2}{2}} (F(f * g))(\omega) \right\|_{L_2(\mathbb{R}^N)} \\ &= c_7 \left\| \left(1 + |\omega|^2_{\vec{a}_2}\right)^{\frac{\rho_2}{2}} (Ff)(\omega)(Fg)(\omega) \right\|_{L_2(\mathbb{R}^N)} , \end{aligned} \tag{15}$$

where $c_7 > 0$ is independent of f and g.

From (13) and (15) we derive inequality (11) with $c_4 = c_5 c_7$. Hence, the theorem is proved. □

3.2. Estimates for the bilinear convolution operator

Theorem 2. *Assume that* $-\infty < \vec{s}_1, \vec{s}_2, \vec{s}_3 < \infty$, *for each* $i = 1, 2, 3$ *all the components of* $\vec{s}_i$ *have the same sign,* $0 < p_1, p_2, p_3 \leq \infty$, $0 < \theta_1, \theta_2, \theta_3 \leq \infty$. *Moreover, let* $f \in B^{\vec{s}_1}_{p_1,\theta_1}(\mathbb{R}^N)$, $g \in B^{\vec{s}_2}_{p_2,\theta_2}(\mathbb{R}^N)$, $f, g \in \tilde{S}'(\mathbb{R}^N)$ *and the pointwise product* $F(f) \cdot F(g) \in \tilde{S}'(\mathbb{R}^N)$.

Then

$$\|f * g\|_{B^{\vec{s}_3}_{p_3,\theta_3}(\mathbb{R}^N)} \leq c_8 \|f\|_{B^{\vec{s}_1}_{p_1,\theta_1}(\mathbb{R}^N)} \|g\|_{B^{\vec{s}_2}_{p_2,\theta_2}(\mathbb{R}^N)} , \tag{16}$$

where $c_8 > 0$ *is independent of* f *and* g, *if the following conditions are satisfied*:

1) $p_3 \geq p_1$ *and* $p_3 \geq p_2$,

2) $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - 1 \geq 0$,

3) *the anisotropy vector* $\vec{a}$ *is the same*[4] *for the spaces* $B^{\vec{s}_1}_{p_1,\theta_1}(\mathbb{R}^N)$, $B^{\vec{s}_2}_{p_2,\theta_2}(\mathbb{R}^N)$, $B^{\vec{s}_3}_{p_3,\theta_3}(\mathbb{R}^N)$,

4) *either*

4a) $s_3 < s_1 + s_2 - N\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - 1\right)$

or

4b) $s_3 = s_1 + s_2 - N\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - 1\right)$ *and* $\frac{1}{\theta_3} \leq \frac{1}{\theta_1} + \frac{1}{\theta_2}$.

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[4] If $\vec{s}_1, \vec{s}_2, \vec{s}_3 \neq 0$, it means that $\vec{s}_1, \vec{s}_2, \vec{s}_3$ are proportional and $\frac{s_1}{\vec{s}_1} = \frac{s_2}{\vec{s}_2} = \frac{s_3}{\vec{s}_3} = \vec{a}$.

Remark 2. In view of condition 3) in the case 4a)

$$s_{3,j} < s_{1,j} + s_{2,j} - N\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - 1\right)$$

for all $j = 1, ..., N$, respectively in the case 4b)

$$s_{3,j} = s_{1,j} + s_{2,j} - N\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - 1\right)$$

for all $j = 1, ..., N$.

Remark 3. In the isotropic case in which $\vec{s}_1 = (s_1, ..., s_1)$, $\vec{s}_2 = (s_2, ..., s_2)$, $\vec{s}_3 = (s_3, ..., s_3)$, inequality (16) reduces to

$$\|f * g\|_{B^{s_3}_{p_3,\theta_3}(\mathbb{R}^N)} \leq c_8 \|f\|_{B^{s_1}_{p_1,\theta_1}(\mathbb{R}^N)} \|g\|_{B^{s_2}_{p_2,\theta_2}(\mathbb{R}^N)} \ . \tag{17}$$

This inequality with various assumptions on the parameters, f and g was proved by Golovkin & Solonnikov [11], Taibleson [15], Herz [12], Triebel [16], Burenkov [5], Batyrov & Burenkov [2], [3]. In this case condition 3) is automatically satisfied. In [2], [3] it was established that conditions 1), 2) and 4) are not only sufficient, but also necessary for the validity of (17).

