New frames of Besov and Triebel-Lizorkin spaces\textsuperscript{1}

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Abstract. Let $s < 0$. The author obtains some new frames for Besov spaces $B_{pq}^s(\mathbb{R}^n)$ with $1 \leq p, q \leq \infty$ and Triebel-Lizorkin spaces $F_{pq}^s(\mathbb{R}^n)$ with $1 < p < \infty$ and $1 < q \leq \infty$ by a dual method via the subatomic characterizations of these spaces when $s > 0$.

1. Introduction

Subatomic (or quarkonial) characterizations for spaces $B_{pq}^s(\mathbb{R}^n)$ and $F_{pq}^s(\mathbb{R}^n)$ are proved to be a very useful tool in many applications. For example, in [7, 8], Triebel obtained the estimates of entropy numbers for the compact embedding between Besov and Triebel-Lizorkin spaces on bounded domains in $\mathbb{R}^n$ by first establishing the subatomic characterizations of these spaces. These characterizations are essential different from the atomic characterizations in that the subatoms are independent of any given $f$, but the atoms do. Thus, the atomic decomposition characterizations are not enough in some applications. In fact, the set of subatoms can be regarded

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as a kind of frame of these function spaces, which behaves like a base in some sense. See also [3] for some new bases of these spaces and their applications in nonlinear approximation.

Let \( s < 0 \). The main purpose of this paper is to show how to obtain new frames of Besov spaces \( B^s_{p,q}(\mathbb{R}^n) \) with \( 1 \leq p < \infty \) and Triebel-Lizorkin spaces \( F^s_{p,q}(\mathbb{R}^n) \) with \( 1 < p < \infty \) and \( 1 < q \leq \infty \) by a dual method via the quarkomial (subatomic) characterizations of these spaces when \( s > 0 \) in [8]. Such a frame characteristic for Besov spaces \( B^s_{p,q}(\mathbb{R}^n) \) with \( 1 < p \leq \infty \) is already obtained by Triebel in [9] and is used to characterize the regularity of the distribution considered. Moreover, using this frame characterization, one can give a new proof for the interpolation theorem between these function spaces and for the estimates of entropy numbers of the compact embedding between corresponding function spaces restricted to a bounded domain in \( \mathbb{R}^n \). We will not give any details on this since they can be proved by the standard procedures in [5, 7]; see also [2, 10] for some details.

Let us now recall the definitions of Besov and Triebel-Lizorkin spaces on \( \mathbb{R}^n \). Let \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) with

\[
\varphi(x) = \begin{cases} 
1, & |x| \leq 1; \\
0, & |x| \geq 3/2.
\end{cases}
\]

We put \( \varphi_0(x) = \varphi(x) \), \( \varphi_1(x) = \varphi(x/2) - \varphi(x) \) and

\[
\varphi_k(x) = \varphi_1(2^{-k+1}x),
\]

where \( x \in \mathbb{R}^n \) and \( k \in \mathbb{N} \). Then,

\[
\sum_{k=0}^{\infty} \varphi_k(x) = 1
\]

for all \( x \in \mathbb{R}^n \). That means that \( \{\varphi_k\}_{k=0}^{\infty} \) forms a dyadic resolution of unity in \( \mathbb{R}^n \). Recall that \( (\varphi_k \hat{f})^\vee \) is an entire analytic function on \( \mathbb{R}^n \) for any \( f \in \mathcal{S}(\mathbb{R}^n) \) by the Paley-Wiener-Schwartz theorem; see [6]. In particular, \( (\varphi_k \hat{f})^\vee(x) \) makes sense pointwise.

**Definition 1.1.** Let \( s \in \mathbb{R} \) and \( 0 < q \leq \infty \). Let \( \{\varphi_j\}_{j=0}^{\infty} \) be as above.

(i) Let \( 0 < p \leq \infty \). Then the space \( B^s_{p,q}(\mathbb{R}^n) \) is the collection of all \( f \in \mathcal{S}(\mathbb{R}^n) \) such that

\[
\|f\|_{B^s_{p,q}(\mathbb{R}^n)} = \left\{ \sum_{k=0}^{\infty} 2^{ksq} \left\| \left( \varphi_k \hat{f} \right)^\vee \right\|^q_{L^p(\mathbb{R}^n)} \right\}^{1/q}
\]

(with the usual modification if \( q = \infty \)) is finite.
(ii) Let $0 < p < \infty$. Then the space $F^s_p(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$
\|f\|_{F^s_p(\mathbb{R}^n)} = \left\| \left( \sum_{k=0}^{\infty} 2^{ksp} \left| \varphi_k f \right|^{q} \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}
$$

(with the usual modification if $q = \infty$) is finite.

We remark that both spaces are quasi-Banach spaces which are independent of the chosen function $\varphi$; see [6]. Moreover, if $p \geq 1$ and $q \geq 1$, then the space $B^s_p(\mathbb{R}^n)$ and the space $F^s_p(\mathbb{R}^n)$ are Banach spaces.

We need the dual spaces of spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. In the following, for $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, we define the space $B^s_{pq}(\mathbb{R}^n)$ to be the completion of $\mathcal{S}(\mathbb{R}^n)$ in $B^s_p(\mathbb{R}^n)$ endowed with the same quasi-norm as $B^s_{pq}(\mathbb{R}^n)$, and the space $F^s_{pq}(\mathbb{R}^n)$ to be the completion of $\mathcal{S}'(\mathbb{R}^n)$ in $F^s_{pq}(\mathbb{R}^n)$ endowed with the same quasi-norm as $F^s_p(\mathbb{R}^n)$. Then we have the following dual theorems; see [6, pp. 178-180] and [4]. In the following, if $1 \leq r \leq \infty$, we define $r'$ by $1/r + 1/r' = 1$, and if $0 < r < 1$, we define $r' = \infty$.

