Traces of multipliers in pairs of weighted Sobolev spaces

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Abstract. We prove that the pointwise multipliers acting in a pair of fractional Sobolev spaces form the space of boundary traces of multipliers in a pair of weighted Sobolev space of functions in a domain.

1. Introduction

By a multiplier acting from one Banach function space $S_1$ into another $S_2$ we call a function $\gamma$ such that $\gamma u \in S_2$ for any $u \in S_1$. By $M(S_1 \to S_2)$ we denote the space of multipliers $\gamma : S_1 \to S_2$ with the norm

$$
\|\gamma\|_{M(S_1 \to S_2)} = \sup\{\|\gamma u\|_{S_2} : \|u\|_{S_1} \leq 1\}.
$$

We write $MS$ instead of $M(S \to S)$, where $S$ is a Banach function space. We shall use the same notation both for spaces of scalar and vector-valued multipliers.

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Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$. It is well known that the fractional Sobolev space $W^s_p(\partial \Omega)$ is the space of traces of the weighted Sobolev space $W^{s, \alpha}_p(\Omega)$ endowed with the norm

$$
\left( \int_{\Omega} (\text{dist}(x, \partial \Omega))^{\rho \alpha} \sum_{\{r \leq \rho \leq s\}} |D^r u|^p \, dx \right)^{1/p},
$$

where $\alpha = 1 - \{l\} - 1/p$, $s = \lfloor l \rfloor + 1$ and $p \in (1, \infty)$ (see [5]). It is straightforward to deduce from this fact that the trace $\gamma$ of the function

$$
(1) \quad \Gamma \in M(W^{l, \beta}_p(\Omega) \rightarrow W^{s, \alpha}_p(\Omega))
$$

belongs to $M(W^m_p(\partial \Omega) \rightarrow W^d_p(\partial \Omega))$. Here $m$ and $l$ are nonintegers, $m \geq l > 0$, $s$ and $\alpha$ are given above, $t = \lfloor m \rfloor + 1$, $\beta = 1 - \lfloor m \rfloor - 1/p$.

In the present paper we prove that the converse assertion is also true showing that there exists an extension $\Gamma$ of $\gamma \in M(W^m_p(\partial \Omega) \rightarrow W^m_p(\partial \Omega))$ subject to (1).

2. The space $M(W^m_p(\mathbb{R}^{n-1}) \rightarrow W^d_p(\mathbb{R}^{n-1}))$

By $B^n_r(x)$ we mean the ball $\{\xi \in \mathbb{R}^{n-1} : |\xi - x| < r\}$ and write $B^n_r$ instead of $B^n_r(0)$.

We need the spaces $S_{loc}$ and $S_{unif}$ of functions on $\mathbb{R}^{n-1}$ defined as follows.

By $S_{loc}$ we denote the space

$$
\{ u : \eta u \in S \text{ for all } \eta \in C_0^\infty(\mathbb{R}^{n-1}) \}
$$

and by $S_{unif}$ we mean the space

$$
\{ u : \sup_\xi \|\eta_\xi u\|_S < \infty \},
$$

where $\eta_\xi(x) = \eta(x - \xi)$, $\eta \in C_0^\infty(\mathbb{R}^{n-1})$, $\eta = 1$ on $B^n_1$. The space $S_{unif}$ is endowed with the norm

$$
\|u\|_{S_{unif}} = \sup_\xi \|\eta_\xi u\|_S.
$$

Let $W^d_p(\mathbb{R}^{n-1})$ denote the fractional Sobolev space with the norm

$$
\|D_{p, q} u; \mathbb{R}^{n-1}\|_{L_p} + \|u; \mathbb{R}^{n-1}\|_{L_p},
$$

where $D_{p, q} u; \mathbb{R}^{n-1}$ denotes the fractional derivatives of $u$ with respect to the boundary $\partial \Omega$. 