Proof. Step 1: Let $\varphi \equiv \{\varphi_k\}_{k=0}^{\infty} \in \Phi^{\vec{a}}\left(\mathbb{R}^N\right)$. Recall that quasinorms (5), (6) are equivalent for different systems $\varphi \in \Phi^{\vec{a}}\left(\mathbb{R}^N\right)$. Moreover, the equivalence holds (Triebel [16]) if we replace the condition $\sum\limits_{k=0}^{\infty} \varphi_k(t) = 1$, $t \in \mathbb{R}^N$, by: For some c_9, $c_{10} > 0$

$$c_9 \leq \sum_{k=0}^{\infty} \varphi_k(t) \leq c_{10},\ t \in \mathbb{R}^N \ ,$$

where c_9, $c_{10} > 0$ are independent of t. This implies that the equivalence also holds if we replace the system φ by the system $\varphi^2 = \left\{\varphi_k^2(t)\right\}_{k=0}^{\infty}$, since

$$\frac{1}{2} = \frac{1}{2}\left(\sum_{k=0}^{\infty} \varphi_k(t)\right)^2 \leq \sum_{k=0}^{\infty} \varphi_k^2(t) \leq \sum_{k=0}^{\infty} \varphi_k(t) = 1 \ ,$$

because the multiplicity of the covering $\{\operatorname{supp} \varphi_k\}_{k=0}^{\infty}$ is equal to 2, and the appropriate estimates for $D^{\alpha}\left(\varphi_k^2\right)$ are satisfied.

Since

$$F^{-1}\varphi_k^2 F(f * g) = F^{-1}\left((\varphi_k F f) \cdot (\varphi_k F f)\right) = F^{-1}\varphi_k F f * F^{-1}\varphi_k F g \ ,$$

we have

$$\begin{aligned} \|f * g\|_{B^{\vec{s}_3}_{p_3,\theta_3}(\mathbb{R}^N)} &\equiv \|f * g\|^{\varphi}_{B^{\vec{s}_3}_{p_3,\theta_3}(\mathbb{R}^N)} \\ &\leq c_{11} \|f * g\|^{\varphi^2}_{B^{\vec{s}_3}_{p_3,\theta_3}(\mathbb{R}^N)} \\ &= c_{11} \left\| 2^{ks_3} \left\| F^{-1}\varphi_k F f * F^{-1}\varphi_k F g \right\|_{L_{p_3}(\mathbb{R}^N)} \right\|_{l_{\theta_3}} \ , \end{aligned} \tag{18}$$

where $c_{11} > 0$ is independent of f and g.

Step 2: We claim that conditions 1) and 2) imply that

$$\begin{aligned} &\left\| F^{-1}\varphi_k Ff * F^{-1}\varphi_k Fg \right\|_{L_{p_3}(\mathbb{R}^N)} \\ &\le c_{12} 2^{kN\left(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}-1\right)} \left\| F^{-1}\varphi_k Ff \right\|_{L_{p_1}(\mathbb{R}^N)} \left\| F^{-1}\varphi_k Fg \right\|_{L_{p_2}(\mathbb{R}^N)} , \end{aligned} \tag{19}$$

where $c_{12} > 0$ is independent of f, g and k.

We shall apply the following inequalities:

If $0 < p < 1$, $g_1, g_2 \in S\left(\mathbb{R}^N\right)$, and $\operatorname{supp} Fg_1$, $\operatorname{supp} Fg_2 \subset B_r$, then

$$\|g_1 * g_2\|_{L_p(\mathbb{R}^N)} \le c_{13} r^{N\left(\frac{1}{p}-1\right)} \|g_1\|_{L_p(\mathbb{R}^N)} \|g_2\|_{L_p(\mathbb{R}^N)} , \tag{20}$$

where $c_{13} > 0$ is independent of g_1, g_2 and r. For $r = 1$ see Triebel [16]. For arbitrary $r > 0$ (20) follows by the scaling argument since $a_1 + \cdots + a_N = N$.

If $0 < p \le q \le \infty$, $g \in S\left(\mathbb{R}^N\right)$ and $\operatorname{supp} Fg \subset B_r$, then

$$\|g\|_{L_q(\mathbb{R}^N)} \le c_{14} r^{N\left(\frac{1}{p}-\frac{1}{q}\right)} \|g\|_{L_p(\mathbb{R}^N)} , \tag{21}$$

where $c_{14} > 0$ is independent of g and r. See Nikol'skiĭ [13], Triebel [16] for $r = 1$. Again, for arbitrary $r > 0$ (21) follows by the scaling argument.