**Lemma 1.1.** (i) Let $s \in \mathbb{R}$, $1 \leq p < \infty$ and $0 < q < \infty$. Then

$$
(B^s_{pq}(\mathbb{R}^n))^* = B^{-s}_{pq}(\mathbb{R}^n).
$$

Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$. Then

$$
(B^s_{pq}(\mathbb{R}^n))^* = B^{-s}_{pq}(\mathbb{R}^n).
$$

(ii) Let $s \in \mathbb{R}$, $1 \leq p < \infty$ and $1 \leq q < \infty$. Then

$$
(F^s_{pq}(\mathbb{R}^n))^* = F^{-s}_{pq}(\mathbb{R}^n).
$$

Let $s \in \mathbb{R}$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Then

$$
(F^s_{pq}(\mathbb{R}^n))^* = F^{-s}_{pq}(\mathbb{R}^n).
$$

Throughout this paper, $C$ denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We denote by $f \sim g$ that there is a constant $C > 0$ independent of the main parameters such that $C^{-1} g < f < C g$.

2. Main results and their proofs

Let $Q_{\nu m}$ be the cube in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centered at $2^{-\nu} m$, and with side length $2^{-\nu}$, where $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$.

We also denote by $\lambda^{(p)}_{\nu m}$ the $p$-normalized characteristic function of the cube $Q_{\nu m}$, which means

$$
\lambda^{(p)}_{\nu m}(x) = \begin{cases} 
2^{\nu n/p}, & x \in Q_{\nu m} ; \\
0, & x \notin Q_{\nu m} ,
\end{cases}
$$
where \( \nu \in \mathbb{N} \cup \{0\} \), \( m \in \mathbb{Z}^n \) and \( 0 < p \leq \infty \). In what follows, we will denote \( \mathbb{N} \cup \{0\} \) simply by \( \mathbb{N}_0 \).

The following definition of quarks (or subatoms) can be found in [8, p. 12]; see also [7].

**Definition 2.1.** Let \( \psi \) be a non-negative \( C^\infty \) function in \( \mathbb{R}^n \) with

\[
\text{supp } \psi \subset \{ y \in \mathbb{R}^n : |y| < 2^r \}
\]

for some \( r \geq 0 \), and

\[
\sum_{m \in \mathbb{Z}^n} \psi(x - m) = 1
\]

for all \( x \in \mathbb{R}^n \). Let \( s \in \mathbb{R} \), \( 0 < \beta \leq \infty \), \( \beta \in \mathbb{N}_0 \) and \( \psi^\beta(x) = x^\beta \psi(x) \), where

\[
x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}
\]

for \( \beta = (\beta_1, \cdots, \beta_n) \). Then

\[
(\beta qu)^{\nu m}(x) = 2^{-(s-n)/p} \psi(2^r x - m), \quad x \in \mathbb{R}^n,
\]

is called to be an \((s, p) - \beta\)-quark (or subatom) related to \( Q_{\nu m} \). Here \( \nu \in \mathbb{N}_0 \) and \( m \in \mathbb{Z}^n \).

**Proposition 2.1.** Let \( s < 0 \), \( 1 < p < \infty \) and \( 1 < q \leq \infty \). There is a set of Schwartz functions, \( \{ \Psi_k^{\beta,l} : \beta \in \mathbb{N}_0^n, k \in \mathbb{N}_0, l \in \mathbb{Z}^n \} \), such that for any \( f \in F_p^s(\mathbb{R}^n) \),

\[
f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \gamma_{k,l}^{\beta} \Psi_k^{\beta,l}
\]

unconditionally in both the norm of \( F_p^s(\mathbb{R}^n) \) and \( S'(\mathbb{R}^n) \) when \( 1 < p, q < \infty \) and only in \( S'(\mathbb{R}^n) \) when \( 1 < p < \infty \) and \( 1 < q \leq \infty \), where, when \( k \in \mathbb{N} \), for all \( \alpha \in \mathbb{N}_0^n \),

\[
\int_{\mathbb{R}^n} \Psi_k^{\beta,l}(x) x^\alpha \, dx = 0,
\]

and \( \{ \gamma_{k,l}^{\beta} : \beta \in \mathbb{N}_0^n, k \in \mathbb{N}_0, l \in \mathbb{Z}^n \} \) is a sequence of numbers linearly depending on \( f \) and satisfying

\[
\sup_{\beta \in \mathbb{N}_0^n} 2^{(\rho - r)|\beta|} \left\| \left( \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \left| \gamma_{k,l}^{\beta}(\cdot) \chi_k(\cdot) \right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \leq C \| f \|_{F_p^s(\mathbb{R}^n)}
\]

with the same \( r \) as in (2.1), \( \rho > r \) and a constant \( C > 0 \) independent of \( f \).

**Proof.** Let \( \varphi \in S(\mathbb{R}^n) \) satisfy (1.1) and (1.2). Let \( Q_k \) be the cube in \( \mathbb{R}^n \) centered at the origin and with side-length \( 2\pi 2^k \). Let \( \kappa \in S(\mathbb{R}^n) \),

\[
\kappa_k(x) = \kappa(2^{-k}x),
\]
\[ g = \sum_{k=0}^{\infty} (\varphi_k \hat{g})^\vee (x) = C \sum_{k=0}^{\infty} \sum_{\beta \in \mathbb{N}_0^n} \sum_{l \in \mathbb{Z}^n} \lambda_{k+\rho,2r^l}^\beta (\beta qu)_{k+\rho,2r^l}(x) 2^{-p(s+n/p')} \cdot \ldots, \]

where \( \ldots \) stands for some similar terms which can be treated in the same way, \( \rho > r \) is a fixed positive number, \( (\beta qu)_{k+\rho,2r^l}(x) \) is a \((s, p') - \beta\)-quark according to Definition 2.1,

\[ \lambda_{k+\rho,2r^l}^\beta = 2^{-\rho|\beta|} 2^{-(k+\rho)(s+n/p')} (g, \psi_{k+\rho,2r^l}^{\beta,p}) \]

and

\[ \psi_{k+\rho,2r^l}^{\beta,p}(x) = \sum_{j \in \mathbb{Z}^n} \frac{D_{\beta}^\vee (l-j)}{\beta!} \varphi_k^\vee (2^{-k} j - x). \]

