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Traces of multipliers in pairs of weighted Sobolev spaces

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Abstract. We prove that the pointwise multipliers acting in a pair of fractional Sobolev spaces form the space of boundary traces of multipliers in a pair of weighted Sobolev space of functions in a domain.

1. Introduction

By a multiplier acting from one Banach function space S_1 into another S_2 we call a function γ such that $\gamma u \in S_2$ for any $u \in S_1$. By $M(S_1 \to S_2)$ we denote the space of multipliers $\gamma : S_1 \to S_2$ with the norm

$$\|\gamma\|_{M(S_1 \to S_2)} = \sup\{\|\gamma u\|_{S_2} : \|u\|_{S_1} \leq 1\}.$$

We write MS instead of $M(S \to S)$, where S is a Banach function space. We shall use the same notation both for spaces of scalar and vector-valued multipliers.

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Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. It is well known that the fractional Sobolev space $W_p^l(\partial\Omega)$ is the space of traces of the weighted Sobolev space $W_p^{s,\alpha}(\Omega)$ endowed with the norm

$$\left(\int_\Omega \big(\mathrm{dist}(x,\partial\Omega)\big)^{p\alpha} \sum_{\{\tau:0\le|\tau|\le s\}} |D^\tau u|^p dx\right)^{1/p},$$

where $\alpha = 1-\{l\}-1/p$, $s=[l]+1$ and $p\in(1,\infty)$ (see [5]). It is straightforward to deduce from this fact that the trace γ of the function

$$\Gamma \in M(W_p^{t,\beta}(\Omega)\to W_p^{s,\alpha}(\Omega)) \tag{1}$$

belongs to $M(W_p^m(\partial\Omega)\to W_p^l(\partial\Omega))$. Here m and l are nonintegers, $m\ge l>0$, s and α are given above, $t=[m]+1$, $\beta=1-\{m\}-1/p$.

In the present paper we prove that the converse assertion is also true showing that there exists an extension Γ of $\gamma\in M(W_p^m(\partial\Omega)\to W_p^m(\partial\Omega))$ subject to (1).

2. The space $M(W_p^m(\mathbb{R}^{n-1})\to W_p^l(\mathbb{R}^{n-1}))$

By $\mathcal{B}_r^{n-1}(x)$ we mean the ball $\{\xi\in\mathbb{R}^{n-1}:|\xi-x|<r\}$ and write $\mathcal{B}_r^{n-1}$ instead of $\mathcal{B}_r^{n-1}(0)$.

We need the spaces S_{loc} and S_{unif} of functions on $\mathbb{R}^{n-1}$ defined as follows. By S_{loc} we denote the space

$$\{u:\ \eta u\in S \ \text{ for all } \ \eta\in C_0^\infty(\mathbb{R}^{n-1})\}$$

and by S_{unif} we mean the space

$$\{u:\ \sup_\xi \|\eta_\xi u\|_S<\infty\},$$

where $\eta_\xi(x)=\eta(x-\xi)$, $\eta\in C_0^\infty(\mathbb{R}^{n-1})$, $\eta=1$ on $\mathcal{B}_1^{n-1}$. The space S_{unif} is endowed with the norm

$$\|u\|_{S_{unif}}=\sup_\xi\|\eta_\xi u\|_S.$$

Let $W_p^l(\mathbb{R}^{n-1})$ denote the fractional Sobolev space with the norm

$$\|D_{p,l}u;\mathbb{R}^{n-1}\|_{L_p}+\|u;\mathbb{R}^{n-1}\|_{L_p},$$

where

$$(2)\qquad (D_{p,l}u)(x) = \left(\int_{\mathbb{R}^{n-1}} |\nabla_{[l]}u(x+h) - \nabla_{[l]}u(x)|^p |h|^{1-n-p\{l\}} dh\right)^{1/p},$$

with $\nabla_{[l]}$ being the gradient of order $[l]$, i.e. $\nabla_{[l]} = \{\partial_{x_1}^{\tau_1}, \dots, \partial_{x_{n-1}}^{\tau_{n-1}}\}$, $\tau_1 + \dots + \tau_{n-1} = [l]$.

In this section we collect some known properties of multipliers between fractional Sobolev spaces $W_p^m(\mathbb{R}^{n-1})$ and $W_p^l(\mathbb{R}^{n-1})$, $m \geq l \geq 0$. The equivalence $a \sim b$ means that a/b is bounded and separated from zero by positive constants depending on n, p, m, and l.

Proposition 1 ([3]). *Let m and l be nonintegers, $m \geq l \geq 0$, and let $p \in (1, \infty)$.*

(i) *There holds*

$$\|\gamma; \mathbb{R}^{n-1}\|_{M(W_p^m \to W_p^l)} \sim \|D_{p,l}\gamma; \mathbb{R}^{n-1}\|_{M(W_p^m \to L_p)} + \|\gamma; \mathbb{R}^{n-1}\|_{M(W_p^{m-l} \to L_p)}.$$

(ii) *If $\gamma \in M(W_p^m(\mathbb{R}^{n-1}) \to W_p^l(\mathbb{R}^{n-1}))$ then for any multi-index σ, $|\sigma| \leq [l]$,*

$$D^\sigma \gamma \in M(W_p^m(\mathbb{R}^{n-1}) \to W_p^{l-|\sigma|}(\mathbb{R}^{n-1})).$$

(iii) *Let $0 < \lambda < \mu$. Then*

$$\|\gamma^{\lambda/\mu}; \mathbb{R}^{n-1}\|_{M(W_p^\lambda \to L_p)} \leq c\, \|\gamma; \mathbb{R}^{n-1}\|^{\lambda/\mu}_{M(W_p^\mu \to L_p)}.$$

Proposition 2 ([3]). *Let m and l be nonintegers, $m \geq l \geq 0$, and let $p \in (1, \infty)$. Then*

$$\|\gamma; \mathbb{R}^{n-1}\|_{M(W_p^m \to W_p^l)} \sim \sup_{\substack{e \subset \mathbb{R}^{n-1} \\ \operatorname{diam}(e) \leq 1}} \frac{\|D_{p,l}\gamma; e\|_{L_p}}{(\operatorname{cap}_{p,m}(e))^{1/p}}$$

$$+ \begin{cases} \sup\limits_{x \in \mathbb{R}^{n-1}} \|\gamma; \mathcal{B}_1^{n-1}(x)\|_{L_1} & \text{for } m > l, \\ \|\gamma; \mathbb{R}^{n-1}\|_{L_\infty} & \text{for } m = l, \end{cases}$$

where e is a compact set in $\mathbb{R}^{n-1}$ and $\operatorname{cap}_{p,m}(e)$ is the (p,m)-capacity of e defined by

$$\operatorname{cap}_{p,m}(e) = \inf\{\|u; \mathbb{R}^{n-1}\|^p_{W_p^m} : u \in C_0^\infty(\mathbb{R}^{n-1}),\ u \geq 1 \text{ on } e\}$$

For $l = 0$ one should replace $D_{p,l}\gamma$ by γ.

Upper estimates for the norm in $M(W_p^m(\mathbb{R}^{n-1}) \to W_p^l(\mathbb{R}^{n-1}))$ are given in the following assertion. By mes_{n-1} we mean the $(n-1)$-dimensional Lebesgue measure of a compact set e.

Proposition 3 ([3]). *Let m and l be nonintegers, $m \geq l \geq 0$, and let $p \in (1, \infty)$.*

(i) *If $mp < n-1$, then*

$$c\|\gamma; \mathbb{R}^{n-1}\|_{M(W_p^m \to W_p^l)} \leq \sup_{\substack{e \subset \mathbb{R}^n \\ \mathrm{diam}(e) \leq 1}} \frac{\|D_{p,l}\gamma; e\|_{L_p}}{(\mathrm{mes}_{n-1}(e))^{1/p - m/(n-1)}} + \begin{cases} \sup\limits_{x \in \mathbb{R}^{n-1}} \|\gamma; \mathcal{B}_1^{n-1}(x)\|_{L_1} & \text{for } m > l, \\ \|\gamma; \mathbb{R}^{n-1}\|_{L_\infty} & \text{for } m = l. \end{cases}$$

(ii) *If $mp = n-1$, then*

$$c\|\gamma; \mathbb{R}^{n-1}\|_{M(W_p^m \to W_p^l)} \leq \sup_{\substack{e \subset \mathbb{R}^{n-1} \\ \mathrm{diam}(e) \leq 1}} \Big(\log \frac{2^{n-1}}{\mathrm{mes}_{n-1}(e)}\Big)^{1-1/p} \|D_{p,l}\gamma; e\|_{L_p} + \begin{cases} \sup\limits_{x \in \mathbb{R}^{n-1}} \|\gamma; \mathcal{B}_1^{n-1}(x)\|_{L_1} & \text{for } m > l, \\ \|\gamma; \mathbb{R}^{n-1}\|_{L_\infty} & \text{for } m = l. \end{cases}$$

Now we list lower estimates for the norm in $M(W_p^m \to W_p^l)$.

