Note on the paper “Regulated domains and Bergman type projections”*

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Abstract. We show that the sufficient condition of the above mentioned paper is also necessary for the boundedness of Bergman type projections on a class of regulated domains.

1. Necessary condition

We continue our analysis (see [3]) of Bergman type projections P from $L^p_\omega(\Omega)$ onto the corresponding Bergman spaces, where Ω is a regulated simply connected complex domain.

In [3] a sufficient, geometric condition (concerning Ω) was given for the boundedness of P . We show below that for a large class of regulated domains, the condition is also necessary. Theorem 1 of our paper actually states that the geometric condition is necessary for a Muckenhoupt-type condition, which again was shown by Békollé to be equivalent to the boundedness of P .

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The reader is urged to consult [3] for all definitions and notations. We only shortly recall some facts concerning regulated domains here; this is necessary in order to introduce a condition for Ω .

So $\Omega \subset \mathbb{C}$ is a simply connected, bounded domain which is regulated. The Riemann map $\psi : \mathbb{D} \rightarrow \Omega$ has a continuous extension to $\overline{\mathbb{D}}$ (still denoted by ψ). We define the curve $w(t) = \psi(e^{it})$, $0 \leq t \leq 2\pi$. Each point of $\partial\Omega$ is attained only finitely often by ψ , and $\beta(t) := \lim_{\tau \rightarrow t^+} \arg(w(\tau) - w(t))$ can be approximated uniformly by step functions, i.e. for every $\varepsilon > 0$ there exist $0 = t_0 < t_1 < \dots < t_n = 2\pi$ and constants $\gamma_1, \dots, \gamma_n$ such that

$$(1.1) \quad |\beta(t) - \gamma_j| < \varepsilon \quad \text{for } t_{j-1} < t < t_j, \quad j = 1, \dots, n.$$

The function β is the direction angle of the forward tangent of $\partial\Omega$ at $w(t)$. We define

$$\delta_1 := \sup_t \lim_{\tau \rightarrow 0^+} (\beta(t + \tau) - \beta(t - \tau)) \geq 0$$

and

$$(1.2) \quad \delta_2 := \inf_t \lim_{\tau \rightarrow 0^+} (\beta(t + \tau) - \beta(t - \tau)) \leq 0;$$

the limits exist by (1.1). It also follows from (1.1) that these “jumps” are actually attained at some boundary points, i.e., the suprema are maxima and so on. Moreover, there can be only finitely many such points, since there are only finitely many steps in (1.1). Let us denote these points by $\tau_1, \dots, \tau_N$ (corresponding to δ_1), and $\rho_1, \dots, \rho_M$ (corresponding to δ_2).

By (1.1), there exist finitely many points $0 = t_0 < t_1 < \dots < t_n = 2\pi$ such that

$$(1.3) \quad |\beta(t) - t - \gamma_j| < \frac{1}{100} \quad \text{for } t_{j-1} < t < t_j,$$

for some real constants γ_j , $j = 1, \dots, n$. We denote by β_1 and β_2 the 2π -periodic extensions to $\mathbb{R}$ of the functions $\beta_1 := \sum_{j=0}^n \gamma_j \chi_j$ and $\beta_2 = \beta - t - \beta_1$, where $\chi_j(t) = 1$ for $t_{j-1} < t < t_j$ and zero elsewhere. Clearly, the modulus of β_2 is bounded by $1/10$.

We make the following assumption (in addition to [3]): There exist a $\tau \in \{\tau_1, \dots, \tau_N\}$ and a $\rho \in \{\rho_1, \dots, \rho_M\}$ such that

$$(1.4) \quad \begin{aligned} &\beta_2 - t \text{ is Dini - continuous in a neighbourhood of } \tau, \\ &\text{and } \lim_{t \rightarrow \tau} \beta_2(t) = 0 \end{aligned}$$

and the same for ρ replacing τ . See [2], Section III, 1 for the definition of Dini-continuity. A condition related to (1.4) is given in Lemma 2 below. The condition has the consequences

$$(1.5) \quad \delta_1 = \gamma_{j+1} - \gamma_j \quad \text{and} \quad \delta_2 = \gamma_{l+1} - \gamma_l$$

where j and l are such that $\tau = t_j$ and $\rho = t_l$.