Moreover,

\[ \left\| \left( \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \lambda_{k+\rho,2r^l}^\beta (\psi_{k+\rho,2r^l}^{\beta,p}) \right)^{1/p'} \right\|_{L^p(\mathbb{R}^n)} \leq C_p 2^{-\rho|\beta|} \| g \|_{F_{\rho}^{-p'}(\mathbb{R}^n)}, \]

where \( C_p > 0 \) is a constant independent of \( g \) and \( \beta \), and \( \| g \|_{F_{\rho}^{-p'}(\mathbb{R}^n)} \) is the usual norm as in Definition 2.1. We also note that the series in (2.4) converge unconditionally in \( L^p(\mathbb{R}^n) \); see [8, p. 14]. In the following, we suppose \( g \in S(\mathbb{R}^n) \), then it is easy to show that the series in (2.4) in this case also converge in \( S(\mathbb{R}^n) \); see [8, pp. 23-24] again. Thus, we have

\[ (f, g) = C \sum_{\beta \in \mathbb{N}_0^n} \sum_{l=0}^{\infty} 2^{-\rho(s+n/p')} \lambda_{k+\rho,2r^l}^\beta (f, \psi_{k+\rho,2r^l}^{\beta,p}) + \ldots \]

\[ = C \sum_{\beta \in \mathbb{N}_0^n} \sum_{l=0}^{\infty} 2^{-\rho(s+n/p')} (f, \psi_{k+\rho,2r^l}^{\beta,p}) \times 2^{-\rho|\beta|} 2^{-(k+\rho)(s+n/p')} (g, \psi_{k+\rho,2r^l}^{\beta,p}) + \ldots \]

\[ = \left( C \sum_{\beta \in \mathbb{N}_0^n} \sum_{l=0}^{\infty} 2^{-\rho(s+n/p')} 2^{-\rho|\beta|} 2^{-(k+\rho)(s+n/p')} \times (f, \psi_{k+\rho,2r^l}^{\beta,p}) \right) + \ldots \]
for all $g \in \mathcal{S}(\mathbb{R}^n)$. Therefore,

$$f = C \sum_{\beta \in \mathbb{N}_0^n} \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} 2^{-\rho(s+n/l')} 2^{-\rho|\beta|} \times 2^{-(k+p)(s+n/l')} (f, (\beta qu)_{k+p,2^l}) \psi^\beta_{k+p,2^l} + \cdots$$

in $\mathcal{S}'(\mathbb{R}^n)$.

Now let

$$\gamma^\beta_{k+p,2^l} = C 2^{-\rho(s+n/l')} 2^{-\rho|\beta|} (f, (\beta qu)_{k+p,2^l})$$

and

$$\Psi^\beta_{k+p,2^l} = 2^{-(k+p)(s+n/l')} \psi^\beta_{k+p,2^l}.$$

We then have

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \gamma^\beta_{k+p,2^l} \Psi^\beta_{k+p,2^l} + \cdots$$

in $\mathcal{S}'(\mathbb{R}^n)$. Let us now verify (2.3). We first establish some estimates. Let $\{\varphi_k\}_{k=0}^\infty$ be the same as in (1.1) and (1.2). Obviously, we have

$$(f, (\beta qu)_{k+p,2^l}) = \sum_{j=0}^{\infty} \left( (\varphi_j \hat{f})^\vee, (\beta qu)_{k+p,2^l} \right).$$

We also have the following trivial estimate, that is, for all $j \in \mathbb{N}_0$,

$$\left| \left( (\varphi_j \hat{f})^\vee, (\beta qu)_{k+p,2^l} \right) \right| \chi^{(p)}_{k+p,2^l}(x)$$

$$= \left| \int_{\mathbb{R}^n} (\varphi_j \hat{f})^\vee(y)(\beta qu)_{k+p,2^l}(y) dy \right| \chi^{(p)}_{k+p,2^l}(x)$$

$$\leq C 2^{k(s+n/l')} 2^{l|\beta|} \int_{|y-2^{l+1}| \leq 2^{-l-1}} \left| (\varphi_j \hat{f})^\vee(y) \right| dy \chi^{(p)}_{k+p,2^l}(x)$$

$$\leq C 2^{k(s+n/l')} 2^{l|\beta|} 2^{-kn} M \left( (\varphi_j \hat{f})^\vee \right)(x) \chi^{(p)}_{k+p,2^l}(x),$$

where $M$ is the Hardy-Littlewood maximal function and $C > 0$ is a constant independent of $k$, $j$, $l$ and $x$.

Now let $\varphi_{-1} \equiv 0$. Then, for all $j \in \mathbb{N}_0$ and all $x \in \mathbb{R}^n$,

$$\sum_{\nu=-1}^{1} \varphi_{j+\nu} \varphi_j(x) = \varphi_j(x).$$
We then have
(2.9) \[
\left( \varphi_{j+\nu} \cdot (\beta qu)_{k+\nu,2^\nu t} \right)
= \sum_{\nu=-1}^{1} \left( \varphi_{j+\nu} \ast \left( \varphi_{j+\nu} \cdot (\beta qu)_{k+\nu,2^\nu t} \right) \right)
= \sum_{\nu=-1}^{1} \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} \varphi_{j+\nu}^y(u-y) \left( \varphi_{j+\nu} \cdot (\beta qu)_{k+\nu,2^\nu t} (u) \right) dy \right] (\beta qu)_{k+\nu,2^\nu t} (u) du
= \sum_{\nu=-1}^{1} \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} \varphi_{j+\nu}^y(u-y)(\beta qu)_{k+\nu,2^\nu t} (u) du \right] \left( \varphi_{j+\nu}^y \cdot (\beta qu)_{k+\nu,2^\nu t} (u) \right) dy.
\]

We now claim that for \( j \geq 2, \nu = -1, 0, 1, \) and any \( L \in \mathbb{N}_0, \)
(2.10) \[
\left| \int_{\mathbb{R}^n} \varphi_{j+\nu}^y(u-y)(\beta qu)_{k+\nu,2^\nu t} (u) du \right|
\leq C_L \frac{2^{(s+n/p')} 2^{(k-j)(L+1)} 2^{|\beta|} 2^{(k-j)\nu} \nu^{|n+1|}}{(1+2^k|y-2^{-k}L|)^{n+1}},
\]
where \( a \vee b = \max(a, b) \) and \( C_L > 0 \) is a constant independent of \( j, k, \beta, l \) and \( y. \)