2a: Let p_1, p_2, $p_3 \ge 1$. If condition 2) holds with the equality sign, then (19) with $c_{12} = 1$ follows by Young's inequality.

If condition 2) holds with the inequality sign, consider $\tilde{p}_3$ defined by

$$\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{\tilde{p}_3} - 1 = 0 .$$

Since p_1, $p_2 \ge 1$, we have $\tilde{p}_3 \ge p_1$, p_2. Also $\tilde{p}_3 < p_3$. Taking into account that

$$\operatorname{supp} F\left(F^{-1}\varphi_k Ff * F^{-1}\varphi_k Fg\right) \subset \operatorname{supp} \varphi_k \subset B_{2^{k+1}} , \tag{22}$$

by inequality (21) we get

$$\begin{aligned} &\left\| F^{-1}\varphi_k Ff * F^{-1}\varphi_k Fg \right\|_{L_{p_3}(\mathbb{R}^N)} \\ &\qquad \le c_{15} 2^{kN\left(\frac{1}{\tilde{p}_3}-\frac{1}{p_3}\right)} \left\| F^{-1}\varphi_k Ff * F^{-1}\varphi_k Fg \right\|_{L_{\tilde{p}_3}(\mathbb{R}^N)} , \end{aligned}$$

where $c_{15} > 0$ is independent of f, g and k.

Hence, by Young's inequality we obtain (19).

2b: Let $0 < p_3 < 1$. Taking into account (22) and applying (20) and (21), we get

$$\begin{aligned} &\left\| F^{-1}\varphi_k Ff * F^{-1}\varphi_k Fg \right\|_{L_{p_3}(\mathbb{R}^N)} \\ &\quad \le c_{16} 2^{kN\left(\frac{1}{p_3}-1\right)} \left\| F^{-1}\varphi_k Ff \right\|_{L_{p_3}(\mathbb{R}^N)} \left\| F^{-1}\varphi_k Fg \right\|_{L_{p_3}(\mathbb{R}^N)} \\ &\quad \le c_{17} 2^{kN\left(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}-1\right)} \left\| F^{-1}\varphi_k Ff \right\|_{L_{p_1}(\mathbb{R}^N)} \left\| F^{-1}\varphi_k Fg \right\|_{L_{p_2}(\mathbb{R}^N)} , \end{aligned}$$

where c_{16}, $c_{17} > 0$ are independent of f, g and k.

2c: Let $p_3 \geq 1$ and one of the numbers p_1 or p_2 is less than 1 and the other is greater than or equal to 1, say $0 < p_1 < 1 \leq p_2$. Consider $\tilde{p}_1$ defined by

$$\frac{1}{\tilde{p}_1} + \frac{1}{p_2} - \frac{1}{p_3} - 1 = 0 \ .$$

Since $1 \leq p_2 \leq p_3$, we have $1 \leq \tilde{p}_1 \leq p_3$. Then by Young's inequality and inequality (21) we obtain

$$\begin{aligned}\left\|F^{-1}\varphi_k Ff * F^{-1}\varphi_k Fg\right\|_{L_{p_3}(\mathbb{R}^N)} &\leq \left\|F^{-1}\varphi_k Ff\right\|_{L_{\tilde{p}_1}(\mathbb{R}^N)} \left\|F^{-1}\varphi_k Fg\right\|_{L_{p_2}(\mathbb{R}^N)} \\ &\leq c_{18} 2^{kN\left(\frac{1}{p_1}-\frac{1}{\tilde{p}_1}\right)} \left\|F^{-1}\varphi_k Ff\right\|_{L_{p_1}(\mathbb{R}^N)} \left\|F^{-1}\varphi_k Fg\right\|_{L_{p_2}(\mathbb{R}^N)} \ ,\end{aligned}$$

where $c_{18} > 0$ is independent of f, g, and k. Hence (19) follows.