Theorem 1. *Let Ω be a regulated domain satisfying the condition (1.4). Let ψ be a Riemann conformal map from $\mathbb{D}$ onto Ω . Let $a > 0$, $1 < p < \infty$, $1/p + 1/p^* = 1$, and let the numbers α , σ and γ (all > -1) satisfy $\alpha = \sigma/p + \gamma/p^*$. If the Muckenhoupt–Békollé–Bonami condition*

$$(1.6) \quad \sup_S \int_S |\psi'|^a dm_\sigma \left(\int_S |\psi'|^{-ap^*/p} dm_\gamma \right)^{p/p^*} m_\alpha(S)^{-p} \leq C$$

is satisfied, then

$$(1.7) \quad 2 + \sigma > \frac{a\delta_1}{\pi} \quad \text{and} \quad \frac{a\delta_2}{\pi} > -\frac{p}{p^*}(\gamma + 2).$$

Remarks. 1) The supremum in (1.6) is taken over Carleson squares $S := S(\theta, \rho) := \{z = re^{it} | 1 - \rho < r < 1, |\theta - t| < 2\pi\rho\}$. The theorem still holds, if $a < 0$. See [3] for details. (There is an obvious “misprint” in (3.2) of [3], compare to (1.6) above.)

2) Combining this result with Theorem 3.1 of [3] we get a necessary and sufficient condition, i.e. (1.6) and (1.7) are equivalent. Moreover, for regulated domains satisfying (1.4), Theorems 4.1 and 4.5 and Proposition 4.3 of that paper become “if and only if” results (characterizing the boundedness of various Bergman type projections by geometry of Ω).

2. Proof of the main result

In view of Theorem 3.1 of [3] we need to prove the following: if equality holds in one of the inequalities (1.7), then (1.6) fails, i.e the “weight function” $|\psi'|^a$ does not satisfy the Muckenhoupt type condition. This function is of a special form; it can be written as a product $\nu_1\nu_2$, where ν_1 is around the critical point, say, $e^{i\tau}$, essentially a rational function, and ν_2 is a perturbation. We will have to consider $\int_S |\psi'|^a dm_\sigma = \int_S \nu_1\nu_2 dm_\sigma$ (together with another factor). Consider for a moment just the integral

$$(2.1) \quad \int_S \nu_1 dm_\sigma.$$

The arguments of [3] (originally due to Solov'ev and Békollé in special cases) imply that (2.1) behaves in the same way as $\int_0^1 \frac{1}{x^b} dx$ with $b := 3 + \sigma - a\delta_1/\pi$. Hence, if $2 + \sigma > a\delta_1/\pi$, then (2.1) converges, and if $2 + \sigma < a\delta_1/\pi$, it diverges. Moreover, the effect of ν_2 is so weak that it does not change this straightforward model: $\int_S \nu_1 \nu_2 dm_\sigma$ behaves in the same way. All of that is contained in [3]. In this paper we are considering the case $2 + \sigma = a\delta_1/\pi$, i.e. (2.1) diverges, but only logarithmically. The main difficulty now is to show that ν_2 is not small enough to make $\int_S \nu_1 \nu_2 dm_\sigma$ converge. (This is an oversimplified explanation, since there is also the other factor in (1.6).)

On the technical level, the main strategy of the proof can be found in part **II**. Also the role of our assumption (1.4) is seen there.