Proposition 4. [3] *Let m and l be nonintegers, $m \geq l \geq 0$, and let $p \in (1, \infty)$.*

(i) *If $mp < n-1$, then*

$$c\|\gamma; \mathbb{R}^{n-1}\|_{M(W_p^m \to W_p^l)} \geq \sup_{\substack{x \in \mathbb{R}^{n-1} \\ r \in (0,1)}} \frac{\|D_{p,l}\gamma; \mathcal{B}_r^{n-1}(x)\|_{L_p}}{r^{(n-1)/p - m}} + \begin{cases} \sup\limits_{x \in \mathbb{R}^{n-1}} \|\gamma; \mathcal{B}_1^{n-1}(x)\|_{L_1} & \text{for } m > l, \\ \|\gamma; \mathbb{R}^{n-1}\|_{L_\infty} & \text{for } m = l. \end{cases}$$

(ii) *If $mp = n-1$, then*

$$c\|\gamma; \mathbb{R}^{n-1}\|_{M(W_p^m \to W_p^l)} \geq \sup_{\substack{x \in \mathbb{R}^n \\ r \in (0,1)}} \Big(\log \frac{2}{r}\Big)^{1-1/p} \|D_{p,l}\gamma; \mathcal{B}_r^{n-1}(x)\|_{L_p} + \begin{cases} \sup\limits_{x \in \mathbb{R}^{n-1}} \|\gamma; \mathcal{B}_1^{n-1}(x)\|_{L_1} & \text{for } m > l, \\ \|\gamma; \mathbb{R}^{n-1}\|_{L_\infty} & \text{for } m = l. \end{cases}$$

3. Multipliers in pairs of weighted Sobolev spaces in $\mathbb{R}^n_+$

3.1 Preliminary facts. Let $\mathbb{R}^n_+$ denote the upper half-space $\{z = (x, y) : x \in \mathbb{R}^{n-1},\ y > 0\}$. We introduce the weighted Sobolev space $W_p^{s,\alpha}(\mathbb{R}^n_+)$ with the norm

$$\|(\min\{1, y\})^\alpha \nabla_s U; \mathbb{R}^n_+\|_{L_p} + \|(\min\{1, y\})^\alpha U; \mathbb{R}^n_+\|_{L_p}, \tag{3}$$

where s is nonnegative integer. We always asume that $-1 < \alpha p < p - 1$.

It is well known that the fractional Sobolev space $W_p^l(\mathbb{R}^{n-1})$, is the space of traces on $\mathbb{R}^{n-1}$ of functions in the space $W_p^{s,\alpha}(\mathbb{R}^n_+)$, where $s = [l] + 1$, $\alpha = 1 - \{l\} - 1/p$, and $p \in (1, \infty)$ (see [5]). We show that a similar result holds for spaces of pointwise multipliers acting in a pair of fractional Sobolev spaces. To be precise, we prove that for all noninteger m and l, $m \geq l > 0$, the multiplier space $M(W_p^m(\mathbb{R}^{n-1}) \to W_p^l(\mathbb{R}^{n-1}))$ is the space of traces on $\mathbb{R}^{n-1}$ of functions in $M(W_p^{t,\beta}(\mathbb{R}^n_+) \to W_p^{s,\alpha}(\mathbb{R}^n_+))$, where s and α are as above and $\beta = 1 - \{m\} - 1/p$, $t = [m] + 1$. Different positive constants depending on n, p, l, m, s, t will be denoted by c. We shall omit $\mathbb{R}^n_+$ in notations of norms.

We introduce the notion of (p, s, α)-capacity of a compact set $e \subset \mathbb{R}^n_+$:

$$\mathrm{cap}_{p,s,\alpha}(e) = \inf\{\|U; \mathbb{R}^n_+\|^p_{W_p^{s,\alpha}} :\ \ U \in C_0^\infty(\mathbb{R}^n_+),\ \ U \geq 1 \ \text{ on } \ e\}.$$

The following result is essentially known (see [2], Sections 8.1, 8.2).

Proposition 5. *Let k be a nonnegative integer, $-1 < \beta p < p - 1$, and let $1 < p < \infty$. Then $\Gamma \in M(W_p^{k,\beta}(\mathbb{R}^n_+) \to W_p^{0,\alpha}(\mathbb{R}^n_+))$ if and only if*

$$\sup_{\substack{e \subset \mathbb{R}^n_+ \\ \mathrm{diam}(e) \leq 1}} \frac{\|(\min\{1, y\})^\alpha \Gamma; e\|_{L_p}}{(\mathrm{cap}_{p,k,\beta}(e))^{1/p}} < \infty\,.$$

The equivalence relation

$$\|\Gamma\|_{M(W_p^{k,\beta} \to W_p^{0,\alpha})} \sim \sup_{\substack{e \subset \mathbb{R}^n_+ \\ \mathrm{diam}(e) \leq 1}} \frac{\|(\min\{1, y\})^\alpha \Gamma; e\|_{L_p}}{(\mathrm{cap}_{p,k,\beta}(e))^{1/p}} \tag{4}$$

is valid.

We shall use some general properties of multipliers. We start with the inequality

$$\|\Gamma\|_{M(W_p^{t-j,\beta} \to W_p^{s-j,\alpha})} \leq c\, \|\Gamma\|^{(s-j)/s}_{M(W_p^{t,\beta} \to W_p^{s,\alpha})} \|\Gamma\|^{j/s}_{M(W_p^{t-s,\beta} \to W_p^{0,\alpha})}, \tag{5}$$

where $0 \le j \le s$, $-1 < \alpha p < p-1$, $-1 < \beta p < p-1$, which follows from the interpolation property of weighted Sobolev spaces (see [4], Section 3.4.2).

The next assertion contains inequalities between multipliers and their mollifiers in variables x.

Lemma 1. *Let Γ_ρ denote a mollifier of a function Γ defined by*

$$\Gamma_\rho(x,y) = \rho^{-n+1} \int_{\mathbb{R}^{n-1}} K(\rho^{-1}(x-\xi))\Gamma(\xi,y)d\xi,$$

where $K \in C_0^\infty(\mathcal{B}_1^{n-1})$, $K \ge 0$, and $\|K;\mathbb{R}^{n-1}\|_{L_1} = 1$. Then

$$\|\Gamma_\rho\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})} \le \|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})} \le \liminf_{\rho\to 0} \|\Gamma_\rho\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})}. \tag{6}$$

Proof. Let $U \in C_0^\infty$. By Minkowski's inequality

$$\left(\int_{\mathbb{R}^n_+} (\min\{1,y\})^{p\alpha} \left|\nabla_{j,z} \int_{\mathbb{R}^{n-1}} \rho^{-n} K(\xi/\rho)\Gamma(x-\xi,y)U(x,y)d\xi\right|^p dz\right)^{1/p}$$

$$\le \int_{\mathbb{R}^{n-1}} \rho^{-n} K(\xi/\rho)\left(\int_{\mathbb{R}^n_+} (\min\{1,y\})^{p\alpha} |\nabla_{j,z}\big(\Gamma(x,y)U(x+\xi,y)\big)|^p dz\right)^{1/p} d\xi,$$

where $j = 0, s$. Therefore,

$$\begin{aligned}
&\|\Gamma_\rho u\|_{W_p^{s,\alpha}} \le \|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})} \\
&\quad \times \int_{\mathbb{R}^{n-1}} \rho^{-n} K(\xi/\rho)\Big\{\left(\int_{\mathbb{R}^n_+} (\min\{1,y\})^{p\beta} |\nabla_{t,z} U(x+\xi,y)|^p dz\right)^{1/p} \\
&\quad + \left(\int_{\mathbb{R}^n_+} (\min\{1,y\})^{p\beta} |U(x+\xi,y)|^p dz\right)^{1/p}\Big\} d\xi \\
&\qquad\qquad \le \|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})} \|U\|_{W_p^{t,\beta}}.
\end{aligned}$$

This gives the left inequality (6). The right inequality (6) follows from

$$\|\Gamma u\|_{W_p^{s,\alpha}} = \liminf_{\rho\to 0} \|\Gamma_\rho U\|_{W_p^{s,\alpha}} \le \liminf_{\rho\to 0} \|\Gamma_\rho\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})} \|U\|_{W_p^{t,\beta}}.$$

The proof is complete. □

Lemma 2. *Let $\Gamma \in L_{p,loc}$, $p \in (1,\infty)$, $-1 < \beta p < p-1$, and let U be an arbitrary function in $C_0^\infty(\mathbb{R}^n_+)$. The best constant in the inequality*

$$\|(\min\{1,y\})^\alpha \Gamma \nabla_s U\|_{L_p} + \|(\min\{1,y\})^\alpha \Gamma U\|_{L_p} \le C\, \|U\|_{W_p^{t,\beta}} \tag{7}$$

is equivalent to the norm $\|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})}$.