We first note that for \( \alpha \in \mathbb{N}_0^n, \)
(2.11) \[
\int_{\mathbb{R}^n} \varphi_{j+\nu}^y(u-y)u^\alpha du = 0.
\]

To prove our claim, without loss of generality, we only show the case \( \nu = 0. \) By (2.11), we have
(2.12) \[
\left| \int_{\mathbb{R}^n} \varphi_j^y(u-y)(\beta qu)_{k+\nu,2^\nu t} (u) du \right|
= 2^{(j-1)n} \left| \int_{\mathbb{R}^n} \varphi_j^y(2^{j-1}u-2^{j-1}y) \left[(\beta qu)_{k+\nu,2^\nu t} (u) \right] \right|
= 2^{j-n} \sum_{\alpha \in \mathbb{N}_0^n, |\alpha| \leq L} \frac{1}{\alpha!} \left| D^\alpha (\beta qu)_{k+\nu,2^\nu t} (y) (u-y)^\alpha \right| du
\leq C 2^{jn} \sum_{\alpha \in \mathbb{N}_0^n, |\alpha| \leq L+1} \int_{\mathbb{R}^n} \left| \varphi_j^y(2^{j-1}u-2^{j-1}y) \right|
\times \left| D^\alpha (\beta qu)_{k+\nu,2^\nu t} ((1-\theta)u+\theta y) \right| |u-y|^{l+1} du,
\]
where \( \theta \in (0, 1). \)

By Definition 2.1, we have that for \( |\alpha| = L + 1, \)
(2.13) \[
\left| D^\alpha (\beta qu)_{k+\nu,2^\nu t} ((1-\theta)u+\theta y) \right| \leq C 2^{(s+n/p')} 2^{(L+1)2^{|\beta|}},
\]
and if $D^a(\beta qu)_{k+p,2^\ell}((1-\theta)u + \theta y) \neq 0$, then
\[
(2.14) \quad |2^{k+\rho}[(1-\theta)u + \theta y] - 2^\rho| < 2^\rho.
\]
From (2.14), it follows that
\[
(2.15) \quad 2^k|y - 2^{-k}t| \leq 2^k|y - u| + 2^k|u - 2^{-k}t|
\]
\[
\leq 2^k|y - u| + 2^k|u - y| + 2^k2^{\rho - k - \rho}
\]
\[
\leq 2^{k+1}|u - y| + 2^\rho 2^{-\rho}.
\]
By (2.13), (2.15) and (2.12), we then have
\[
\left| \int_{\mathbb{R}^n} \varphi^\vee_j(u - y)(\beta qu)_{k+p,2^\ell}(u) \, du \right|
\]
\[
\leq C \frac{2^{j\rho}2^{\rho/2}2^{k(L+1)}2^{(s+n/p')}}{(1 + 2^k|y - 2^{-k}t|)^{n+1}}
\]
\[
\times \int_{\mathbb{R}^n} \varphi^\vee_j(2^{j-1}u - 2^{j-1}y) \ |u - y|^{L+1} \ (1 + 2^k|u - y|)^{n+1} \, du
\]
\[
= C \frac{2^{j\rho}2^{k(L+1)}2^{h(s+n/p')2^{-j(L+1)}}}{(1 + 2^k|y - 2^{-k}t|)^{n+1}}
\]
\[
\times \int_{\mathbb{R}^n} \varphi^\vee_j(u) \ |u|^{L+1} \ (1 + 2^k|u|)^{n+1} \, du
\]
\[
\leq C \frac{2^{j\rho}2^{k(L+1)}2^{h(s+n/p')2^{-j(L+1)}}}{(1 + 2^k|y - 2^{-k}t|)^{n+1}}.
\]
Thus, our claim holds.

By (2.10) and (2.9), for $j \geq k$, we have
\[
(2.16) \quad \left| \left( \varphi_j \hat{f} \right)^\vee, (\beta qu)_{k+p,2^\ell}(x) \right|^{\lambda(k)^p}_{k+p,2^\ell}(x)
\]
\[
\leq C_L 2^{k-s-n/p'}2^{(k-\beta)(L+1)}2^{n\|\lambda(p)^p\|_{k+p,2^\ell}} \int_{\mathbb{R}^n} \left| \left( \varphi_j \hat{f} \right)^\vee(y) \right| (1 + 2^k|y - x|)^{n+1} \, dy
\]
\[
\leq C_L 2^{k-s-n/p'}2^{(k-\beta)(L+1)}2^{n\|\lambda(p)^p\|_{k+p,2^\ell}} M \left( \left( \varphi_j \hat{f} \right)^\vee(x) \right) \chi^{(p)}_{k+p,2^\ell}(x),
\]
where $C_L > 0$ is a constant independent of $k, j, \beta, l$ and $x$.

We now have
\[
\left\{ \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \left[ \varphi_j \hat{f} \right]_{k+p,2^\ell} \chi^{(p)}_{k+p,2^\ell}(x) \right\}^{1/q}
\]
\[
= C \left\{ \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} 2^{-\rho/2} \left| \left[ \left( \varphi_j \hat{f} \right)_{k+p,2^\ell} \right] \chi^{(p)}_{k+p,2^\ell}(x) \right|^{q} \right\}^{1/q}
\]
\[
= C \left\{ \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} 2^{-\rho |\beta| l} \left[ \sum_{j=0}^{\infty} \left( (\varphi_j \tilde{f}) \right)^{\vee} \right] \chi_{k+\rho, 2^q l}(x) \right\}^{\frac{q}{\mu}}
\]
\[
\leq C \left\{ \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} 2^{-\rho |\beta| l} \left[ \sum_{j=0}^{k+1} \left( (\varphi_j \tilde{f}) \right)^{\vee} \right] \chi_{k+\rho, 2^q l}(x) \right\}^{\frac{q}{\mu}}
\]
\[
+ C \left\{ \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} 2^{-\rho |\beta| l} \left[ \sum_{j=k+2}^{\infty} \left( (\varphi_j \tilde{f}) \right)^{\vee} \right] \chi_{k+\rho, 2^q l}(x) \right\}^{\frac{q}{\mu}}
\]
\[
= I_1 + I_2.
\]

