A sharpness result for powers of Besov functions

Grard Bourdaud

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Abstract. A recent result of Kateb asserts that $f \in B_{p,q}^s(\mathbb{R}^n)$ implies $|f|^\mu \in B_{p,q}^s(\mathbb{R}^n)$ as soon as the following three conditions hold : (1) $0 < s < \mu + (1/p)$, (2) f is bounded, (3) $\mu > 1$. By means of counterexamples, we prove that those conditions are optimal.

1. Introduction and main results

According to Kateb [4], the implication

$$f \in B_{p,q}^s(\mathbb{R}^n) \implies |f|^\mu \in B_{p,q}^s(\mathbb{R}^n)$$

holds as soon as the following three conditions are satisfied :

- (1) $0 < s < \mu + (1/p)$,
- (2) $f \in L^\infty(\mathbb{R}^n)$,
- (3) $\mu > 1$.

We denote here by $B_{p,q}^s(\mathbb{R}^n)$, the Besov space — see for instance [9] for the definition. Throughout the paper, we assume that $p, q \in [1, +\infty]$ and $s, \mu > 0$. In the following three theorems, we shall see that the above conditions are the best possible.

Theorem 1. *Assume that μ is not an even integer. Then there exists $f \in \mathcal{D}(\mathbb{R}^n)$ such that $|f|^\mu \notin B_{p,q}^{\mu+(1/p)}(\mathbb{R}^n)$, for every $q < \infty$.*

Remark. If μ is an even integer, then $t \mapsto |t|^\mu$ is a C^∞ function. Such functions are known to act on $B_{p,q}^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ (cf. [6, Theorem 5.3.6(1), p. 336]).

Theorem 2. *Let $\mu > 1$. Assume that $s < n/p$ or that $s = n/p$ and $q > 1$. Then there exists a positive unbounded function f such that $f \in B_{p,q}^s(\mathbb{R}^n)$ but $f^\mu \notin B_{p,q}^s(\mathbb{R}^n)$.*

Now we turn to the sharpness of the condition $\mu > 1$. If $\mu < 1$, the function $t \mapsto |t|^\mu$ is not locally Lipschitz continuous; then by a general result of Bourdaud [1], there exist functions f in $B_{p,q}^s(\mathbb{R}^n)$ such that $|f|^\mu \notin B_{p,q}^s(\mathbb{R}^n)$. Here we want to go further : *the same phenomenon can appear even if the function f is bounded and positive.*

Theorem 3. *If $\mu < 1$, there exists a positive function $f \in B_{p,q}^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $f^\mu \notin B_{p,q}^s(\mathbb{R}^n)$.*

In Section 2, we construct some specific test functions in Besov spaces while in Section 3, we give the proofs of the above three theorems. We denote by $c, c_1, c_2, \dots$ various strictly positive constants.

2. Some functions in Besov spaces

Let ρ be a C^∞ function on $\mathbb{R}$ such that $\rho(x) = 1$ for $x \leq e^{-3}$ and $\rho(x) = 0$ for $x \geq e^{-2}$. We denote by Δ the Laplace operator on $\mathbb{R}^n$.

Proposition 1. *Let $(\alpha, \sigma) \in \mathbb{R}^2$. Then the Fourier transform of the function f defined by*

$$(1) \quad f(x) := |\log |x||^\alpha (\log |\log |x||)^{-\sigma} \rho(|x|),$$

is indefinitely differentiable on $\mathbb{R}^n$ and satisfies

$$\Delta^k(\widehat{f})(\xi) = O\left(|\xi|^{-n-2k} (\log |\xi|)^{\alpha-1} (\log(\log |\xi|))^{-\sigma}\right),$$

as $|\xi| \rightarrow \infty$, for any $k \in \mathbb{N}$. In case $\alpha = 0$, the above estimation can be improved as follows :

$$\Delta^k(\widehat{f})(\xi) = O\left(|\xi|^{-n-2k} (\log |\xi|)^{-1} (\log(\log |\xi|))^{-\sigma-1}\right).$$

Proof. This is probably a known result. We outline the proof, following the method of S. Wainger [10, Theorem 3, p. 27]. We need a few notations :

- J_μ is the classical Bessel function (see e.g. [11]) ;
- $h_0(t) := |\log t|^\alpha (\log |\log t|)^{-\sigma}$, for $0 < t < 1/e$;
- $u_k(r) := \int_0^\infty J_{\frac{n}{2}-1}(rt) h_0(t) t^{\frac{n}{2}+2k} \rho(t) dt$;
- $h_m(t) := t h'_{m-1}(t)$ ($m = 1, 2, \dots$) .