Proof. The estimate $C \le c\|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})}$ is obvious. To derive the converse estimate, we introduce a function $x \to \sigma$ which is positive on $[0,\infty)$ and is equal to x for $x > 1$. For any $U \in C_0^\infty(\mathbb{R}_+^n)$ there holds

$$U = (-\Delta)^s\big(\sigma(-\Delta)\big)^{-[l]-1}u + T(-\Delta)u,$$

where T is a function in $C_0^\infty([0,\infty))$. Since

$$(-\Delta)^s = (-1)^s \sum_{|\tau|=s} \frac{s!}{\tau!} D^{2\tau},$$

it follows from (7) and the theorem on the boundedness of convolution operators in weighted L_p spaces (see [1]) that

$$\int_{\mathbb{R}_+^n} (\min\{1,y\})^{p\alpha} |\Gamma(z)U(z)|^p dz$$

$$\le c\,C\big(\|\nabla_s(\sigma(-\Delta))^{-s}U\|^p_{W_p^{t,\beta}} + \|TU\|^p_{W_p^{t,\beta}}\big) \le c\,C\|U\|^p_{W_p^{t-s,\beta}}.$$

The proof is complete. □

4. Characterisation of the space $M(W_p^{t,\beta} \to W_p^{s,\alpha})$

Here we derive necessary and sufficient conditions for a function to belong to the space $M(W_p^{t,\beta} \to W_p^{s,\alpha})$ for $p \in (1,\infty)$ with α and β satisfying

$$-1 < \alpha p < p-1, \quad -1 < \beta p < p-1, \quad t \ge s. \tag{8}$$

These inequalities will be assumed everywhere. We start with an assertion on derivatives of multipliers. We shall omit $\mathbb{R}_+^{n+1}$ in notations of spaces, norms, and integrals.

Lemma 3. *Suppose*

$$\Gamma \in M(W_p^{t,\beta} \to W_p^{s,\alpha}) \cap M(W_p^{t-s,\beta} \to W_p^{0,\alpha}), \quad p \in (1,\infty).$$

Then $D^\sigma\Gamma \in M(W_p^{t,\beta} \to W_p^{s-|\sigma|,\alpha})$ *for any multiindex* σ *of order* $|\sigma| \le s$ *and*

$$\|D^\sigma\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s-|\sigma|,\alpha})}$$

$$\le \varepsilon\,\|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})} + c(\varepsilon)\,\|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})}, \tag{9}$$

where ε *is an arbitrary positive number.*

Proof. Let $U \in W_p^{s,\alpha}$ and let φ be an arbitrary function in C_0^∞. Applying Leibniz formula

$$D^\sigma(\varphi U) = \sum_{\{\tau:\sigma\geq\tau\geq 0\}} \frac{\sigma!}{\tau!(\sigma-\tau)!} D^\tau \varphi D^{\sigma-\tau} U,$$

we find

$$\begin{aligned}\int \varphi U(-D)^\sigma \Gamma dz &= \int \Gamma D^\sigma(\varphi U) dz \\ &= \sum_{\{\tau:\sigma\geq\tau\geq 0\}} \frac{\sigma!}{\tau!(\sigma-\tau)!} \Gamma D^\tau \varphi D^{\sigma-\tau} U dz \\ &= \int \varphi \sum_{\{\beta:\sigma\geq\tau\geq 0\}} \frac{\sigma!}{\tau!(\sigma-\tau)!} (-D)^\tau (\Gamma D^{\sigma-\tau} U) dz.\end{aligned}$$

Therefore,

$$UD^\sigma\Gamma = \sum_{\{\tau:\sigma\geq\tau\geq 0\}} \frac{\sigma!}{\tau!(\sigma-\tau)!} (D)^\tau (\Gamma(-D)^{\sigma-\tau} U),$$

which implies the estimate

$$\|UD^\sigma\Gamma\|_{W_p^{s-|\sigma|,\alpha}} \leq c \sum_{\{\tau:\sigma\geq\tau\geq 0\}} \|\Gamma D^{\sigma-\tau} U\|_{W_p^{s-|\sigma|+|\tau|,\alpha}}.$$

Hence, it suffices to prove (9) for $|\sigma| = 1$. We have

$$\begin{aligned}\|U\nabla\Gamma\|_{W_p^{s-1,\alpha}} &\leq \|U\Gamma\|_{W_p^{s,\alpha}} + \|\Gamma\nabla U\|_{W_p^{s-1,\alpha}} \\ &\leq \left(\|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})} + \|\Gamma\|_{M(W_p^{t-1,\beta}\to W_p^{s-1,\alpha})}\right)\|U\|_{W_p^{t,\beta}}.\end{aligned}$$

Estimating the norm $\|\Gamma\|_{M(W_p^{t-1,\beta}\to W_p^{s-1,\alpha})}$ by (5) we arrive at (9). □

We pass now to two-sided estimates of norms in $M(W_p^{t,\beta} \to W_p^{s,\alpha})$, $p \in (1,\infty)$, given in terms of the spaces $M(W_p^{k,\beta} \to W_p^{0,\alpha})$. We start with lower estimates.

Lemma 4. *Let* $\Gamma \in M(W_p^{t,\beta} \to W_p^{s,\alpha})$. *Then*

$$\|\nabla_s\Gamma\|_{M(W_p^{t,\beta}\to W_p^{0,\alpha})} + \|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})} \leq c\|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})}. \tag{10}$$

Proof. Suppose first that $\Gamma \in M(W_p^{t-s,\beta} \to W_p^{0,\alpha})$. We have

$$\|\Gamma\nabla_s U\|_{W_p^{0,\alpha}} \leq \|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})}\|U\|_{W_p^{t,\beta}} + c \sum_{\substack{|\sigma|+|\tau|=s,\\ \tau\neq 0}} \|D^\sigma U D^\tau \Gamma\|_{W_p^{0,\alpha}}$$

$$(11)\qquad \le \left(\|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})} + c\sum_{j=1}^{s} \|\nabla_j\Gamma\|_{M(W_p^{t-s+j}\to W_p^{0,\alpha})} \right) \|U\|_{W_p^{t,\beta}}.$$

By Lemma 3,

$$\|\nabla_j\Gamma\|_{M(W_p^{t-s+j,\beta}\to W_p^{0,\alpha})}$$

$$(12)\qquad \le \varepsilon\, \|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})} + c(\varepsilon)\, \|\Gamma\|_{M(W_p^{t-s+j,\beta}\to W_p^{j,\alpha})}.$$

Estimating the last norm by (5) we obtain

$$\|\nabla_j\Gamma\|_{M(W_p^{t-s+j,\beta}\to W_p^{0,\alpha})}$$

$$\le \varepsilon\, \|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})} + c(\varepsilon)\, \|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})}.$$

Substitution of this into (11) gives

$$\|\Gamma\nabla_s U\|_{W_p^{0,\alpha}}$$

$$(13)\qquad \le \left(\varepsilon\, \|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})} + c(\varepsilon)\, \|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})}\right) \|U\|_{W_p^{t,\beta}}.$$

Besides,

$$(14)\qquad \|\Gamma U\|_{W_p^{0,\alpha}} \le \|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})} \|U\|_{W_p^{t,\beta}}.$$

Summing up two last estimates and applying Lemma 2 we arrive at

$$\|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})} \le \varepsilon\, \|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})} + c(\varepsilon)\, \|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})}.$$

Hence,

$$(15)\qquad \|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})} \le c\, \|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})}.$$

Now, we are going to remove the assumption $\Gamma \in M(W_p^{t-s,\beta} \to W_p^{0,\alpha})$. Since $\Gamma \in M(W_p^{t,\beta} \to W_p^{s,\alpha})$, then

$$\|\Gamma\eta\|_{W_p^{s,\alpha}} \le c\, \|\eta\|_{W_p^{t,\beta}}\ ,$$

where $\eta \in C_0^\infty(\mathcal{B}_2^n(z))$, $\eta = 1$ on $\mathcal{B}_1^n(z)$, and z is an arbitrary point in $\mathbb{R}_+^n$. Hence $\Gamma \in W_{p,\mathrm{unif}}^{s,\alpha}(\mathbb{R}_+^n)$ which implies that for any $(n-1)$-dimensional multiindex τ the derivative $D_x^\tau\Gamma_\rho$ belongs to $W_{p,\mathrm{unif}}^{s,\alpha}(\mathbb{R}_+^n)$. Therefore, $\Gamma_\rho \in L_\infty(\mathbb{R}_+^n)$ which in its turn guarantees that $\Gamma_\rho \in M(W_p^{t-s,\beta} \to W_p^{0,\alpha})$.

Thus, we may put Γ_ρ into (15) in order to obtain

$$\|\Gamma_\rho\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})} \le c\,\|\Gamma_\rho\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})}.$$

Letting $\rho\to 0$ and using Lemma 1 we arrive at (15) for all $\Gamma\in M(W_p^{t,\beta}\to W_p^{s,\alpha})$.

To estimate the first term in the right-hand side of (10), we combine (15) with (12) for $j=s$. □

The estimate converse to (10) is contained in the following lemma.

Lemma 5. *Let $\Gamma\in M(W_p^{t-s,\beta}\to W_p^{0,\alpha})$ and let $\nabla_s\Gamma\in M(W_p^{t,\beta}\to W_p^{0,\alpha})$. Then $\Gamma\in M(W_p^{t,\beta}\to W_p^{s,\alpha})$ and the estimate*

$$\|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})} \le c\left(\|\nabla_s\Gamma\|_{M(W_p^{t,\beta}\to W_p^{0,\alpha})} + \|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})}\right) \tag{16}$$

is valid.