For \( I_1 \), by (2.8), Hölder’s inequality and \( s < 0 \), we have

\[
(2.17)
\]
\[
I_1 \leq C \left\{ \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} 2^{-\rho |\beta| l} \left[ \sum_{j=0}^{k+1} M \left( 2^j \left( \varphi_j \tilde{f} \right)^{\vee} \right)(x) 2^{j-k}s \right] \right\}^{\frac{q}{\mu}}
\]
\[
= C 2^{(r-\rho) |\beta| l} \left\{ \sum_{k=0}^{\infty} \left[ \sum_{j=0}^{k+1} \left( 2^j \left( \varphi_j \tilde{f} \right)^{\vee} \right)(x) 2^{j-k}s \right]^{\frac{q}{\mu}} \right\}
\]
\[
\leq C 2^{(r-\rho) |\beta| l} \left\{ \sum_{k=0}^{\infty} \left[ \sum_{j=0}^{k+1} \left( 2^j \left( \varphi_j \tilde{f} \right)^{\vee} \right)(x) 2^{j-k}s \right]^{\frac{q}{\mu}} \right\}
\]
\[
\leq C 2^{(r-\rho) |\beta| l} \left\{ \sum_{j=0}^{\infty} \left[ \sum_{k=0}^{\infty} 2^{(k+1)j} \left( \varphi_j \tilde{f} \right)^{\vee} \chi_{k+\rho, 2^q l}(x) \right]^{\frac{q}{\mu}} \right\}
\]

where \( C > 0 \) is a constant independent of \( \beta \) and \( x \).

For \( I_2 \), we choose \( L \in \mathbb{N}_0 \) such that \( L + 1 > -s \). We then, by (2.16) and Hölder’s inequality, have that

\[
(2.18)
\]
\[
I_2 \leq C_L \left\{ \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} 2^{-\rho |\beta| l} \left[ \sum_{j=k+2}^{\infty} 2^{(k+1)(j+1) + L} 2^{(j+1)q} |\beta| 2^{kn/p - kn} \right] \right\}^{\frac{q}{\mu}}
\]
\[
\times M \left( \left( \varphi_j \tilde{f} \right)^{\vee} \right)(x) \chi_{k+\rho, 2^q l}(x) \right\}^{\frac{q}{\mu}}
\]
\[
\leq C_L 2^{(r-\rho) |\beta| l} \left\{ \sum_{k=0}^{\infty} \left[ \sum_{j=k+2}^{\infty} M \left( 2^j \left( \varphi_j \tilde{f} \right)^{\vee} \right)(x) 2^{(j+1)(L+1+s)} \right]^{\frac{q}{\mu}} \right\}
\]
\[
\begin{align*}
&\leq C_L 2^{(r-\rho)|\beta|}\left\{\sum_{k=0}^{\infty} \sum_{j=k+2}^{\infty} M\left(2^{i(j-\beta)} f\right)(x) 2^{(k-j)(L+1)+\rho} / q\right\}^{1/q} \\
&\leq C_L 2^{(r-\rho)|\beta|}\left\{\sum_{j=0}^{\infty} M\left(2^{i(j-\beta)} f\right)(x)\right\}^{1/q},
\end{align*}
\]
where \( C_L > 0 \) is a constant independent of \( \beta \) and \( x \).

By (2.17), (2.18) and the Fefferman-Stein vector-valued maximal inequality in [1], we have that the right hand side of (2.3) is controlled by

\[
C \left\| \sum_{j=0}^{\infty} \left\{ M\left(2^{i(j-\beta)} f\right)(\cdot)\right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| f \right\|_{F^p_{pq}(\mathbb{R}^n)},
\]
where \( C > 0 \) is a constant independent of \( f \). Thus, (2.3) holds.

We still need to verify that the series in (2.2) converge in the norm of \( F_{pq}^s(\mathbb{R}^n) \) when \( 1 < p, q < \infty \). By (2.3) and the following Proposition 2.2, we know that the series in (2.2) converge in the norm of \( F_{pq}^s(\mathbb{R}^n) \) to some, say, \( h \in F_{pq}^s(\mathbb{R}^n) \). Moreover, by Proposition 2.2 below, we know that the series in (2.2) also converge in \( S'\) to \( h \). Thus, \( h = f \) in \( S'\) and \( h \in F_{pq}^s(\mathbb{R}^n) \). Then, by Definition 1.1, we have also \( h = f \) in the norm of \( F_{pq}^s(\mathbb{R}^n) \). This just means that the series in (2.2) converge to \( f \) in the norm of \( F_{pq}^s(\mathbb{R}^n) \).

This finishes the proof of Proposition 2.1.

\[\square\]

**Proposition 2.2.** Let \( s < 0, 1 < p < \infty, 1 < q \leq \infty \) and \( \{\psi_{k,l}^{\beta} : \beta \in \mathbb{N}^n_0, k \in \mathbb{N}_0, l \in \mathbb{Z}^n\} \) be the same as in Proposition 2.1. If \( \{\gamma_{k,l}^{\beta} : \beta \in \mathbb{N}^n_0, k \in \mathbb{N}_0, l \in \mathbb{Z}^n\} \) is a sequence of complex numbers and

\[
(2.19) \quad \sup_{\beta \in \mathbb{N}^n_0} 2^{(r-\rho)|\beta|} \left\| \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \left| \gamma_{k,l}^{\beta} \Psi_{k,l}^{\beta} \right|^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty,
\]

where \( \rho > r \) and \( r \) is the same as in (2.1), then

\[
(2.20) \quad \sum_{\beta \in \mathbb{N}^n_0} \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \gamma_{k,l}^{\beta} \psi_{k,l}^{\beta},
\]

unconditionally converge in both \( S'(\mathbb{R}^n) \) and the norm of \( F_{pq}^s(\mathbb{R}^n) \) when \( 1 < p, q < \infty \) and only in \( S'(\mathbb{R}^n) \) when \( 1 < p < \infty \) and \( 1 < q \leq \infty \) to some \( f \in F_{pq}^s(\mathbb{R}^n) \).
\[ (2.21) \quad \|f\|_{F_p^s(\mathbb{R}^n)} \leq C \sup_{\beta \in \mathbb{N}_0^n} 2^{(\rho-\rho_\beta)|\beta|} \left\| \left( \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \left| \gamma_{k+l}^{(p)} \beta \chi_{k+l}^{(p)} \right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}, \]

where \( C > 0 \) is a constant independent of \( f \).