An easy computation yields the formula

$$(2) \quad h_m(t) = (-1)^m |\log t|^{\alpha-m} (\log |\log t|)^{-\sigma} \left(\sum_{j=0}^m b_{\alpha, \sigma, j, m} (\log |\log t|)^{-j} \right),$$

with $b_{\alpha, \sigma, 0, m} = \alpha(\alpha-1) \cdots (\alpha-m+1)$. Then we have

$$\Delta^k(\widehat{f})(\xi) = (-1)^k (2\pi)^{n/2} |\xi|^{1-\frac{n}{2}} u_k(|\xi|) \quad , \quad \forall \xi \neq 0.$$

First of all, we establish the following alternative formula for u_k :

$$(3) \quad u_k(r) = r^{-\nu} \int_0^\infty t^{\frac{n}{2}+2k} J_{\frac{n}{2}+\nu-1}(rt) \left(\sum a_{\nu, m, j, l} t^{-m-l} h_m(t) \rho^{(j)}(t) \right) dt,$$

for any integer $\nu \geq k+1$, where the summation is extended to all the triples $(m, j, l) \in \mathbb{N}^3$ such that $m+j+l = \nu$ and such that

$$(4) \quad m = 0 \quad \implies \quad j \geq 1.$$

Proof of (3). It is essentially the Lemma 8 of Wainger [10]. We first prove (3), for any $\nu \geq 1$ and without the restriction (4), by using repeated integrations by parts and the identity

$$(5) \quad J_\nu(tr) = \frac{1}{r} t^{-\nu-1} \frac{d}{dt} (t^{\nu+1} J_{\nu+1}(tr))$$

(see [11, p. 45]). If we take $\nu = k$, the formula becomes

$$u_k(r) = r^{-k} \int_0^\infty t^{\frac{n}{2}+2k} J_{\frac{n}{2}+k-1}(rt) \left(\sum a_{k, m, j, l} t^{-m-l} h_m(t) \rho^{(j)}(t) \right) dt.$$

In the above formula, we consider the term corresponding to $m = j = 0$, i.e. $l = k$, and we perform a further integration by parts; by (5), we obtain

$$\begin{aligned}
& \int_0^\infty t^{\frac{n}{2}+k} J_{\frac{n}{2}+k-1}(rt) \rho(t) h_0(t) dt \\
&= \frac{1}{r} \int_0^\infty \frac{d}{dt} (t^{\frac{n}{2}+k} J_{\frac{n}{2}+k}(rt)) \rho(t) h_0(t) dt \\
&= -\frac{1}{r} \int_0^\infty t^{\frac{n}{2}+k} J_{\frac{n}{2}+k}(rt) (\rho'(t) h_0(t) + \rho(t) h_0'(t)) dt.
\end{aligned}$$

By this way, we obtain the desired formula for $\nu = k+1$ and with condition (4). To obtain the general case $\nu > k+1$, we pursue the integrations by parts.

It remains to estimate the various terms of (3). We choose $\nu > (n/2) + 2k + (1/2)$ and we recall the classical properties of Bessel functions :

$$J_\mu(t) = O(t^{-1/2}) \quad (t \rightarrow +\infty), \quad J_\mu(t) = O(t^\mu) \quad (t \rightarrow 0).$$

In the formula (3), we first consider the terms such that $j = 0$. From the formula (2), we deduce the following estimations :

$$(6) \quad h_m(t) = O\left(|\log t|^{\alpha-m} (\log |\log t|)^{-\sigma}\right) \quad (t \rightarrow 0),$$

where σ can be replaced by $\sigma + 1$ in case $\alpha = 0$. By using the conditions