Proof of Theorem 1. I. Some preliminaries. Let us for example assume $2 + \sigma = a\delta_1/\pi$; we want to show that (1.6) fails. By (2.3) of [3], the function $|\psi'|^a$ has the representation

$$(2.2) \quad |\psi'|^a = C \exp\left(-\frac{a}{2\pi} \operatorname{Re} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \tilde{\beta}(t) dt\right),$$

where $\tilde{\beta}$ is the harmonic conjugate of $\beta - t$. Let β_1 and β_2 be defined as above. Let us define for $j = 1, 2$

$$(2.3) \quad \begin{aligned} \nu_j(z) &:= \exp\left(-\frac{a}{2\pi} \operatorname{Re} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \tilde{\beta}_j(t) dt\right) \\ &= \exp\left(-\frac{a}{2\pi} \operatorname{Im} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \beta_j(t) dt\right); \end{aligned}$$

we thus have $|\psi'|^a = e^C \nu_1 \nu_2$. In [3], proof of Theorem 3.1, (3.23), it is shown that ν_1 is of the form

$$(2.4) \quad \nu_1(z) := \nu_1(re^{i\theta}) = \hat{\nu}_1(z) \prod_{\iota=0}^{n-1} \left((1-r)^2 + r(\theta - t_\iota)^2 \right)^{-a(\gamma_\iota + 1 - \gamma_\iota)/(2\pi)},$$

where $\hat{\nu}_1$ is a bounded function which is also bounded away from 0. We do not reprove this result here. Using (2.4) and (1.5) we can find a fixed $0 < h_0 < 1/2$ such that

$$(2.5) \quad |\nu_1(z)| \cong \left(|1-r| + |\theta - \tau| \right)^{-a\delta_1/\pi}$$

for $z \in S(\tau, h_0)$.

II. The main reduction of the proof. Let the number h , $0 < h < h_0 < 1/2$, be arbitrarily small and consider the set $S := S(\tau, h)$. Suppose by antithesis

that (1.6) holds. Then $m_\alpha(S)^p \cong h^{p(\alpha+2)}$ and we obtain from (2.5)

$$\begin{aligned}
 Ch^{p(\alpha+2)} &\geq \int_S \nu_1 \nu_2 dm_\sigma \left(\int_S (\nu_1 \nu_2)^{p^*/p} dm_\gamma \right)^{p/p^*} \\
 &\geq C \int_{1-h}^1 r dr \int_{\tau-h}^{\tau+h} \left((1-r) + |\tau-\theta| \right)^{-a\delta_1/\pi} (1-r)^\sigma \nu_2 d\theta \\
 (2.6) \quad &\cdot \left(\int_{1-h}^1 r dr \int_{\tau-h}^{\tau+h} \left((1-r) + |\tau-\theta| \right)^{\frac{-a\delta_1}{\pi} - \frac{p^*}{p}} (1-r)^\gamma \nu_2^{\frac{p^*}{p}} d\theta \right)^{p/p^*}
 \end{aligned}$$

Making the linear change of variables $(1-r)e^{i(\tau-\theta)} \rightarrow re^{i\theta}$ we find that (2.6) is larger than a constant times

$$\begin{aligned}
 &\int_0^h dr \int_{-h}^h \left(r + |\theta| \right)^{-a\delta_1/\pi} r^\sigma V(r, \theta) d\theta \\
 &\cdot \left(\int_0^h dr \int_{-h}^h \left(r + |\theta| \right)^{\frac{-a\delta_1}{\pi} - \frac{p^*}{p}} r^\gamma V(r, \theta)^{\frac{p^*}{p}} d\theta \right)^{p/p^*} \\
 (2.7) \quad &=: \mathbf{I}_1(h) \cdot (\mathbf{I}_2(h))^{p/p^*},
 \end{aligned}$$

where $V(r, \theta) := \nu_2((1-r)e^{i(\tau+\theta)})$.