**Proof.** Obviously, when \( 1 < p, q < \infty \), we only need to show that the series in (2.20) converge in the norm of \( F_p^s(\mathbb{R}^n) \) since \( F_p^s(\mathbb{R}^n) \subset S'(\mathbb{R}^n) \); see [6, Theorem 2.3.3]. By noting (2.7), where \( \{ \Psi_{k,l}^{(p)} : \beta \in \mathbb{N}_0^n, \ k \in \mathbb{N}_0, \ l \in \mathbb{Z}^n \} \) comes from, and by using the same notation as in (2.7), we can re-write (2.20) into

\[ (2.22) \quad \sum_{\beta \in \mathbb{N}_0^n} \sum_{l \in \mathbb{Z}^n} \sum_{k=0}^{\infty} \gamma_{k+l}^{(p)} \beta \Psi_{k+l}^{(p)}, \]

and (2.19) into

\[ (2.23) \quad \sup_{\beta \in \mathbb{N}_0^n} 2^{(\rho-\rho_\beta)|\beta|} \left\| \left( \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \left| \gamma_{k+l}^{(p)} \beta \chi_{k+l}^{(p)} \right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty. \]

Let us now show the conclusions for the series in (2.22) under the assumption (2.23). To prove that the first series in (2.22) converges in the norm of \( F_p^s(\mathbb{R}^n) \) when \( 1 < p, q < \infty \), for \( L \in \mathbb{N} \), we set

\[ S_L = \sum_{l \in \mathbb{Z}^n, |l| \leq L} \gamma_{k+l}^{(p)} \beta \Psi_{k+l}^{(p)}. \]

We now use a dual argument. Let \( g \in S(\mathbb{R}^n) \). By (2.5) and (2.6), we then have that for \( L_1, L_2 \in \mathbb{N} \) and \( L_1 < L_2 \),

\[ (2.24) \quad \| (S_{L_2} - S_{L_1}, g) \|
\]

\[ = \left| \sum_{l \in \mathbb{Z}^n, L_1 < |l| \leq L_2} \gamma_{k+l}^{(p)} \beta \Psi_{k+l}^{(p)} \right| (2.22)
\]

\[ = C \left| \sum_{l \in \mathbb{Z}^n, L_1 < |l| \leq L_2} 2^{(\rho-\rho_\beta)(k+|l|)} \gamma_{k+l}^{(p)} \beta \Psi_{k+l}^{(p)} \right| (2.23)
\]

\[ = C 2^{\rho|\beta|} \left| \sum_{l \in \mathbb{Z}^n, L_1 < |l| \leq L_2} \gamma_{k+l}^{(p)} \beta \Psi_{k+l}^{(p)} \int_{\mathbb{R}^n} \chi_{k+p,2\nu_1}^{(p)}(x) \chi_{k+p,2\nu_1}^{(p)}(x) dx \right|
\]

\[ \leq C 2^{\rho|\beta|} \left[ \sum_{l \in \mathbb{Z}^n, L_1 < |l| \leq L_2} \left| \gamma_{k+l}^{(p)} \beta \chi_{k+l}^{(p)}(x) \right|^q \right]^{1/q}, \]
\[
\times \left[ \sum_{l \in \mathbb{Z}^n, L_1 < |l| \leq L_2} \left| \frac{\lambda^{\rho}_{k+\rho, 2n} \chi_{k+\rho, 2n}^{(p)}(x)}{\gamma_{k+\rho, 2n}^{(p)} \chi_{k+\rho, 2n}^{(p)}(\cdot)} \right|^q \right]^{1/q} dx \\
\leq C 2^{qL_2} \left[ \sum_{l \in \mathbb{Z}^n, L_1 < |l| \leq L_2} \left| \frac{\gamma_{k+\rho, 2n}^{(p)} \chi_{k+\rho, 2n}^{(p)}(\cdot)}{\gamma_{k+\rho, 2n}^{(p)} \chi_{k+\rho, 2n}^{(p)}(\cdot)} \right|^q \right]^{1/q} \\
\times \left[ \sum_{l \in \mathbb{Z}^n, L_1 < |l| \leq L_2} \left| \frac{\lambda^{\rho}_{k+\rho, 2n} \chi_{k+\rho, 2n}^{(p)}(\cdot)}{\gamma_{k+\rho, 2n}^{(p)} \chi_{k+\rho, 2n}^{(p)}(\cdot)} \right|^q \right]^{1/q} \\
\leq C \|g\|_{F^{s}_{pq}(\mathbb{R}^n)} \left[ \sum_{l \in \mathbb{Z}^n, L_1 < |l| \leq L_2} \left| \frac{\gamma_{k+\rho, 2n}^{(p)} \chi_{k+\rho, 2n}^{(p)}(\cdot)}{\gamma_{k+\rho, 2n}^{(p)} \chi_{k+\rho, 2n}^{(p)}(\cdot)} \right|^q \right]^{1/q} ,
\]

By Lemma 1.1, when \(1 < p < \infty\) and \(1 < q \leq \infty\), we further have
\[
(2.25) \quad \|S_{L_2} - S_{L_1}\|_{F^{s}_{pq}(\mathbb{R}^n)} = \sup_{\|g\|_{F^{s}_{pq}(\mathbb{R}^n)} \leq 1} \|(S_{L_2} - S_{L_1}, g)\| \\
\leq C \left[ \sum_{l \in \mathbb{Z}^n, L_1 < |l| \leq L_2} \left| \frac{\gamma_{k+\rho, 2n}^{(p)} \chi_{k+\rho, 2n}^{(p)}(\cdot)}{\gamma_{k+\rho, 2n}^{(p)} \chi_{k+\rho, 2n}^{(p)}(\cdot)} \right|^q \right]^{1/q} ,
\]

where \(C > 0\) is a constant independent of \(L_1\) and \(L_2\). By (2.23), we know that if \(1 < p, q < \infty\), as \(L_1, L_2 \to \infty\),
\[
(2.26) \quad \|S_{L_2} - S_{L_1}\|_{F^{s}_{pq}(\mathbb{R}^n)} \to 0.
\]