2d: Let $p_3 \geq 1$ and both numbers p_1 and p_2 are less than 1. Choose $\tilde{p}_1$ and $\tilde{p}_2$ such that $1 \leq \tilde{p}_1, \tilde{p}_2 \leq p_3$ and

$$\frac{1}{\tilde{p}_1} + \frac{1}{\tilde{p}_2} - \frac{1}{p_3} - 1 = 0 \ ,$$

in particular $\tilde{p}_1 = 1$, $\tilde{p}_2 = p_3$. By Young's inequality and inequality (21) we get

$$\begin{aligned}\left\|F^{-1}\varphi_k Ff * F^{-1}\varphi_k Fg\right\|_{L_{p_3}(\mathbb{R}^N)} &\leq \left\|F^{-1}\varphi_k Ff\right\|_{L_{\tilde{p}_1}(\mathbb{R}^N)} \left\|F^{-1}\varphi_k Fg\right\|_{L_{\tilde{p}_2}(\mathbb{R}^N)} \\ &\leq c_{19} 2^{kN\left(\frac{1}{p_1}-\frac{1}{\tilde{p}_1}+\frac{1}{p_2}-\frac{1}{\tilde{p}_2}\right)} \left\|F^{-1}\varphi_k Ff\right\|_{L_{p_1}(\mathbb{R}^N)} \left\|F^{-1}\varphi_k Fg\right\|_{L_{p_2}(\mathbb{R}^N)} \ ,\end{aligned}$$

where $c_{19} > 0$ is independent of f, g and k. Therefore (19) follows.

Step 3: Let condition 4b) be satisfied and choose $\tilde{\theta}_3 \leq \theta_3$ such that $\dfrac{1}{\tilde{\theta}_3} = \dfrac{1}{\theta_1} + \dfrac{1}{\theta_2}$. Then by (18), (19), Jensen's and Hölder's inequalities we obtain (16):

$$\begin{aligned}&\|f * g\|_{B^{\vec{s}_3}_{p_3,\theta_3}(\mathbb{R}^N)} \\ &\leq c_8 \left\| 2^{k\left(s_3+N\left(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}-1\right)\right)} \left\|F^{-1}\varphi_k Ff\right\|_{L_{p_1}(\mathbb{R}^N)} \left\|F^{-1}\varphi_k Fg\right\|_{L_{p_2}(\mathbb{R}^N)} \right\|_{l_{\theta_3}} \\ &\leq c_8 \left\| 2^{ks_1} \left\|F^{-1}\varphi_k Ff\right\|_{L_{p_1}(\mathbb{R}^N)} \cdot 2^{ks_2} \left\|F^{-1}\varphi_k Fg\right\|_{L_{p_2}(\mathbb{R}^N)} \right\|_{l_{\tilde{\theta}_3}} \\ &\qquad \leq c_8 \left\| 2^{ks_1} \left\|F^{-1}\varphi_k Ff\right\|_{L_{p_1}(\mathbb{R}^N)} \right\|_{l_{\theta_1}} \left\| 2^{ks_2} \left\|F^{-1}\varphi_k Fg\right\|_{L_{p_2}(\mathbb{R}^N)} \right\|_{l_{\theta_2}} \\ &\qquad\qquad = c_8 \|f\|_{B^{\vec{s}_1}_{p_1,\theta_1}(\mathbb{R}^N)} \|g\|_{B^{\vec{s}_2}_{p_2,\theta_2}(\mathbb{R}^N)} \ ,\end{aligned} \tag{23}$$

where $c_8 = c_{11} \cdot c_{12}$.

Step 4: Let condition 4a) be satisfied. Then

$$\varepsilon = s_1 + s_2 - s_3 - N\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - 1\right) > 0 \ .$$

Consider the space $B^{\vec{s}_4}_{p_3,\theta_4}\left(\mathbb{R}^N\right)$, with the same vector $\vec{a}$ (see *Step* 1), where $\vec{s}_4 = \vec{s}_3 + \dfrac{\varepsilon}{\vec{a}} = \dfrac{s_3+\varepsilon}{\vec{a}}$ and θ_4 is defined by $\dfrac{1}{\theta_4} = \dfrac{1}{\theta_1} + \dfrac{1}{\theta_2}$. Then $s_4 = s_3 + \varepsilon$ and

$$s_4 = s_1 + s_2 - N\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - 1\right) \ .$$

Hence by embedding (8) and *Step* 3

$$\|f * g\|_{B^{\vec{s}_3}_{p_3,\theta_3}(\mathbb{R}^N)} \le c_{20} \|f * g\|_{B^{\vec{s}_4}_{p_3,\theta_4}(\mathbb{R}^N)} \le c_{21} \|f\|_{B^{\vec{s}_1}_{p_1,\theta_1}(\mathbb{R}^N)} \|g\|_{B^{\vec{s}_2}_{p_2,\theta_2}(\mathbb{R}^N)} \ ,$$

where c_{20}, $c_{21} > 0$ are independent of f and g.