Proof. By Lemma 4 and (5) we have

$$\|\nabla_j\Gamma\|_{M(W_p^{t-s+j,\beta}\to W_p^{0,\alpha})} \le c\,\|\Gamma\|_{M(W_p^{t-s+j,\beta}\to W_p^{j,\alpha})}$$

$$\le c\,\|\Gamma\|^{j/s}_{M(W_p^{t,\beta}\to W_p^{s,\alpha})}\|\Gamma\|^{1-j/s}_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})}, \tag{17}$$

where $j=1,\dots,s$. For any $U\in C_0^\infty$,

$$\begin{aligned}\|(\min\{1,y\})^\alpha\nabla_s(\Gamma U)\|_{L_p} &\le c\sum_{j=0}^{s}\|(\min\{1,y\})^\alpha|\nabla_j\Gamma|\,|\nabla_{s-j}U|\,\|_{L_p}\\ &\le c\Bigg(\|\nabla_s\Gamma\|_{M(W_p^{t,\beta}\to W_p^{0,\alpha})}+\|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})}\\ &\quad+\sum_{j=1}^{s-1}\|\nabla_j\Gamma\|_{M(W_p^{t-s+j,\beta}\to W_p^{0,\alpha})}\Bigg)\|U\|_{W_p^{t,\beta}}.\end{aligned}$$

This and (17) imply

$$\begin{aligned}&\|(\min\{1,y\})^\alpha\nabla_s(\Gamma U)\|_{L_p}\\ &\le c\,\left(\|\nabla_s\Gamma\|_{M(W_p^{t,\beta}\to W_p^{0,\alpha})}+\|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})}\right)\|U\|_{W_p^{t,\beta}}.\end{aligned}$$

It remains to note that

$$\|(\min\{1,y\})^\alpha\Gamma U\|_{L_p} \le \|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})}\|U\|_{W_p^{t-s,\beta}}.$$

The proof is complete. □

Using Lemmas 4 and 5 we arrive at the following description of the space $M(W_p^{t,\beta}(\mathbb{R}^n_+) \to W_p^{s,\alpha}(\mathbb{R}^n_+))$.

Theorem 1. *A function Γ belongs to the space $M(W_p^{t,\beta} \to W_p^{s,\alpha})$ if and only if $\Gamma \in W^{s,\alpha}_{p,\mathrm{loc}}$, $\Gamma \in M(W_p^{t-s,\beta} \to W_p^{0,\alpha})$, and $\nabla_s \Gamma \in M(W_p^{t,\beta} \to W_p^{0,\alpha})$. Moreover,*

$$\|\Gamma\|_{M(W_p^{t,\beta} \to W_p^{s,\alpha})} \sim \|\nabla_s \Gamma\|_{M(W_p^{t,\beta} \to W_p^{0,\alpha})} + \|\Gamma\|_{M(W_p^{t-s,\beta} \to W_p^{0,\alpha})}.$$

The equivalence relation (4) enables one to reformulate Theorem 1.

Theorem 2. *A function Γ belongs to the space $M(W_p^{t,\beta} \to W_p^{s,\alpha})$ if and only if $\Gamma \in W^{s,\alpha}_{p,\mathrm{loc}}$ and for any compact set $e \subset \mathbb{R}^n_+$*

$$\|(\min\{1,y\})^\alpha \nabla_s \Gamma; e\|^p_{L_p} \le c \operatorname{cap}_{p,t,\beta}(e),$$

$$\|(\min\{1,y\})^\alpha \Gamma; e\|^p_{L_p} \le c \operatorname{cap}_{p,t-s,\beta}(e).$$

Moreover,

$$\|\Gamma\|_{M(W_p^{t,\beta} \to W_p^{s,\alpha})}$$

$$\sim \sup_{\substack{e \subset \mathbb{R}^n_+ \\ \operatorname{diam}(e) \le 1}} \left(\frac{\|(\min\{1,y\})^\alpha \nabla_s \Gamma; e\|_{L_p}}{(\operatorname{cap}_{p,t,\beta}(e))^{1/p}} + \frac{\|(\min\{1,y\})^\alpha \Gamma; e\|_{L_p}}{(\operatorname{cap}_{p,t-s,\beta}(e))^{1/p}} \right). \tag{18}$$

An important particular case of Theorem 2 is $t = s$.

Corollary 1. *A function Γ belongs to the space $MW_p^{s,\alpha}$ if and only if $\Gamma \in W^{s,\alpha}_{p,\mathrm{loc}}$ and for any compact set $e \subset \mathbb{R}^n_+$*

$$\|(\min\{1,y\})^\alpha \nabla_s \Gamma; e\|^p_{L_p} \le c \operatorname{cap}_{p,s,\alpha}(e),$$

Moreover,

$$\|\Gamma\|_{MW_p^{s,\alpha}} \sim \sup_{\substack{e \subset \mathbb{R}^n_+ \\ \operatorname{diam}(e) \le 1}} \frac{\|(\min\{1,y\})^\alpha \nabla_s \Gamma; e\|_{L_p}}{(\operatorname{cap}_{p,s,\alpha}(e))^{1/p}} + \|\Gamma\|_{L_\infty}. \tag{19}$$

5. Trace theorems for multipliers in weighted Sobolev spaces

We start with the following simple fact concerning traces of multipliers.

Theorem 3. *Let m and l be positive noninteger, $m \geq l$ and let*

$$\Gamma \in M(W_p^{t,\beta}(\mathbb{R}^n_+) \ \to \ W_p^{s,\alpha}(\mathbb{R}^n_+))$$

where $t = [m] + 1$, $s = [l] + 1$, $\beta = 1 - \{m\} - 1/p$, and $\alpha = 1 - \{l\} - 1/p$. If γ is the trace of Γ on $\mathbb{R}^{n-1}$, then

$$\gamma \in M(W_p^m(\mathbb{R}^{n-1}) \ \to \ W_p^l(\mathbb{R}^{n-1}))$$

and the estimate

$$\|\gamma; \mathbb{R}^{n-1}\|_{M(W_p^m \to W_p^l)} \leq c\|\Gamma; \mathbb{R}^n_+\|_{M(W_p^{t,\beta} \to W_p^{s,\alpha})} \tag{20}$$

holds.

Proof. Let $U \in W_p^{t,\beta}(\mathbb{R}^n_+)$ and let u be the trace of U on $\mathbb{R}^{n-1}$. By setting ΓU and γu instead of U and u, respectively, in the inequality

$$\|u; \mathbb{R}^{n-1}\|_{W_p^l} \leq c\|U; \mathbb{R}^n_+\|_{W_p^{s,\alpha}}$$

we arrive at the estimate

$$\|\gamma u; \mathbb{R}^{n-1}\|_{W_p^l} \leq c\ \|\Gamma; \mathbb{R}^n_+\|_{M(W_p^{t,\beta} \to W_p^{s,\alpha})}\|U; \mathbb{R}^n_+\|_{W_p^{t,\beta}}\,.$$

Minimizing the right-hand side over all extensions U of u we obtain

$$\|\gamma u; \mathbb{R}^{n-1}\|_{W_p^l} \leq c\ \|\Gamma; \mathbb{R}^n_+\|_{M(W_p^{t,\beta} \to W_p^{s,\alpha})}\|u; \mathbb{R}^{n-1}\|_{W_p^m}$$

which gives (20). □

We state an extension theorem for functions in $M(W_p^m(\mathbb{R}^{n-1}) \ \to \ W_p^l(\mathbb{R}^{n-1}))$ to be proved in Section 7.