We will soon show (with some abuse of notation) that (2.6)–(2.7) implies

$$(2.8) \quad \nu_2(1-h_n, \tau) := \nu_2((1-h_n)e^{i\tau}) \leq Ch_n$$

for a strictly positive sequence (h_n) tending to 0. We claim that (2.8) means an impossibly fast convergence to 0 for $\nu_2(1-h_n, \tau)$. We get by (2.4) and [2], Lemma III.1.2,

$$\left| \log(\nu_2(1-h_n, \tau)) \right| \cong \left| \int_{|\theta|>h_n} \frac{\beta_2(\tau-\theta)}{\theta} d\theta \right|$$

Now (2.8) would mean (uniformly for $n \in \mathbb{N}$)

$$(2.9) \quad \infty > C \geq \frac{|\log h_n|}{|\log \nu_2(1-h_n, \tau)|} \geq C' \frac{\int_{|\theta|>h_n} \theta^{-1} d\theta}{\int_{|\theta|>h_n} \beta_2(\tau-\theta) \theta^{-1} d\theta},$$

a contradiction, since $\lim_{\theta \rightarrow 0} \beta_2(\tau-\theta) = 0$ by (1.4)–(1.5). As a conclusion, (1.6) fails.

We still need to prove that (2.6)–(2.7) implies (2.8).

III. Estimate for $\mathbf{I}_1$. We first show that for some $C > 0$

$$(2.10) \quad C\mathbf{I}_1 \geq \int_0^h r^\sigma \left[(r + \theta)^{-1-\sigma} V(r, \theta) \right]_{\theta=0}^r dr =: \mathbf{B}_1.$$

Notice that we shall use in the course of the proof the fact that V is non-negative. Since $2 + \sigma = a\delta_1/\pi$ by assumption, we have

$$(2.11) \quad \mathbf{I}_1 \geq \int_0^h r^\sigma \int_0^r (r + \theta)^{-2-\sigma} V(r, \theta) d\theta dr =: \mathbf{A}.$$

Moreover, in our integration domain (recall also $|\beta_2| < 1$ by its definition)

$$\begin{aligned} (r + \theta)^{-1} &\cong r^{-1} \\ &\geq C \left(\left| \int_{|t-\theta|>r} \frac{1}{(t-\theta)^2} dt \right| + \left| \left[\frac{1}{t-\theta} \right]_{t=\theta+r}^\infty \right| + \left| \left[\frac{1}{t-\theta} \right]_{t=-\theta-r}^{-\infty} \right| \right) \\ &\geq C' \left| \int_{|t-\theta|>r} \frac{\beta_2(t)}{(t-\theta)^2} dt + \left[\frac{\beta_2(t)}{(t-\theta)^2} \right]_{t=\theta+r}^\infty + \left[\frac{\beta_2(t)}{(t-\theta)^2} \right]_{t=-\theta-r}^{-\infty} \right| \\ (2.12) \quad &\geq C' \left| \frac{\partial}{\partial \theta} \int_{|t-\theta|>r} \frac{\beta_2(t)}{t-\theta} dt \right| \end{aligned}$$

Hence,

$$\begin{aligned} \mathbf{A} &\geq C \int_0^h r^\sigma \int_0^r (r + \theta)^{-1-\sigma} \left| \frac{\partial}{\partial \theta} \int_{|t-\theta|>r} \frac{\beta_2(t)}{t-\theta} dt \right| V(r, \theta) d\theta dr \\ &\geq C \left| \int_0^h r^\sigma \int_0^r (r + \theta)^{-1-\sigma} \frac{\partial}{\partial \theta} \int_{|t-\theta|>r} \frac{\beta_2(t)}{t-\theta} dt V(r, \theta) d\theta dr \right| \\ &= C' \left| \int_0^h r^\sigma \int_0^r (r + \theta)^{-1-\sigma} \frac{\partial V}{\partial \theta} d\theta dr \right| \end{aligned}$$

Denoting

$$\int_0^h r^\sigma \int_0^r (r + \theta)^{-1-\sigma} \frac{\partial V}{\partial \theta} d\theta dr =: \mathbf{B}_2$$

we thus obtain

$$(2.13) \quad \mathbf{A} \geq C |\mathbf{B}_2|.$$

Integration by parts with respect to θ implies $\mathbf{A} = -b\mathbf{B}_1 + b\mathbf{B}_2$ where $b := 1/(1 + \sigma)$, hence the positivity of $\mathbf{A}$, (2.13) and the triangle inequality yield $\mathbf{A} \geq C|\mathbf{B}_1|$. So, (2.10) follows.