Thus we obtained (16) and the theorem is proved. □

3.3. Equivalent quasinorms on a cone of functions with a regular Fourier transform. In this section we formulate the result proved in Burenkov & García Almeida [9], which will be used in the next section.

Theorem 3. *Assume that $0 < p$, $\theta \le \infty$; $-\infty < \vec{s} < \infty$, and all the components of $\vec{s}$ have the same sign. Let $\vec{a} = \dfrac{s}{\vec{s}}$ if $\vec{s} \ne 0$. If $\vec{s} = 0$, let $\vec{a}$ be an arbitrary vector with positive components, satisfying $\sum\limits_{j=1}^{N} a_j = N$. Moreover, for $\lambda > 1$, let $\Lambda(\lambda, \vec{a}, p)$ denote the cone of all functions $f \in \tilde{S}'\left(\mathbb{R}^N\right)$ satisfying the following conditions:*

1) $(Ff)(\xi) > 0$ for all $\xi \in \mathbb{R}^N$,

2) for ξ, $\eta \in \mathbb{R}^N$

$$\frac{1}{2} \le \frac{|\xi|_{\vec{a}}}{|\eta|_{\vec{a}}} \le 2 \Rightarrow \frac{1}{\lambda} \le \frac{(Ff)(\xi)}{(Ff)(\eta)} \le \lambda \ , \tag{24}$$

3) *for all $\alpha \in \mathbb{N}_0^N$ satisfying*

$$|\alpha| \le \frac{1}{p} + 1 \tag{25}$$

the derivatives $D^\alpha Ff$ are continuous on $\mathbb{R}^N$ and

$$|(D^\alpha Ff)(\xi)| \le \lambda \left(1 + |\xi|^2_{\vec{a}}\right)^{-\frac{\langle \vec{a}, \alpha \rangle}{2}} (Ff)(\xi) \ . \tag{26}$$

Then for all $\lambda > 1$

$$\|f\|_{B^{\vec{s}}_{p,\theta}(\mathbb{R}^N)} \sim \left\| \left(1 + |\xi|^2_{\vec{a}}\right)^{\frac{1}{2}\left(s - \frac{N}{p} + \frac{N}{\theta'}\right)} (Ff)(\xi) \right\|_{L_\theta(\mathbb{R}^N)} \tag{27}$$

on $\Lambda(\lambda, \vec{a}, p)$, where $\dfrac{1}{\theta} + \dfrac{1}{\theta'} = 1$.

In the isotropic case $\vec{s} = (s, ..., s)$, hence $\vec{a} = \vec{1} \equiv (1, ..., 1)$, and (27) reduces to the equivalence

$$\|f\|_{B^s_{p,\theta}(\mathbb{R}^N)} \sim \left\| \left(1 + |\xi|^2\right)^{\frac{1}{2}\left(s - \frac{N}{p} + \frac{N}{\theta'}\right)} (Ff)(\xi) \right\|_{L_\theta(\mathbb{R}^N)} \tag{28}$$

on $\Lambda\left(\lambda, \vec{1}, p\right)$, which has been established in Batyrov & Burenkov [2], [3].

3.3. Estimates for convolutions based on Sections 3.2 and 3.3

Theorem 4. *Assume that $-\infty < \vec{s}_1, \vec{s}_2 < \infty$, for each $i = 1, 2$ all the components of $\vec{s}_i$ have the same sign, $0 < p_1 \le p_2 \le \infty$, $0 < \theta_1 \le \theta_2 \le \infty$. Moreover, assume that the anisotropy vector $\vec{a}$ is the same*[5] *for the spaces $B^{\vec{s}_1}_{p_1,\theta_1}\left(\mathbb{R}^N\right)$ and $B^{\vec{s}_2}_{p_2,\theta_2}\left(\mathbb{R}^N\right)$.*