Theorem 4. *Let m and l be positive nonintegers, $m \geq l$, $p \in (1, \infty)$, and let*

$$\gamma \in M(W_p^m(\mathbb{R}^{n-1}) \to W_p^l(\mathbb{R}^{n-1}))\,.$$

Then the Dirichlet problem

$$\Delta\Gamma = 0 \ \ on \ \ \mathbb{R}^n_+\,, \ \ \Gamma|_{\mathbb{R}^{n-1}} = \gamma \tag{21}$$

has a unique solution in $M(W_p^{t,\beta}(\mathbb{R}^n_+) \to W_p^{s,\alpha}(\mathbb{R}^n_+))$, where $t = [m] + 1$, $s = [l] + 1$, $\beta = 1 - \{m\} - 1/p$, and $\alpha = 1 - \{l\} - 1/p$. There holds the estimate

$$\|\Gamma; \mathbb{R}^n_+\|_{M(W_p^{t,\beta} \to W_p^{s,\alpha})} \leq c\ \|\gamma; \mathbb{R}^{n-1}\|_{M(W_p^m \to W_p^l)}\,. \tag{22}$$

6. Auxiliary estimates for an extension operator

6.1 Pointwise estimate for $T\gamma$ and $\nabla T\gamma$. For functions $\gamma \in L_{1,unif}(\mathbb{R}^{n-1})$, we introduce the operator

$$(T\gamma)(x,y) = y^{1-n}\int_{\mathbb{R}^{n-1}} \zeta\left(\frac{x-\xi}{y}\right)\gamma(\xi)d\xi, \quad (x,y)\in\mathbb{R}^n_+\,, \tag{23}$$

where ζ is a continuously differentiable function defined on $\overline{\mathbb{R}^n_+}$ outside the origin. We assume that

$$(|z|+1)|\nabla\zeta(z)| + |\zeta(z)| \le C\ (|z|+1)^{-n}\,. \tag{24}$$

Lemma 6. *Let $\gamma \in M(W_p^{m-l}(\mathbb{R}^{n-1}) \to L_p(\mathbb{R}^{n-1}))$, where $m \ge l$ and $1<p<\infty$. Then*

$$|T\gamma(z)| + y|\nabla(T\gamma(z))| \ \le\ c(1+y^{l-m})\|\gamma;\mathbb{R}^{n-1}\|_{M(W_p^{m-l}\to L_p)}\,.$$

Proof. In view of (24)

$$\begin{aligned}&|T\gamma(z)| + y|\nabla(T\gamma(z))|\\ &\le cy^{1-n}\left(\int_{\mathcal{B}_y^{n-1}(x)}|\gamma(\xi)|d\xi + y^n\int_{\mathbb{R}^{n-1}\setminus\mathcal{B}_y^{n-1}(x)}\frac{|\gamma(\xi)|d\xi}{|\xi-x|^n}\right).\end{aligned} \tag{25}$$

By Hölder's inequality,

$$\int_{\mathcal{B}_y^{n-1}(x)}|\gamma(\xi)|d\xi \le cy^{(n-1)(p-1)/p}\|\gamma;\mathcal{B}_y^{n-1}(x)\|_{L_p}. \tag{26}$$

Let $y\in(0,1)$. The right-hand side in (26) does not exceed

$$\begin{aligned}&cy^{-m+l+n-1}\sup_{\substack{r\in(0,1)\\ x\in\mathbb{R}^{n-1}}}(1+r^{m-l-\frac{n-1}{p}})\ \|\gamma;\mathcal{B}_r^{n-1}(x)\|_{L_p}\\ &\le cy^{-m+l+n-1}\sup_{\substack{r\in(0,1)\\ x\in\mathbb{R}^{n-1}}}\left(\mathrm{cap}_{p,m-l}(\mathcal{B}_r^{n-1}(x))\right)^{-1/p}\|\gamma;\mathcal{B}_r^{n-1}(x)\|_{L_p}.\end{aligned} \tag{27}$$

This and Proposition 2 show that for $y\in(0,1)$

$$\int_{\mathcal{B}_y^{n-1}(x)}|\gamma(\xi)|d\xi \le cy^{-m+l+n-1}\|\gamma;\mathbb{R}^{n-1}\|_{M(W_p^{m-l}\to L_p)}\,. \tag{28}$$

Suppose $y>1$. Since

$$\mathrm{cap}_{p,m-l}(\mathcal{B}_r^{n-1}(x)) \ \sim\ r^{n-1} \ \text{ for } \ r>1\,, \tag{29}$$

it follows that the right-hand side of (26) is dominated by

$$cy^{n-1}\big(\mathrm{cap}_{p,m-l}(\mathcal{B}_y^{n-1}(x))\big)^{-1/p}\|\gamma;\mathcal{B}_y^{n-1}(x)\|_{L_p}\,.$$

Combining this with (27) and Proposition 2 we conclude that

$$\int_{\mathcal{B}_y^{n-1}(x)}|\gamma(\xi)|d\xi\le cy^{n-1}(1+y^{l-m})\|\gamma;\mathbb{R}^{n-1}\|_{M(W_p^{m-l}\to L_p)}\,. \tag{30}$$

We now estimate the second integral in the right-hand side of (25). Clearly,

$$\int_{\mathbb{R}^{n-1}\setminus\mathcal{B}_y^{n-1}(x)}\frac{|\gamma(\xi)|d\xi}{|\xi-x|^n}\le n\int_y^\infty\frac{d\rho}{\rho^{n+1}}\int_{\mathcal{B}_\rho^{n-1}(x)}|\gamma(\xi)|d\xi\,. \tag{31}$$

By Hölder's inequality the right-hand side of (31) admits the majorant

$$c\int_y^\infty\rho^{-2-\frac{n-1}{p}}\|\gamma;\mathcal{B}_\rho^{n-1}(x)\|_{L_p}d\rho\,. \tag{32}$$

Using (29) we see that the function (32), for $y>1$, does not exceed

$$cy^{-1}\sup_{\substack{r\in(0,1)\\x\in\mathbb{R}^{n-1}}}\big(\mathrm{cap}_{p,m-1}(\mathcal{B}_r^{n-1}(x))\big)^{-1/p}\,\|\gamma;\mathcal{B}_r^{n-1}(x)\|_{L_p}$$

which in view of Proposition 2 is dominated by

$$cy^{-1}\|\gamma;\mathbb{R}^{n-1}\|_{M(W_p^{m-l}\to L_p)}. \tag{33}$$

Let $y<1$. Then

$$\begin{aligned}\int_y^1&\rho^{-2-\frac{n-1}{p}}\|\gamma;\mathcal{B}_\rho^{n-1}(x)\|_{L_p}d\rho\\&\le cy^{-m+l-1}\sup_{\substack{r\in(0,1)\\x\in\mathbb{R}^{n-1}}}(1+r^{m-1-\frac{n-1}{p}})\,\|\gamma;\mathcal{B}_r^{n-1}(x)\|_{L_p}\\&\le cy^{-m+l-1}\sup_{\substack{r\in(0,1)\\x\in\mathbb{R}^{n-1}}}\big(\mathrm{cap}_{p,m-l}(\mathcal{B}_r^{n-1}(x))\big)^{-1/p}\,\|\gamma;\mathcal{B}_r^{n-1}(x)\|_{L_p}.\end{aligned} \tag{34}$$

Furthermore, by (29)

$$\int_1^\infty \rho^{-2-\frac{n-1}{p}} \|\gamma; \mathcal{B}_\rho^{n-1}(x)\|_{L_p} d\rho$$

$$\leq c \int_1^\infty \rho^{-2} \big(\mathrm{cap}_{p,m-l}(\mathcal{B}_\rho^{n-1}(x))\big)^{-1/p} \|\gamma; \mathcal{B}_\rho^{n-1}(x)\|_{L_p} d\rho$$

$$\leq c \sup_{\substack{r>0 \\ x\in\mathbb{R}^{n-1}}} \big(\mathrm{cap}_{p,m-l}(\mathcal{B}_r^{n-1}(x))\big)^{-1/p} \|\gamma; \mathcal{B}_r^{n-1}(x)\|_{L_p}.$$

Summing up this inequality and (34), and using Proposition 2 we conclude that the integral (32) is majorized, for $y<1$, by

$$cy^{-m+l-1}\|\gamma; \mathbb{R}^{n-1}\|_{M(W_p^{m-l}\to L_p)}.$$

This, together with (33), imply that for all $y>0$ the integral (32) does not exceed

$$cy^{-1}(1+y^{l-m})\|\gamma; \mathbb{R}^{n-1}\|_{M(W_p^{m-l}\to L_p)}.$$

Hence, the result follows from (30), (31), and (25). □

6.2 Weighted L_p-estimates for $T\gamma$ and $\nabla T\gamma$.

Lemma 7. *Let the extension operator T be defined by* (23) *and suppose that $\gamma \in M(W_p^{m-l}(\mathbb{R}^{n-1}) \to L_p(\mathbb{R}^{n-1}))$, where $l\in(0,1)$, $[m]\geq 1$, $1<p<\infty$. Then, for $k=1,\dots,[m]$,*

$$\left(\int_0^1 y^{p(k-l)-1}\big(|T\gamma(z)| + y|\nabla(T\gamma)(z)|\big)^p dy\right)^{1/p}$$

$$\leq c\|\gamma; \mathbb{R}^{n-1}\|_{M(W_p^{m-l}\to L_p)}^{\frac{k-l}{m-l}} [(\mathcal{M}\gamma)(x)]^{\frac{m-k}{m-l}}, \tag{35}$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal operator in $\mathbb{R}^{n-1}$.