$$n + 2k - 1 > -1, \quad \frac{n}{2} + 2k - \frac{1}{2} - \nu < -1,$$

and the estimations (6), we obtain

$$\begin{aligned}
& \left| \int_0^\infty t^{\frac{n}{2}+2k-\nu} J_{\frac{n}{2}+\nu-1}(rt) \rho(t) h_m(t) dt \right| \\
& \leq c_1 \left(r^{\frac{n}{2}+\nu-1} \int_0^{1/r} t^{n+2k-1} |\log t|^{\alpha-m} (\log |\log t|)^{-\sigma} dt \right. \\
& \quad \left. + r^{-1/2} \int_{1/r}^{1/e^3} t^{\frac{n}{2}+2k-\frac{1}{2}-\nu} |\log t|^{\alpha-m} (\log |\log t|)^{-\sigma} dt \right) \\
& \leq c_2 r^{\nu-\frac{n}{2}-1-2k} (\log r)^{\alpha-m} (\log(\log r))^{-\sigma},
\end{aligned}$$

for r sufficiently large. If we turn to the terms such that $j > 0$, we have the following trivial estimation :

$$\begin{aligned}
& \left| \int_0^\infty t^{\frac{n}{2}+2k-m-l} J_{\frac{n}{2}+\nu-1}(rt) \rho^{(j)}(t) h_m(t) dt \right| \\
& \leq c_1 r^{-1/2} \int_{1/e^3}^{1/e^2} t^{\frac{n}{2}+2k-m-l-\frac{1}{2}} |h_m(t)| dt \leq c_2 r^{-1/2}.
\end{aligned}$$

That ends up the Proof of Proposition 1. $\square$

We give now sharp conditions on the couple (α, σ) such that the fonction defined by (1) belongs to the critical Besov spaces. For the sake of completeness, we discuss the same question in the critical Lizorkin-Triebel spaces $F_{p,q}^{n/p}$ (cf. [9] for their definition). Let us define U_q as the set of $(\alpha, \sigma) \in \mathbb{R}^2$ such that :

- $\alpha = 1 - \frac{1}{q}$ and $\sigma > 1/q$, or $\alpha < 1 - \frac{1}{q}$, in case $1 < q < \infty$,
- $\alpha = 0$ and $\sigma > 0$, or $\alpha < 0$, in case $q = 1$,
- $\alpha = 1$ and $\sigma \geq 0$, or $\alpha < 1$, in case $q = \infty$.

Proposition 2. *Let f be the function defined by (1).*

1. *If $p, q \in [1, +\infty]$, then $f \in B_{p,q}^{n/p}(\mathbb{R}^n)$ if and only if $(\alpha, \sigma) \in U_q$.*
2. *If $p \in]1, +\infty[$ and $q \in [1, +\infty]$, then $f \in F_{p,q}^{n/p}(\mathbb{R}^n)$ if and only if $(\alpha, \sigma) \in U_p$.*

Proof. To each couple (α, σ) , we associate the sequence $(\varepsilon_j)_{j \geq 2}$ defined by

$$\varepsilon_j := j^{\alpha-1} (\log j)^{-\sigma} \quad \text{if } \alpha \neq 0, \quad \varepsilon_j := j^{-1} (\log j)^{-\sigma-1} \quad \text{if } \alpha = 0,$$

which belongs to l^q if and only if $(\alpha, \sigma) \in U_q$.

Step 1. Let us assume $(\alpha, \sigma) \in U_q$. We are going to prove that $f \in B_{1,q}^n(\mathbb{R}^n)$, which the smallest relevant Besov space. We use the Littlewood-Paley setting, so we consider a function $\psi \in \mathcal{D}(\mathbb{R}^n)$, with support in the annulus $1 \leq |\xi| \leq 3$, such that

$$\sum_{j \in \mathbb{Z}} \psi(2^j \xi) = 1 \quad (\forall \xi \neq 0),$$

and we define the operators L_j by

$$(L_j f)^\wedge(\xi) := \psi(2^{-j} \xi) \widehat{f}(\xi).$$

Since f is clearly integrable, it suffices to prove the following :

$$(7) \quad \left(\sum_{j \geq 2} (2^{jn} \|L_j f\|_1)^q \right)^{1/q} < \infty.$$

To do so, we consider the functions g_j defined by $\widehat{g_j}(\xi) := \psi(\xi) \widehat{f}(2^j \xi)$. By Proposition 1, we have

$$(8) \quad |\Delta^k(\widehat{g_j})(\xi)| \leq c_k 2^{-jn} \varepsilon_j, \quad \forall \xi \in \mathbb{R}^n.$$

Since $\widehat{g_j}$ is supported by the annulus $1 \leq |\xi| \leq 3$, the estimation (8) implies

$$(9) \quad |x|^{2k} |g_j(x)| \leq c'_k 2^{-jn} \varepsilon_j, \quad \forall x \in \mathbb{R}^n.$$

By taking $k > n/2$ and $k = 0$ in (9), we obtain $\|g_j\|_1 \leq c 2^{-jn} \varepsilon_j$. Since $L_j f(x) = 2^{jn} g_j(2^j x)$, we obtain (7).