Let $g \in B^{\vec{s}_1}_{p_1,\theta_1}\left(\mathbb{R}^N\right)$, $f, g \in \tilde{S}'\left(\mathbb{R}^N\right)$ and the pointwise product $F(f) \cdot F(g) \in \tilde{S}'\left(\mathbb{R}^N\right)$. Moreover, let $f \in \Lambda(\lambda, \vec{a}, p)$, where $\lambda > 1$ and p is defined by $\dfrac{1}{p} = 1 - \dfrac{1}{p_1} + \dfrac{1}{p_2}$, and $\left(1 + |\omega|^2_{\vec{a}}\right)^{\frac{1}{2}(s_2 - \tau_1)} (Ff)(\omega) \in L_\infty\left(\mathbb{R}^N\right)$, where $\tau_1 = s_1 - N\left(\dfrac{1}{p_1} - \dfrac{1}{p_2}\right)$. Then

$$\|f * g\|_{B^{\vec{s}_2}_{p_2,\theta_2}(\mathbb{R}^N)} \le c_{22} \left\| \left(1 + |\omega|^2_{\vec{a}}\right)^{\frac{1}{2}(s_2 - \tau_1)} (Ff)(\omega) \right\|_{L_\infty(\mathbb{R}^N)} \|g\|_{B^{\vec{s}_1}_{p_1,\theta_1}(\mathbb{R}^N)} , \tag{29}$$

where $c_{22} > 0$ is independent of f and g.

Remark 4. This is the same inequality as in Remark 1, but without assumptions $p_1 \le 2 \le p_2$, $\theta_1 \le 2 \le \theta_2$. Instead it is assumed that $\vec{s}_1$ and $\vec{s}_2$ are proportional and there is an additional assumption on f: $f \in \Lambda(\lambda, \vec{a}, p)$.

Proof. We apply Theorem 2 in which the parameters are chosen in the following way: $\vec{s}_3$, p_3, θ_3 are replaced by $\vec{s}_2$, p_2, θ_2 of Theorem 4, $\vec{s}_2$, p_2, θ_2 are replaced by $\vec{s}_1$, p_1, θ_1 of Theorem 4 and, finally, $\vec{s}_1$, p_1, θ_1 are replaced by $\vec{s}_2 - \vec{s}_1$, $p = \left(1 - \dfrac{1}{p_1} + \dfrac{1}{p_2}\right)^{-1}$, ∞, where $\vec{s}_1$, $\vec{s}_2$, p_1, p_2 are from Theorem 4. Note that the conditions 1), 2) and 4b) of Theorem 2 are satisfied. Hence

$$\|f * g\|_{B^{\vec{s}_2}_{p_2,\theta_2}(\mathbb{R}^N)} \le c_{13} \|f\|_{B^{\vec{s}_2 - \vec{s}_1}_{p,\theta}(\mathbb{R}^N)} \|g\|_{B^{\vec{s}_1}_{p_1,\theta_1}(\mathbb{R}^N)} , \tag{30}$$

where $c_{13} > 0$ is independent of f and g.

All components of $\vec{s}_2 - \vec{s}_1$ have the same sign, the mean smoothness of this vector is equal to $s_2 - s_1$. If $\vec{s}_2 - \vec{s}_1 \ne 0$, then $\dfrac{s_2 - s_1}{\vec{s}_2 - \vec{s}_1} = \dfrac{s_1}{\vec{s}_1} = \dfrac{s_2}{\vec{s}_2} = \vec{a}$.

[5] If $\vec{s}_1$, $\vec{s}_2 \ne 0$, this means that $\vec{s}_1$ and $\vec{s}_2$ are proportional and $\dfrac{s_1}{\vec{s}_1} = \dfrac{s_2}{\vec{s}_2} = \vec{a}$.

If $\vec{s}_2 - \vec{s}_1 = \vec{0}$, we assume that the anisotropy vector for $B^{\vec{0}}_{p,\theta}\left(\mathbb{R}^N\right)$ is again $\vec{a}$. Moreover, $s_2 - s_1 - \dfrac{N}{p} + N = s_2 - \tau_1$. Therefore by Theorem 3 we have

$$\|f\|_{B^{\vec{s}_2-\vec{s}_1}_{p,\theta}(\mathbb{R}^N)} \leq c_{14} \left\| \left(1 + |\omega|^2_{\vec{a}}\right)^{\frac{1}{2}(s_2-\tau_1)} (Ff)(\omega) \right\|_{L_\infty(\mathbb{R}^N)},$$

where $c_{14} > 0$ is independent of f and (29) follows. Thus, theorem 4 is proved. □

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