Proof. Let δ be a number in (0,1] to be chosen later. We set

$$\int_0^1 y^{p(k-l)-1}\big(|T\gamma| + y|\nabla(T\gamma)(z)|\big)^p dy \;=\; \int_0^\delta \dots dy \;+\; \int_\delta^1 \dots dy\,.$$

In view of (25)

$$\int_0^\delta \dots dy \leq c\int_0^\delta y^{p(k+1-l-n)-1}\left(\int_{\mathcal{B}_y^{n-1}(x)} |\gamma(\xi)|d\xi\right)^p dy$$

$$+ c \int_0^\delta y^{p(k+1-l)-1} \left(\int_{\mathbb{R}^{n-1} \setminus \mathcal{B}_y^{n-1}(x)} \frac{|\gamma(\xi)| d\xi}{|\xi - x|^n} \right)^p dy.$$

By the definition of $\mathcal{M}$,

$$\int_0^\delta y^{p(k+1-l-n)-1} \left(\int_{\mathcal{B}_y^{n-1}(x)} |\gamma(\xi)| d\xi \right)^p dy \le c[(\mathcal{M}\gamma)(x)]^p \delta^{p(k-l)}. \tag{36}$$

Using (31) we obtain

$$\int_0^\delta y^{p(k+1-l)-1} \left(\int_{\mathbb{R}^{n-1} \setminus \mathcal{B}_y^{n-1}(x)} \frac{|\gamma(\xi)| d\xi}{|\xi - x|^n} \right)^p dy \le c[(\mathcal{M}\gamma)(x)]^p \delta^{p(k-l)}. \tag{37}$$

Combining (36) and (37) we conclude that

$$\int_0^\delta \dots dy \ \le c[(\mathcal{M}\gamma)(x)]^p \delta^{p(k-l)}. \tag{38}$$

By Lemma 6,

$$\int_\delta^1 y^{p(k-l)-1} \big(|T\gamma| + y |\nabla (T\gamma)(z)| \big)^p dy$$

$$\le c \, \|\gamma; \mathbb{R}^{n-1}\|^p_{M(W_p^{m-l} \to L_p)} \delta^{p(k-m)} . \tag{39}$$

Summing up (38) and (39) we find

$$\int_0^1 y^{p(k-l)-1} \big(|T\gamma| + y |\nabla (T\gamma)(z)| \big)^p dy$$

$$\le c \big([(\mathcal{M}\gamma)(x)]^p \delta^{p(k-l)} + \|\gamma; \mathbb{R}^{n-1}\|^p_{M(W_p^{m-l} \to L_p)} \delta^{p(k-m)} \big).$$

The right-hand side in this inequality attains its minimum value for

$$\delta \ = \ \left(\frac{\|\gamma; \mathbb{R}^{n-1}\|_{M(W_p^{m-l} \to L_p)}}{(\mathcal{M}\gamma)(x)} \right)^{1/(m-l)} .$$

The proof is complete. □

Lemma 8. *Let the operators T and $D_{p,l}$ be defined by* (23) *and* (2). *Then*

$$\int_0^1 y^{p(1-l)-1} |\nabla (T\gamma)(z)|^p dy \ \le \ c \, \big((D_{p,l}\gamma)(x) \big)^p.$$

Proof. Let $R(\xi,x)=\gamma(\xi)-\gamma(x)$. Using the identity

$$y^{-n+1}\int_{\mathbb{R}^{n-1}}\zeta\left(\frac{\xi-x}{y}\right)d\xi=const$$

we have

$$\frac{\partial T\gamma}{\partial y}(x,y)=\frac{\partial}{\partial y}\left(y^{-n+1}\int_{\mathbb{R}^{n-1}}\zeta\left(\frac{\xi-x}{y}\right)R(\xi,x)d\xi\right). \tag{40}$$

Furthermore, it is clear that

$$\frac{\partial T\gamma}{\partial x_j}(x,y)=y^{-n+1}\int_{\mathbb{R}^{n-1}}R(\xi,x)\frac{\partial}{\partial x_j}\zeta\left(\frac{\xi-x}{y}\right)d\xi\,.$$

Therefore,

$$|\nabla(T\gamma)(x,y)|\le cy^{-n}\sum_{k=0}^{1}\int_{\mathbb{R}^{n-1}}\left|\nabla_k\zeta\left(\frac{\xi-x}{y}\right)\right|\left(1+\frac{|\xi-x|}{y}\right)^k|R(\xi,x)|d\xi.$$

This estimate and (24) imply

$$|\nabla(T\gamma)(x,y)|\le cy^{-n}\int_{\mathbb{R}^{n-1}}\left(1+\frac{|\xi-x|}{y}\right)^{-n}|R(\xi,x)|d\xi$$

$$=cy^{-1/p}\int_{\mathbb{R}^{n-1}}\left(\frac{|\xi-x|}{y}\right)^{n-1/p}\left(1+\frac{|\xi-x|}{y}\right)^{-n}\frac{|R(\xi,x)|}{|\xi-x|^{n-1/p}}d\xi.$$

Consequently,

$$\int_0^1 y^{p(1-l)-1}|\nabla(T\gamma)(x,y)|^p dy$$

$$\le c\int_0^1\left(\int_{\mathbb{R}^{n-1}}f\left(\frac{|\xi-x|}{y}\right)\frac{|R(\xi,x)|}{|\xi-x|^{n-1/p}}d\xi\right)^p y^{p(1-l)-1}\frac{dy}{y},$$

where $f(\eta)=\eta^{n-1/p}(1+\eta)^{-n}$. We write the last integral over $(0,1)$ as

$$\int_0^1\left(\int_0^\infty f(\frac{t}{y})g(t,x)\frac{dt}{t}\right)^p y^{p(1-l)-1}\frac{dy}{y}$$

$$=\int_0^1\left(\int_0^\infty f(s)g(sy,x)\frac{ds}{s}\right)^p y^{p(1-l)-1}\frac{dy}{y}, \tag{41}$$

with

$$g(t,x) = t^{1/p-1} \int_{\partial \mathcal{B}_1^{n-1}} |R(t\theta + x, x)| d\theta.$$

By Minkowski's inequality, the right-hand side of (41) does not exceed

$$\left(\int_0^\infty \left(\int_0^1 (f(s))^p (g(sy,x))^p y^{p(1-l)-1} \frac{dy}{y} \right)^{1/p} \frac{ds}{s} \right)^p$$

$$= \left(\int_0^\infty f(s) \left(\int_0^s (g(\tau,x))^p \tau^{p(1-l)-1} \frac{d\tau}{\tau} \right)^{1/p} \frac{ds}{s^{2-l-1/p}} \right)^p$$

$$\leq \left(\int_0^\infty f(s) \frac{ds}{s^{2-l-1/p}} \right)^p \int_0^\infty (g(\tau,x))^p \tau^{p(1-l)-1} \frac{d\tau}{\tau}. \tag{42}$$

Therefore,

$$\int_0^1 y^{p(1-l)-1} |\nabla (T\gamma)(x,y)|^p dy \leq c \int_0^\infty (g(\tau,x))^p \tau^{p(1-l)-1} \frac{d\tau}{\tau}.$$

It remains to note that

$$\begin{aligned}
\int_0^\infty (g(\tau,x))^p \tau^{p(1-l)-1} \frac{d\tau}{\tau} &= \int_0^\infty \tau^{-pl} \left(\int_{\partial \mathcal{B}_1^{n-1}} |\gamma(\tau\theta + x) - \gamma(x)| d\theta \right)^p \frac{d\tau}{\tau} \\
&\leq c \int_0^\infty \int_{\partial \mathcal{B}_1^{n-1}} |\gamma(\tau\theta + x) - \gamma(x)|^p d\theta \frac{d\tau}{\tau^{pl+1}} \\
&\leq c \int_{\mathbb{R}^{n-1}} \frac{|\gamma(x+h) - \gamma(x)|^p}{|h|^{pl+n-1}} dh \\
&= c \left(\left(D_{p,l}\gamma \right)(x) \right)^p .
\end{aligned}$$

The result follows. □

7. Proof of Theorem 4

7.1 The case $l < 1$. Our aim now is to prove that for $l < 1$ and $s = 1$ the operator T defined by (23) maps $M(W_p^m(\mathbb{R}^{n-1}) \to W_p^l(\mathbb{R}^{n-1}))$ into $M(W_p^{[m]+1,\beta}(\mathbb{R}^n_+) \to W_p^{1,\alpha}(\mathbb{R}^n_+))$ with $\alpha = 1 - l - 1/p$, $\beta = 1 - \{m\} - 1/p$ and there holds the estimate

$$\|T\gamma; \mathbb{R}^n_+\|_{M(W_p^{t,\beta} \to W_p^{1,\alpha})} \leq c\, C \|\gamma; \mathbb{R}^{n-1}\|_{M(W_p^m \to W_p^l)}, \tag{43}$$

where C is the constant in (24).

We have

$$\|(\min\{1,y\})^\alpha \nabla(UT\gamma);\mathbb{R}^n_+\|^p_{L_p} \le c\int_0^1 y^{p\alpha}\int_{\mathbb{R}^{n-1}}(|\nabla(T\gamma)|^p|U|^p+|T\gamma|^p|\nabla U|^p)dz$$

$$+\ c\int_1^\infty\int_{\mathbb{R}^{n-1}}(|\nabla(T\gamma)|^p|U|^p+|T\gamma|^p|\nabla U|^p)dz$$

$$= c\int_{0<y<1}\dots dz + c\int_{y>1}\dots dz. \tag{44}$$

By Lemma 1, for $y>1$

$$y|\nabla(T\gamma)(z)|+|(T\gamma)(z)| \le c\|\gamma;\mathbb{R}^{n-1}\|_{M(W_p^{m-l}\to L_p)}.$$

Hence,

$$\int_{y>1}\dots dz \le c\|\gamma;\mathbb{R}^{n-1}\|^p_{M(W_p^{m-l}\to L_p)}\|U;\mathbb{R}^n_+\|^p_{W_p^{1,\alpha}}. \tag{45}$$

It remains to refer to the estimate

$$\|U;\mathbb{R}^n_+\|_{W_p^{1,\alpha}} \le c\|U;\mathbb{R}^n_+\|_{W_p^{t,\beta}}$$

which follows from the one dimensional Hardy inequality.