Step 2. Let us assume $(\alpha, \sigma) \notin U_q$. We are going to prove that $f \notin B_{\infty, q}^0(\mathbb{R}^n)$, which is the biggest relevant Besov space. Let us consider a positive function $\varphi \in \mathcal{D}(]0, +\infty[)$, such that $\varphi(t) = 1$ for $1 \leq t \leq 2$, and define

$$\theta(x) := |x|^{1-n} \varphi'(|x|), \quad \forall x \in \mathbb{R}^n.$$

Then $\theta \in \mathcal{D}(\mathbb{R}^n)$ and $\int \theta(x) dx = 0$. If f would belong to $B_{\infty, q}^0(\mathbb{R}^n)$, then, by a theorem of Peetre [5, Theorem 4, p.164] (see also [2, Proposition 19]), the sequence defined by

$$f_j(x) := 2^{nj} \int \theta(2^j(x-y)) f(y) dy, \quad \forall j \in \mathbb{N},$$

would satisfy the following property :

$$(10) \quad \left(\sum_{j \geq 0} \|f_j\|_{\infty}^q \right)^{1/q} < \infty.$$

Now we are going to prove that

$$(11) \quad \left(\sum_{j \geq 0} |f_j(0)|^q \right)^{1/q} = \infty,$$

a property which contradicts (10). If ω denotes the volume of the unit sphere in $\mathbb{R}^n$, we have

$$\omega^{-1} f_j(0) = \int_0^\infty \varphi(2^j t) (-h'_0(t)) \rho(t) dt - \int_0^\infty \varphi(2^j t) h_0(t) \rho'(t) dt.$$

The second above term plays no role, since it is a $O(2^{-j})$. On the other hand, by formula (2) we have

$$\int_0^\infty \varphi(2^j t) (-h'_0(t)) \rho(t) dt \geq c_1 \int_{2^{-j}}^{2^{-j+1}} |h_1(t)| \frac{dt}{t} \geq c_2 \varepsilon_j,$$

for sufficiently large j 's.

That ends up the proof of (11) and of Proposition 2. $\square$

Step 3. According to Jawerth [3] (see also [6, Theorem 2.2.3, p. 31]), the following embeddings hold :

$$B_{1,p}^n(\mathbb{R}^n) \hookrightarrow F_{p,q}^{n/p}(\mathbb{R}^n) \hookrightarrow B_{\infty,p}^0(\mathbb{R}^n).$$

Then assertion concerning Lizorkin-Triebel spaces is a consequence of the two preceding steps.

Remark. The case of $B_{p,p}^{n/p}(\mathbb{R}^n)$, $1 < p < +\infty$, in Proposition 2, was previously obtained by Triebel [8].

Proposition 3. *Let $\beta > 0$, $0 < \alpha + \frac{n}{p} < 2(\beta + 1)$. Let us define*

$$\sigma := \frac{1}{\beta + 1} \left(\alpha + \frac{n}{p} \right).$$

If $g \in B_{\infty,\infty}^\gamma(\mathbb{R})$ for some $\gamma > \sigma$, then the function

$$(12) \quad f(x) := |x|^\alpha g(|x|^{-\beta}) \rho(|x|)$$

belongs to $B_{p,\infty}^\sigma(\mathbb{R}^n)$. If moreover $g(t) = \sin^2 \frac{t}{2}$, then f does not belong to $B_{p,q}^\sigma(\mathbb{R}^n)$ for $q < \infty$.