Next we claim that

$$(2.14) \quad |\mathbf{B}_1| \geq \int_0^h r^{-1} V(r, 0) dr.$$

This is a consequence of Lemma 2 (see also the choice of h and recall $-1 - \sigma \neq 0$):

$$(2.15) \quad \mathbf{B}_1 = \int_0^h r^{-1} V(r, 0) \left(c \frac{V(r, r)}{V(r, 0)} - 1 \right) dr = \int_0^h r^{-1} V(r, 0) \cdot \left(c \exp \left(-\frac{a}{2\pi} \int_0^{2\pi} \left(Q_{(1-r)e^{i\tau+r}}(\theta) - Q_{(1-r)e^{i\tau}}(\theta) \right) \beta_2(\theta) d\theta \right) - 1 \right) dr,$$

where $Q_z(\theta) = \operatorname{Im} \left((e^{i\theta} + z)/(e^{i\theta} - z) \right)$ is the conjugate Poisson kernel. Here $c = 2^{-1-\sigma} < 1$ by the definition of $\mathbf{B}_1$ in (2.10), hence the modulus of (2.15) is larger than a constant times

$$\begin{aligned} \left| \int_0^h r^{-1} V(r, 0) \left(c(1 + o(h)) - 1 \right) dr \right| &\geq C \left| \int_0^h r^{-1} V(r, 0) dr \right| \\ &= C \int_0^h r^{-1} V(r, 0) dr \end{aligned}$$

for every small enough h .

As a consequence, (2.10) and (2.14) imply (for small enough h)

$$(2.16) \quad \mathbf{I}_1 \geq C \int_0^h r^{-1} V(r, 0) dr.$$

IV. Estimate for $\mathbf{I}_2$. We prove that

$$(2.17) \quad \mathbf{I}_2^{p/p^*} \geq Ch^{\alpha p + 2p} V(h, 0)^{-1}.$$

Since

$$\left| \frac{\partial}{\partial \theta} V^{-p^*/p} \right| = C \left| V^{-p^*/p} \frac{\partial}{\partial \theta} \int_{|t-\theta|>r} \frac{\beta_2(t)}{t-\theta} dt \right|,$$

we get by considerations as around (2.11) and (2.13) (recall that V is positive and that $2 + \sigma = a\delta_1/\pi$)

$$(2.18) \quad \begin{aligned} \mathbf{I}_2 &\geq \int_0^h r^\gamma \int_0^r (r+\theta)^{(-2-\sigma)\frac{-p^*}{p}} V(r, \theta)^{\frac{-p^*}{p}} d\theta dr \\ &\geq C \left| \int_0^h r^\gamma \int_0^r (r+\theta)^{(2+\sigma)\frac{p^*}{p}+1} \frac{\partial}{\partial \theta} V(r, \theta)^{\frac{-p^*}{p}} d\theta dr \right|, \end{aligned}$$

and we can deduce as before that

$$(2.19) \quad \mathbf{I}_2 \geq C \left| \int_0^h r^\gamma \left[(r+\theta)^{(2+\sigma)\frac{p^*}{p}+1} V(r, \theta)^{\frac{-p^*}{p}} \right]_{\theta=0}^r dr \right|.$$

Analogously to (2.14)–(2.16) we obtain

$$(2.20) \quad \mathbf{I}_2 \geq C \int_0^h r^{\alpha p^*+1+\frac{2p^*}{p}} V(r, 0)^{\frac{-p^*}{p}} dr.$$