Introducing the notation

$$\mathcal{R}_0U(z) = U(z) - \sum_{k=0}^{[m]}\frac{\partial^k}{\partial y^k}U(x,0)\frac{y^k}{k!},$$

$$\mathcal{R}_1U(z) = \begin{cases}\nabla U(z) - \displaystyle\sum_{k=0}^{[m]-1}\frac{\partial^k}{\partial y^k}\nabla U(x,0)\frac{y^k}{k!} & \text{for } m>1\\ \nabla U(z) & \text{for } m<1\end{cases}$$

we have

$$\int_{0<y<1}\dots dz \le c\int_{0<y<1} y^{p(1-l)-1}\sum_{j=0}^{1}|\nabla_j(T\gamma)|^p|\mathcal{R}_{1-j}U(z)|^p dz$$

$$+\ c\int_{0<y<1} y^{-pl-1}\big(|T\gamma(z)|+y|\nabla(T\gamma)(z)|\big)^p\sum_{k=1}^{[m]}|\nabla_kU(x,0)|^p y^{pk}dz$$

$$+c\int_{0<y<1} y^{p(1-l)-1}|\nabla T\gamma(z)|^p|U(x,0)|^p dz \tag{46}$$

for $m > 1$. In case $m < 1$ the second integral in the right hand side of (46) should be omitted.

By Lemma 6, for $0 < y < 1$

$$|T\gamma(z)| + y|\nabla(T\gamma)(z)| \le c\, y^{l-m} \|\gamma; \mathbb{R}^{n-1}\|_{M(W_p^{m-l} \to L_p)}. \tag{47}$$

Since for $j = 0, 1$

$$|\mathcal{R}_{1-j} U(z)| \le \frac{y^{[m]+j-1}}{([m]+j-1)!} \int_0^y |\nabla_t U(x,t)| dt, \tag{48}$$

we have

$$\int_{0<y<1} y^{p(1-\{m\})-1} |\mathcal{R}_{1-j} U(z)|^p dz$$

$$\le c \int_{0<y<1} y^{-p\{m\}-1} \left(\int_0^y |\nabla_{[m]+1} U(x,t)| dt \right)^p dz\,.$$

By Hardy's inequality the right-hand side does not exceed $c\, \|U; \mathbb{R}^n_+\|^p_{W_p^{[m]+1,\beta}}$. Combining this with (47) we obtain that the first integral in the right-hand side of (46) does not exceed

$$\|\gamma; \mathbb{R}^{n-1}\|^p_{M(W_p^{m-l} \to L_p)} \|U; \mathbb{R}^n_+\|^p_{W_p^{[m]+1,\beta}}\,. \tag{49}$$

We now pass to the estimate of the second integral in the right-hand side of (46) for $k = 1, \dots, [m]$, $m > 1$. Applying Lemma 7, we find

$$\int_{0<y<1} y^{p(k-l)-1} \big(|T\gamma(z)| + y|\nabla(T\gamma)(z)|\big)^p |\nabla_k U(x,0)|^p dz$$

$$\le c \|\gamma; \mathbb{R}^{n-1}\|^{p\frac{k-l}{m-l}}_{M(W_p^{m-l} \to L_p)} \int_{\mathbb{R}^{n-1}} \big(\mathcal{M}\gamma(x)\big)^{p\frac{m-k}{m-l}} |\nabla_k U(x,0)|^p dx\,. \tag{50}$$

The last integral is not greater than

$$\|(\mathcal{M}\gamma)^{\frac{m-k}{m-l}}; \mathbb{R}^{n-1}\|^p_{M(W_p^{m-k} \to L_p)} \|\nabla_k U(\cdot,0); \mathbb{R}^{n-1}\|^p_{W_p^{m-k}}\,. \tag{51}$$

Using Proposition 2 with $\lambda = m-k$, $\mu = m-l$ and Verbitsky's theorem on the boundedness of the maximal operator $\mathcal{M}$ in the space $M(W_p^{m-l}(\mathbb{R}^{n-1}) \to L_p(\mathbb{R}^{n-1}))$ (see [3], Ch.2), we find that (51) is dominated by

$$c \|\gamma, \mathbb{R}^{n-1}\|^{\frac{p(m-k)}{m-l}}_{M(W_p^{m-l} \to L_p)} \|U(\cdot,0); \mathbb{R}^{n-1}\|^p_{W_p^m}.$$

Hence and by (50)

$$\int_{0<y<1} y^{p(k-l)-1}\big(|T\gamma(z)| + y|\nabla(T\gamma)(z)|\big)^p |\nabla_k U(x,0)|^p dz$$

$$\leq c\|\gamma, \mathbb{R}^{n-1}\|^p_{M(W_p^{m-l}\to L_p)} \|U; \mathbb{R}^n_+\|^p_{W_p^{[m]+1,\beta}}. \tag{52}$$

By Lemma 8, the integral

$$\int_{0<y<1} y^{p(1-l)-1}|\nabla(T\gamma)(z)|^p |U(x,0)|^p dz \tag{53}$$

does not exceed

$$c\int_{\mathbb{R}^{n-1}} (D_{p,l}\gamma(x))^p |U(x,0)|^p dx$$

$$\leq c\|D_{p,l}\gamma; \mathbb{R}^{n-1}\|^p_{M(W_p^m\to L_p)} \|U(\cdot,0); \mathbb{R}^{n-1}\|^p_{W_p^m}$$

$$\leq c\|\gamma; \mathbb{R}^{n-1}\|^p_{M(W_p^m\to W_p^l)} \|U; \mathbb{R}^n_+\|^p_{W_p^{[m]+1,\beta}}. \tag{54}$$

Thus we arrive at the inequality

$$\int_{0<y<1} y^{p\alpha}|\nabla(UT\gamma)(z)|^p dz \leq c\|\gamma; \mathbb{R}^{n-1}\|^p_{M(W_p^m\to W_p^l)} \|U; \mathbb{R}^n_+\|^p_{W_p^{[m]+1,\beta}}.$$

It remains to estimate the integral

$$\int_{0<y<1} y^{p(1-l)-1}|(T\gamma)(z)|^p |U(z)|^p dz.$$

Clearly,

$$\int_{0<y<1} y^{p(1-l)-1}|(T\gamma)(z)|^p |U(z)|^p dz \leq \int_{0<y<1} y^{p(1-l)-1}|(T\gamma)(z)|^p |\mathcal{R}_0 U(z)|^p dz$$

$$+\sum_{k=0}^{[m]} \int_{0<y<1} y^{pk} y^{p(1-l)-1}|(T\gamma)(z)|^p |\nabla_k U(x,0)|^p dz. \tag{55}$$

By (47) and (48) with $j = 1$ we have

$$\int_{0<y<1} y^{p(1-l)-1}|(T\gamma)(z)|^p|\mathcal{R}_0 U(z)|^p dz$$
$$\leq c\|\gamma;\mathbb{R}^{n-1}\|^p_{M(W_p^{m-l}\to L_p)}$$
$$\times \int_{0<y<1} y^{p(1-\{m\})-1}\left(\int_0^y |\nabla_{[m]+1}U(x,t)|dt\right)^p dz$$

which by Hardy's inequality is dominated by (49). In view of (52)

$$\int_{0<y<1} y^{p(k-l)-1}|(T\gamma)(z)|^p|\nabla_k U(x,0)|^p dz$$
$$\leq c\|\gamma;\mathbb{R}^{n-1}\|^p_{M(W_p^{m-l}\to L_p)}\|U;\mathbb{R}^n_+\|^p_{W_p^{[m]+1,\beta}}.$$

Thus we arrive at the estimate

$$\int_{0<y<1}\ldots dz \leq c\|\gamma;\mathbb{R}^{n-1}\|^p_{M(W_p^m\to W_p^l)}\|U;\mathbb{R}^n_+\|^p_{W_p^{[m]+1,\beta}}.$$

Since the Poisson kernel satisfies condition (24), Theorem 4 with $l < 1$ follows.