Proof. Step 1. Case $\sigma < 1$. Of course we may assume that $\gamma < 1$. Thanks to known properties of power functions, we are reduced to estimate

$$I(h) := \int_{|x| \leq 1} |x|^{\alpha p} |g(|x+h|^{-\beta}) - g(|x|^{-\beta})|^p dx,$$

when $h \rightarrow 0$. From the inequality

$$(13) \quad \left| |x+h|^{-\beta} - |x|^{-\beta} \right| \leq c|h| |x|^{-\beta-1}, \quad \text{for } |h| \leq \frac{|x|}{2},$$

we deduce

$$I(h) \leq c \left(|h|^{\gamma p} \int_{|h|^{\frac{1}{\beta+1}} \leq |x| \leq 1} |x|^{\alpha p - \gamma p(\beta+1)} dx + \int_{|x| < |h|^{\frac{1}{\beta+1}}} |x|^{\alpha p} dx \right).$$

By assumptions, we have $\alpha p - \gamma p(\beta+1) < -n$ and $\alpha p > -n$. Hence we have

$$I(h) \leq c|h|^{\frac{\alpha p + n}{\beta+1}},$$

the wished estimation.

Step 2. Case $\sigma \geq 1$. Now we have to estimate

$$J(h) := \int_{|x| \leq 1} |x|^{\alpha p} \left| g(|x+h|^{-\beta}) + g(|x-h|^{-\beta}) - 2g(|x|^{-\beta}) \right|^p dx.$$

We shall use the following *identity* :

$$\begin{aligned} & 2 \left(g(|x+h|^{-\beta}) + g(|x-h|^{-\beta}) - 2g(|x|^{-\beta}) \right) \\ &= g(|x+h|^{-\beta}) - g(2|x|^{-\beta} - |x-h|^{-\beta}) + g(|x-h|^{-\beta}) \\ &\quad - g(2|x|^{-\beta} - |x+h|^{-\beta}) + g(|x+h|^{-\beta}) + g(2|x|^{-\beta} - |x+h|^{-\beta}) \\ &\quad - 2g(|x|^{-\beta}) + g(|x-h|^{-\beta}) + g(2|x|^{-\beta} - |x-h|^{-\beta}) - 2g(|x|^{-\beta}). \end{aligned}$$

By combining this identity with (13) and with

$$\left| |x+h|^{-\beta} + |x-h|^{-\beta} - 2|x|^{-\beta} \right| \leq c|h|^2 |x|^{-\beta-2}, \quad \text{for } |h| \leq \frac{|x|}{2},$$

we see that $J(h)$ is estimated by

$$\begin{aligned} & |h|^{2p} \int_{|h|^{\frac{1}{\beta+1}} \leq |x| \leq 1} |x|^{\alpha p - p(\beta+2)} dx \\ &+ |h|^{\gamma p} \int_{|h|^{\frac{1}{\beta+1}} \leq |x| \leq 1} |x|^{\alpha p - \gamma p(\beta+1)} dx + \int_{|x| < |h|^{\frac{1}{\beta+1}}} |x|^{\alpha p} dx. \end{aligned}$$

As in *Step 1*, the two above last terms are estimated by $|h|^{\frac{\alpha p + n}{\beta+1}}$. Moreover a short discussion shows that

$$|h|^{2p} \int_{|h|^{\frac{1}{\beta+1}} \leq |x| \leq 1} |x|^{\alpha p - p(\beta+2)} dx = O(|h|^{rp}),$$

for some $r > \sigma$.

Step 3. Now we prove the estimation from below in case of $g(t) = \sin^2 \frac{t}{2}$. We give the proof for $\sigma \geq 1$ (the case $\sigma < 1$ is similar and easier). Assume $|h| \leq c_1|x|^{\beta+1}$, with c_1 sufficiently small. Then by (13) we have

$$\begin{aligned} & |g(|x+h|^{-\beta}) + g(2|x|^{-\beta} - |x+h|^{-\beta}) - 2g(|x|^{-\beta}) \\ & + g(|x-h|^{-\beta}) + g(2|x|^{-\beta} - |x-h|^{-\beta}) - 2g(|x|^{-\beta})| \\ & = |\cos|x|^{-\beta}| |\cos(|x+h|^{-\beta} - |x|^{-\beta}) + \cos(|x-h|^{-\beta} - |x|^{-\beta}) - 2| \\ & \geq c_2 |\cos|x|^{-\beta}| |h|^2 |x|^{-2(\beta+1)}. \end{aligned}$$