This leads to (2.17): integration by parts of the right hand side yields the terms

$$(2.21) \quad \left[r^{\alpha p^*+2+\frac{2p^*}{p}} V(r, 0)^{\frac{-p^*}{p}} \right]_{r=0}^h \quad \text{and} \quad \int_0^h r^{\alpha p^*+2+\frac{2p^*}{p}} \frac{\partial}{\partial r} V(r, 0)^{\frac{-p^*}{p}} dr$$

(multiplied by some constants). The exponent $\alpha p^* + 2 + 2p^*/p$ is strictly positive, and the argument following (2.7) implies that $V(r, 0)$ approaches 0 more slowly than r^ε (as $r \rightarrow 0$) for any $\varepsilon > 0$. Hence, $\left[\cdot \right]_{r=0} = 0$ in (2.21). Arguments similar to (2.12)–(2.13) show that the right hand side of (2.20) majorizes the modulus of $\int_0^h$ in (2.21). From the $\left[\cdot \right]_{r=h}$ term in (2.21) we thus obtain

$$(2.22) \quad \mathbf{I}_2 \geq C h^{\alpha p^*+2p^*} V(h, 0)^{-p^*/p}$$

for small enough h .

V. The final steps. The inequalities (2.6)–(2.7), (2.16) and (2.22) imply

$$(2.23) \quad \begin{aligned} &\int_0^h r^{-1} V(r, 0) dr V(h, 0)^{-1} \leq C \\ \Leftrightarrow &\int_{-\infty}^{\log h} V(e^y, 0) dy \leq C V(h, 0), \end{aligned}$$

and denoting $\log h =: y$, $V(e^y, 0) =: F(y)$, this is equivalent to

$$(2.24) \quad \int_{-\infty}^y F \leq CF(y).$$

This implies $F(y) \leq e^{Cy}$ for $y < 0$ (*). (Well known, or, approximating F by a smooth function in the sup-norm in (2.24), we may assume that F is actually continuously differentiable. Then (2.24) implies $\int_{-\infty}^y F = Cf(y)F(y)$ for an unknown function f , $0 \leq f \leq 1$. Differentiate this identity and solve the resulting differential equation for F .) Finally, (2.8) follows from (*) actually for every small enough h .

The case $a\delta_2/\pi = -pp^{*-1}(\gamma+2)$ is similar, working with δ_2 and ρ instead of δ_1 and τ . It leads to the inequality ($V(r, \theta) := \nu_2(1-r, \rho+\theta)$)

$$V(h, 0)^{\frac{-p^*}{p}} \geq C \int_0^h r^{-1} V(r, 0)^{\frac{-p^*}{p}} dr,$$

which has the consequence $V(h, 0) > Ch^{\frac{-p^*}{p}}$. As after (2.7), this is seen to mean too quick a divergence for $V(h, 0)$ as $h \rightarrow 0$. $\square$

Let us denote the conjugate Poisson kernel by $Q_z(\theta) := \text{Im}((e^{i\theta} + z)/(e^{i\theta} - z))$.

Lemma 2. *The condition (1.4) implies:*

$$(2.25) \quad \lim_{r \rightarrow 1} \int_0^{2\pi} \left(Q_{(1-r)e^{i\tau+r}}(\theta) - Q_{(1-r)e^{i\tau}}(\theta) \right) \beta_2(\theta) d\theta = 0.$$

Proof. Let $I \subset \partial\mathbb{D}$ be a neighbourhood of τ such that β_2 is Dini-continuous there. Corollary III.1.4 of [2] implies that also $\tilde{\beta}_2$ is continuous on I . Hence, the Poisson extension $F(z)$ of $\tilde{\beta}_2$ is continuous in a neighbourhood $U \subset \overline{\mathbb{D}}$ of $e^{i\tau}$. But $F(z) = \int_0^{2\pi} Q_z(\theta) \beta_2(\theta) d\theta$, hence, the Lemma follows, interpreting (2.25) as a continuity statement for the conjugate function of β_2 . $\square$

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