7.2 The case $l > 1$.

Lemma 9. *Let m and l be nonintegers, $m \geq l > 0$, and let T be the extension operator* (23). *Suppose that $\gamma \in M(W_p^{m-l}(\mathbb{R}^{n-1}) \to L_p(\mathbb{R}^{n-1}))$. Then*

$$T\gamma \in M(W_p^{[m]-[l],\beta}(\mathbb{R}^n_+) \to W_p^{0,\alpha}(\mathbb{R}^n_+))$$

and

$$\|T\gamma;\mathbb{R}^n_+\|_{M(W_p^{[m]-[l],\beta}\to W_p^{0,\alpha})} \leq c\|\gamma;\mathbb{R}^{n-1}\|_{M(W_p^{m-l}\to L_p)}. \tag{56}$$

Proof. To begin with, let $[m] = [l]$. Then by (47)

$$\int_{0<y<1} y^{p(1-\{l\})-1}|U(z)(T\gamma)(z)|^p dz$$
$$\leq c\|\gamma;\mathbb{R}^{n-1}\|^p_{M(W_p^{m-l}\to L_p)}\int_{0<y<1} y^{p(1-\{m\})-1}|U(z)|^p dz$$

which gives the result.

Suppose $[m] \geq [l]+1$. We introduce the function

$$\mathcal{R}U = U(z) - \sum_{j=0}^{[m]-[l]-1} \frac{\partial^j U}{\partial y^j}(x,0)\frac{y^j}{j!}$$

which, clearly, satisfies

$$|\mathcal{R}U(z)| \leq \frac{y^{[m]-[l]-1}}{([m]-[l]-1)!}\int_0^y |\nabla_{[m]-[l]}U(x,t)|dt.$$

This and (47) imply

$$\int_{0<y<1} y^{p(1-\{l\})-1}|T\gamma(z)|^p|\mathcal{R}U(z)|^p dz$$

$$\leq c\|\gamma;\mathbb{R}^{n-1}\|^p_{M(W_p^{m-l}\to L_p)}\int_{0<y<1} y^{-p\{m\}-1}\left(\int_0^y |\nabla_{[m]-[l]}U(x,t)|dt\right)^p dz.$$

By Hardy's inequality the right-hand side is dominated by

$$c\|\gamma;\mathbb{R}^{n-1}\|^p_{M(W_p^{m-l}\to L_p)}\|U;\mathbb{R}^n_+\|^p_{W_p^{[m]-[l],\beta}}.$$

Furthermore, by Lemma 7 with m replaced by $m-[l]$, l replaced by $\{l\}$and $k=j+1$ we have for $j=0,\dots,[m]-[l]-1$

$$\int_{0<y<1} y^{p(j+1-\{l\})-1}|T\gamma(z)|^p|\nabla_j U(x,0)|^p dz$$

$$\leq c\|\gamma;\mathbb{R}^{n-1}\|^{p\frac{j+1-\{l\}}{m-l}}_{M(W_p^{m-l}\to L_p)}\int_{\mathbb{R}^{n-1}}\big(\mathcal{M}\gamma(x)\big)^{p\frac{m-[l]-j-1}{m-l}}|\nabla_j U(x,0)|^p dx. \tag{57}$$

The last integral is dominated by

$$\|\big(\mathcal{M}\gamma\big)^{p\frac{m-[l]-j-1}{m-l}};\mathbb{R}^{n-1}\|^p_{M(W_p^{m-[l]-j-1}\to L_p)}\|U(\cdot,0);\mathbb{R}^{n-1}\|^p_{W_p^{m-[l]-1}}$$

which by Proposition 3 does not exceed

$$\|\mathcal{M}\gamma;\mathbb{R}^{n-1}\|^{p\frac{m-[l]-j-1}{m-l}}_{M(W_p^{m-l}\to L_p)}\|U;\mathbb{R}^n_+\|^p_{W_p^{[m]-[l],\beta}}.$$

Hence and by (57)

$$\int_{0<y<1} y^{p(j+1-\{l\})-1}|T\gamma(z)|^p|\nabla_j U(x,0)|^p dz$$

$$\leq c\|\gamma;\mathbb{R}^{n-1}\|^p_{M(W_p^{m-l}\to L_p)}\|U;\mathbb{R}^n_+\|^p_{W_p^{[m]-[l],\beta}}.$$

The result follows. □

Proof of Theorem 4 for $l>1$. Suppose Theorem has been proved for $[l]=1,\dots,\mathcal{L}-1$, where $\mathcal{L}\geq 2$. Let $[l]=\mathcal{L}$ and let

$$\gamma\in M(W_p^m(\mathbb{R}^{n-1})\to W_p^l(\mathbb{R}^{n-1}))\ \text{ for }\ m\geq\mathcal{L}. \tag{58}$$

Let $T\gamma$ denote the Poisson integral. Since by Proposition 1 one has

$$\gamma\in M(W_p^{m-l}(\mathbb{R}^{n-1})\to L_p(\mathbb{R}^{n-1})),$$

it follows from Lemma 9 that

$$T\gamma\in M(W_p^{[m]-[l],\beta}(\mathbb{R}^n_+)\to W_p^{0,\alpha}(\mathbb{R}^n_+))$$

and (56) holds. Next we show that

$$\nabla_{\mathcal{L}+1}(T\gamma)\in M(W_p^{[m]+1,\beta}(\mathbb{R}^n_+)\to W_p^{0,\alpha}(\mathbb{R}^n_+)). \tag{59}$$

Using Proposition 1, we obtain

$$\frac{\partial\gamma}{\partial x_k}\in M(W_p^m(\mathbb{R}^{n-1})\to W_p^{l-1}(\mathbb{R}^{n-1})),\quad k=1,\dots,n-1.$$

Then, by the induction hypothesis applied to $\partial\gamma/\partial x_k$,

$$\frac{\partial}{\partial x_k}(T\gamma)=T\frac{\partial\gamma}{\partial x_k}\in M(W_p^{[m]+1,\beta}(\mathbb{R}^n_+)\to W_p^{\mathcal{L},\alpha}(\mathbb{R}^n_+)). \tag{60}$$

By Lemma 3,

$$\nabla_{\mathcal{L}}\frac{\partial}{\partial x_k}(T\gamma)\in M(W_p^{[m]+1,\beta}(\mathbb{R}^n_+)\to W_p^{0,\alpha}(\mathbb{R}^n_+)). \tag{61}$$

Using the harmonicity of $T\gamma$ and (61) we find

$$\frac{\partial^{\mathcal{L}+1}(T\gamma)}{\partial y^{\mathcal{L}+1}}=-\frac{\partial^{\mathcal{L}-1}(\Delta_x(T\gamma))}{\partial y^{\mathcal{L}-1}}\in M(W_p^{[m]+1,\beta}(\mathbb{R}^n_+)\to W_p^{0,\alpha}(\mathbb{R}^n_+))$$

which together with (61) implies the inclusion (59). Combining this with (56) we find that $T\gamma\in M(W_p^{[m]+1,\beta}(\mathbb{R}^n_+)\to W_p^{[l]+1,\alpha}(\mathbb{R}^n_+))$. It remains to note that all above inclusions are accompanied by the corresponding estimates. The result follows. □

8. Extension of multipliers defined on $\partial\Omega$

We return to the assertion stated in Introduction.

Theorem 5. *Let $\gamma \in M(W_p^m(\partial\Omega) \to W_p^l(\partial\Omega))$, where m and l are nonintegers, $m \geq l > 0$, $p \in (1,\infty)$. There exists a linear extension operator*

$$\gamma \to \Gamma \in M(W_p^{t,\beta}(\Omega) \to W_p^{s,\alpha}(\Omega)),$$

where $t = [m]+1$, $s = [l]+1$, $\beta = 1 - \{m\} - 1/p$, and $\alpha = 1 - \{l\} - 1/p$.

Proof. It suffices to construct an extension Γ only for γ with sufficiently small support. To be precise, we assume that $\gamma = 0$ outside the ball $\mathcal{B}_\rho^n$ centered at $0 \in \partial\Omega$, where ρ is small enough. We introduce a cut off function $\varphi \in C_0^\infty(\mathcal{B}_{3\rho}^n)$, equal to one on $\mathcal{B}_{2\rho}^n$. Let us define cartesian coordinates $\zeta = (\xi,\eta)$ with the origin 0, where $\xi \in \mathbb{R}^{n-1}$ and $\eta \in \mathbb{R}^1$. Let $\Omega \cap \mathcal{B}_{3\rho}^n = \{\zeta : \xi \in \mathcal{B}_{3\rho}^{n-1}, \eta > f(\xi)\}$, where f is a smooth function. We make the standard change of variables $\kappa : \zeta \to (x,y)$, where $x = \xi$, $y = \eta - f(\xi)$. The diffeomorphism κ maps $\Omega \cap \mathcal{B}_{3\rho}^n$ into the half space $\mathbb{R}_+^n = \{(x,y) : x \in \mathbb{R}^{n-1}, y > 0\}$. Clearly, the function $\tilde{\gamma} = \gamma \circ \kappa^{-1}$ belongs to $M(W_p^m(\mathbb{R}^{n-1}) \to W_p^l(\mathbb{R}^{n-1}))$. Its harmonic extension to $\mathbb{R}_+^n$, denoted by $\tilde{\Gamma}$, is in $M(W_p^{t,\beta}(\mathbb{R}_+^n) \to W_p^{s,\alpha}(\mathbb{R}_+^n))$ and satisfies the estimate (22) according to Theorem 4. Hence the function $\gamma = \big(\tilde{\Gamma} \circ \kappa\big)\varphi$ is a desired extension. The proof is complete. $\square$

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