By using again the *identity*, we see that $J(h)$ is greater than

$$\begin{aligned} & c_3 |h|^{2p} \int_{c_2|h|^{\frac{1}{\beta+1}} \leq |x| \leq 1} |x|^{\alpha p - 2p(\beta+1)} |\cos(|x|^{-\beta})|^p dx - c_4 |h|^{2p} \\ & \times \int_{c_2|h|^{\frac{1}{\beta+1}} \leq |x| \leq 1} |x|^{\alpha p - p(\beta+2)} dx. \end{aligned}$$

It is easily seen that

$$\int_{\varepsilon \leq |x| \leq 1} |x|^a |\cos(|x|^{-\beta})|^p dx \approx \varepsilon^{a+n} \quad (\varepsilon \rightarrow 0),$$

for any $a < -n$. Then $J(h) \geq c_5 |h|^{\frac{\alpha p + n}{\beta+1}}$, for $|h|$ sufficiently small.

3. Proofs of the main results

3.1 Proof of Theorem 1. This theorem is a classical result. We outline the proof, for the sake of completeness. The function $t \mapsto \rho(|t|)^\mu |t|^\mu$ does not belong to $B_{p,q}^{\mu+(1/p)}(\mathbb{R})$, for $q < +\infty$ (cf. for instance [6, 2.3.1, p. 44]). Hence the function defined by $f(x) := x_1 \rho(|x|)$, for $x \in \mathbb{R}^n$, belongs to $\mathcal{D}(\mathbb{R}^n)$, but $|f|^\mu$ does not belong to $B_{p,q}^{\mu+(1/p)}(\mathbb{R}^n)$ for $q < +\infty$. $\square$

3.2 Proof of Theorem 2. Step 1. Assume that $s < n/p$. Let us take α such that

$$\frac{1}{\mu} \left(\frac{n}{p} - s \right) < \alpha < \frac{n}{p} - s.$$

According to [6, 2.3.1, p. 44], the searched function is $f(x) := |x|^{-\alpha} \rho(|x|)$.

Step 2. Assume that $s = n/p$ and $q > 1$. We consider the function f defined by (1), with $\alpha = 1 - \frac{1}{q}$, $\sigma q > 1$. According to Proposition 2, one has $f \in B_{p,q}^{n/p}(\mathbb{R}^n)$ and $f^\mu \notin B_{p,q}^{n/p}(\mathbb{R}^n)$. $\square$

3.3 Proof of Theorem 3. Following a remark of Sickel [7, Par. 3.4], we can assume that $n = 1$. Indeed tensorizing with a C^∞ compactly supported function of the remaining variables gives the general case.

Step 1. For $s > 1/p$, we use the same test functions as in the proof of Theorem 2. We take α such that

$$s - \frac{1}{p} < \alpha < \frac{1}{\mu} \left(s - \frac{1}{p} \right),$$

then we set $f(x) := |x|^\alpha \rho(|x|)$. We obtain $f \in B_{p,q}^s(\mathbb{R})$ whereas $f^\mu \notin B_{p,q}^s(\mathbb{R})$.

Step 2. Now we assume $s \leq 1/p$. Then the relevant power functions are not bounded. Instead we use the bounded functions with great oscillations at 0 constructed in Proposition 3. Let us take

$$\beta > \max \left(0, \frac{1}{sp} - 1 \right), \quad s(\beta + 1) - \frac{1}{p} < \alpha < \frac{1}{\mu} \left(s(\beta + 1) - \frac{1}{p} \right).$$

Let us consider the function f defined by (12), with $n = 1$ and

$$g(t) := \left(\sin^2 \frac{t}{2} \right)^{1/\mu}.$$

Since $g \in B_{\infty,\infty}^{1/\mu}(\mathbb{R})$, we obtain $f \in B_{p,q}^s(\mathbb{R})$. Since we have

$$f(x)^\mu = |x|^{\alpha\mu} \sin^2 \left(\frac{1}{2} |x|^{-\beta} \right)$$

in a neighborhood of 0, the second part of Proposition 3 yields $f^\mu \notin B_{p,q}^s(\mathbb{R})$. $\square$

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Institut de Mathématiques de Jussieu
 Équipe d'Analyse Fonctionnelle
 Case 186
 4 place Jussieu, 75252 Paris Cedex 05
 France
 (E-mail : bourdaud@ccr.jussieu.fr)

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