Thus, \(\{S_L\}_{L \in \mathbb{N}}\) is a Cauchy sequence in \(F^{s}_{pq}(\mathbb{R}^n)\) when \(1 < p, q < \infty\). Since \(F^{s}_{pq}(\mathbb{R}^n)\) is a Banach space, we then know that \(\{S_L\}_{L \in \mathbb{N}}\) converges as \(L \to \infty\) in the norm of \(F^{s}_{pq}(\mathbb{R}^n)\). This means that the interior series in
\[
(2.22) \quad \sum_{l \in \mathbb{Z}^n, |l| \leq L_2} \left| \frac{\gamma_{k+\rho, 2n}^{(p)} \chi_{k+\rho, 2n}^{(p)}(\cdot)}{\gamma_{k+\rho, 2n}^{(p)} \chi_{k+\rho, 2n}^{(p)}(\cdot)} \right|^q \\
\to 0,
\]
as \(L_1, L_2 \to \infty\). From (2.27), it still follows that for any given \(g \in \mathcal{S}(\mathbb{R}^n)\),
\[
\|(S_{L_2} - S_{L_1}, g)\| \to 0,
\]
as \(L_1, L_2 \to \infty\). This just means that the first series in (2.22) converges in \(\mathcal{S}'(\mathbb{R}^n)\) when \(1 < p < \infty\) and \(q = \infty\). Similarly, we can show the other series in (2.22) also converge in the norm of \(F^{s}_{pq}(\mathbb{R}^n)\) when \(1 < p, q < \infty\).
and in $S'(\mathbb{R}^n)$ when $1 < p < \infty$, $1 < q \leq \infty$. Thus, the series in (2.22) converge in the norm of $F_{pq}^\ast(\mathbb{R}^n)$ when $1 < p$, $q < \infty$ and in $S'(\mathbb{R}^n)$ when $1 < p < \infty$ and $1 < q \leq \infty$ to some, say, $f \in F_{pq}^\ast(\mathbb{R}^n)$. We now verify (2.21). Let $g \in S(\mathbb{R}^n)$. By (2.5) and (2.6), we then have

$$
|\langle f, g \rangle| = \left| \sum_{\beta \in \mathbb{N}_0^n} \sum_{\ell = 0}^{\infty} \sum_{i \in \mathbb{Z}^n} \gamma_{\beta+k+\rho,2^{\ell}i} \left( \Psi_{\beta+k+\rho,2^{\ell}i} g \right) \right|
$$

$$
= C \sum_{\beta \in \mathbb{N}_0^n} 2^{\beta |\beta|} \sum_{\ell = 0}^{\infty} \sum_{i \in \mathbb{Z}^n} \gamma_{\beta+k+\rho,2^{\ell}i} \left( \Psi_{\beta+k+\rho,2^{\ell}i} g \right) \left( \int_{\mathbb{R}^n} \chi_{\beta+k+\rho,2^{\ell}i}(x) \chi_{\beta+k+\rho,2^{\ell}i}(x) \, dx \right)^{1/q}
$$

$$
\leq C \sum_{\beta \in \mathbb{N}_0^n} 2^{\beta |\beta|} \left[ \sum_{i \in \mathbb{Z}^n} \left( \int_{\mathbb{R}^n} \left( \sum_{\ell = 0}^{\infty} \gamma_{\beta+k+\rho,2^{\ell}i} \chi_{\beta+k+\rho,2^{\ell}i}(x) \right)^q \right)^{1/q} \right]^{1/q} \left[ \sum_{\ell = 0}^{\infty} \sum_{i \in \mathbb{Z}^n} \chi_{\beta+k+\rho,2^{\ell}i}(x) \right]^{1/q} \left[ \sum_{\ell = 0}^{\infty} \sum_{i \in \mathbb{Z}^n} \chi_{\beta+k+\rho,2^{\ell}i}(x) \right]^{1/q} \left[ \sum_{i \in \mathbb{Z}^n} \left( \int_{\mathbb{R}^n} \left( \sum_{\ell = 0}^{\infty} \gamma_{\beta+k+\rho,2^{\ell}i} \chi_{\beta+k+\rho,2^{\ell}i}(x) \right)^q \right)^{1/q} \right]^{1/q}
$$

$$
\leq C \|g\|_{F_{pq}^\ast(\mathbb{R}^n)} \sum_{\beta \in \mathbb{N}_0^n} \left[ \sum_{i \in \mathbb{Z}^n} \left( \int_{\mathbb{R}^n} \left( \sum_{\ell = 0}^{\infty} \gamma_{\beta+k+\rho,2^{\ell}i} \chi_{\beta+k+\rho,2^{\ell}i}(x) \right)^q \right)^{1/q} \right]^{1/q},
$$

where $C > 0$ is a constant independent of $f$ and $g$. By Lemma 1.1 again, we have

$$
\|\langle f, g \rangle\| = \|g\|_{F_{pq}^\ast(\mathbb{R}^n)} \sup_{\|g\|_{F_{pq}^\ast(\mathbb{R}^n)} \leq 1} \|\langle f, g \rangle\|
$$

$$
\leq C \sum_{\beta \in \mathbb{N}_0^n} \left[ \sum_{i \in \mathbb{Z}^n} \left( \int_{\mathbb{R}^n} \left( \sum_{\ell = 0}^{\infty} \gamma_{\beta+k+\rho,2^{\ell}i} \chi_{\beta+k+\rho,2^{\ell}i}(x) \right)^q \right)^{1/q} \right]^{1/q} \left[ \sum_{\ell = 0}^{\infty} \sum_{i \in \mathbb{Z}^n} \chi_{\beta+k+\rho,2^{\ell}i}(x) \right]^{1/q} \left[ \sum_{\ell = 0}^{\infty} \sum_{i \in \mathbb{Z}^n} \chi_{\beta+k+\rho,2^{\ell}i}(x) \right]^{1/q} \left[ \sum_{i \in \mathbb{Z}^n} \left( \int_{\mathbb{R}^n} \left( \sum_{\ell = 0}^{\infty} \gamma_{\beta+k+\rho,2^{\ell}i} \chi_{\beta+k+\rho,2^{\ell}i}(x) \right)^q \right)^{1/q} \right]^{1/q},
$$

where $C > 0$ is a constant independent of $f$. 
This finishes the proof of Proposition 2.2.

Combining Proposition 2.1 with Proposition 2.2 leads us to the following theorem.

**Theorem 2.1.** Let \( s < 0 \), \( 1 < p < \infty \), \( 1 < q \leq \infty \) and \( \{ \Psi^\beta_k : \beta \in \mathbb{N}_0^n, k \in \mathbb{N}_0, l \in \mathbb{Z}_n \} \) be the same as in Proposition 2.1. Then \( f \in \mathcal{S}'(\mathbb{R}^n) \) is an element of \( F_p^s(\mathbb{R}^n) \) if, and only if, it can be represented as in (2.2) with a sequence of complex numbers \( \{ \gamma^\beta_{k,l} : \beta \in \mathbb{N}_0^n, k \in \mathbb{N}_0, l \in \mathbb{Z}_n \} \) satisfying (2.19), where \( \rho > r \) and \( r \) is the same as in (2.1), and (2.2) holds unconditionally in \( \mathcal{S}'(\mathbb{R}^n) \). Furthermore,

\[
\|f\|_{F_p^s(\mathbb{R}^n)} \sim \inf \left\{ \sup_{\beta \in \mathbb{N}_0^n} 2^{(\rho - r)|\beta|} \left\| \left( \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} |\gamma^\beta_{k,l} x_k|^p \right)^{1/p} \right\|_{L_p(\mathbb{R}^n)} \right\},
\]

where the infimum is taken over all admissible representations (2.2).

By a proof similar to that for Proposition 2.1 and Proposition 2.2, respectively, we can establish the following theorems on the spaces \( B_p^s(\mathbb{R}^n) \). We omit the details.

**Proposition 2.3.** Let \( s < 0 \) and \( 1 \leq p, q \leq \infty \). There is a set of Schwartz functions, \( \{ \Psi^\beta_k : \beta \in \mathbb{N}_0^n, k \in \mathbb{N}_0, l \in \mathbb{Z}_n \} \), such that for any \( f \in B_p^s(\mathbb{R}^n) \), (2.2) holds unconditionally in both the norm of \( B_p^s(\mathbb{R}^n) \) and \( \mathcal{S}'(\mathbb{R}^n) \) when \( 1 \leq p, q < \infty \), and only in \( \mathcal{S}'(\mathbb{R}^n) \) when \( 1 \leq p, q \leq \max(p,q) = \infty \), where, when \( k \in \mathbb{N} \), for all \( \alpha \in \mathbb{N}_0^n \),

\[
\int_{\mathbb{R}^n} \Psi^\beta_k(x) x^{\alpha} \, dx = 0,
\]

and \( \{ \gamma^\beta_{k,l} : \beta \in \mathbb{N}_0^n, k \in \mathbb{N}_0, l \in \mathbb{Z}_n \} \) is a sequence of numbers linearly depending on \( f \) and satisfying

\[
\sup_{\beta \in \mathbb{N}_0^n} 2^{(\rho - r)|\beta|} \left\{ \left( \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} |\gamma^\beta_{k,l}|^p \right)^{1/p} \right\} \leq C \|f\|_{B_p^s(\mathbb{R}^n)} \]

with the same \( r \) as in (2.1), \( \rho > r \) and a constant \( C > 0 \) independent of \( f \).

**Proposition 2.4.** Let \( s < 0 \), \( 1 \leq p, q \leq \infty \) and \( \{ \Psi^\beta_k : \beta \in \mathbb{N}_0^n, k \in \mathbb{N}_0, l \in \mathbb{Z}_n \} \) be the same as in Proposition 2.3. If \( \{ \gamma^\beta_{k,l} : \beta \in \mathbb{N}_0^n, k \in \mathbb{N}_0, l \in \mathbb{Z}_n \} \) is a sequence of complex numbers and

\[
(2.28) \quad \sup_{\beta \in \mathbb{N}_0^n} 2^{(\rho - r)|\beta|} \left\{ \left( \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} |\gamma^\beta_{k,l}|^p \right)^{1/p} \right\} < \infty,
\]

where \( \rho > r \) and \( r \) is the same as in (2.1), then (2.20) unconditionally converge in both \( \mathcal{S}'(\mathbb{R}^n) \) and the norm of \( B_p^s(\mathbb{R}^n) \) when \( 1 \leq p, q < \infty \) and...
only in $\mathcal{S}'(\mathbb{R}^n)$ when $1 \leq p, q \leq \max(p, q) = \infty$ to some $f \in B_{p,q}^s(\mathbb{R}^n)$. Moreover,

$$
\|f\|_{B_{p,q}^s(\mathbb{R}^n)} \leq C \sup_{\beta \in \mathbb{N}_0^n} 2^{(p-r)|\beta|} \left\{ \left( \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} |\gamma_{k,l}^\beta|^p \right)^{q/p} \right\}^{1/q},
$$

where $C > 0$ is a constant independent of $f$.

Combining Proposition 2.1 with Proposition 2.2 yields the following result.

**Theorem 2.2.** Let $s < 0$, $1 \leq p, q \leq \infty$ and $\{\Psi_k^\beta : \beta \in \mathbb{N}_0^n, k \in \mathbb{N}_0, l \in \mathbb{Z}^n\}$ be the same as in Proposition 2.3. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ is an element of $B_{p,q}^s(\mathbb{R}^n)$ if, and only if, it can be represented as in (2.2) with a sequence of complex numbers $\{\gamma_{k,l}^\beta : \beta \in \mathbb{N}_0^n, k \in \mathbb{N}_0, l \in \mathbb{Z}^n\}$ satisfying (2.28), where $p > r$ and $r$ is the same as in (2.1), and (2.2) holds unconditionally in $\mathcal{S}'(\mathbb{R}^n)$. Furthermore,

$$
\|f\|_{B_{p,q}^s(\mathbb{R}^n)} \sim \inf \left\{ \sup_{\beta \in \mathbb{N}_0^n} 2^{(p-r)|\beta|} \left\{ \left( \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} |\gamma_{k,l}^\beta|^p \right)^{q/p} \right\}^{1/q} \right\},
$$

where the infimum is taken over all admissible representations (2.2).

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