where
\begin{align*}
(2) \quad (D_{p,\gamma} u)(x) = \left( \int_{\mathbb{R}^{n-1}} |\nabla_{[\gamma]} u(x + h) - \nabla_{[\gamma]} u(x)|^p |h|^{1-n-p(l)} \, dh \right)^{1/p},
\end{align*}
with $\nabla_{[\gamma]}$ being the gradient of order $[\gamma]$, i.e. $\nabla_{[\gamma]} = \{\partial_{x_1}^{\tau_1}, \ldots, \partial_{x_{n-1}}^{\tau_{n-1}}\}$, 
$\tau_1 + \ldots + \tau_{n-1} = [\gamma]$.

In this section we collect some known properties of multipliers between fractional Sobolev spaces $W^m_p(\mathbb{R}^{n-1})$ and $W^l_p(\mathbb{R}^{n-1})$, $m \geq l \geq 0$. The equivalence $a \sim b$ means that $a/b$ is bounded and separated from zero by positive constants depending on $n$, $p$, $m$, and $l$.

**Proposition 1** ([3]). Let $m$ and $l$ be nonintegers, $m \geq l \geq 0$, and let $p \in (1, \infty)$.

(i) There holds
\begin{align*}
||\gamma; \mathbb{R}^{n-1}||_{M(W^m_p(\mathbb{R}^{n-1}) \rightarrow W^l_p(\mathbb{R}^{n-1}))} \sim ||D_{p,\gamma}; \mathbb{R}^{n-1}||_{M(W^m_p(\mathbb{R}^{n-1}) \rightarrow L_p)} + ||\gamma; \mathbb{R}^{n-1}||_{M(W^m_p(\mathbb{R}^{n-1}) \rightarrow L_p)}.
\end{align*}

(ii) If $\gamma \in M(W^m_p(\mathbb{R}^{n-1}) \rightarrow W^l_p(\mathbb{R}^{n-1}))$ then for any multi-index $\sigma$, $|\sigma| \leq [\gamma]$,
\begin{align*}
D^\sigma \gamma \in M(W^m_p(\mathbb{R}^{n-1}) \rightarrow W^{l-|\sigma|}_p(\mathbb{R}^{n-1})).
\end{align*}

(iii) Let $0 < \lambda < \mu$. Then
\begin{align*}
||\gamma; \mathbb{R}^{n-1}||_{M(W^m_p(\mathbb{R}^{n-1}) \rightarrow L_p)} \leq c ||\gamma; \mathbb{R}^{n-1}||_{M(W^m_p(\mathbb{R}^{n-1}) \rightarrow L_p)}^{\lambda/\mu}.
\end{align*}

**Proposition 2** ([3]). Let $m$ and $l$ be nonintegers, $m \geq l \geq 0$, and let $p \in (1, \infty)$. Then
\begin{align*}
||\gamma; \mathbb{R}^{n-1}||_{M(W^m_p(\mathbb{R}^{n-1}) \rightarrow L_p)} \sim \sup_{e \in \mathbb{R}^{n-1}, \text{dim}(e) \leq 1} \frac{||D_{p,\gamma} e||_{L_p}}{(\text{cap}_{p,m}(e))^{1/p}}
\end{align*}
\begin{align*}
&+ \left\{ \begin{array}{ll}
\sup_{x \in \mathbb{R}^{n-1}} ||\gamma; \mathbb{R}^{n-1}(x)||_{L_1}, & \text{for } m > l, \\
||\gamma; \mathbb{R}^{n-1}||_{L_\infty}, & \text{for } m = l,
\end{array} \right.
\end{align*}
where $e$ is a compact set in $\mathbb{R}^{n-1}$ and $\text{cap}_{p,m}(e)$ is the $(p,m)$-capacity of $e$ defined by
\begin{align*}
\text{cap}_{p,m}(e) = \inf \{ ||u; \mathbb{R}^{n-1}||_{W^m_p}^p : u \in C_0^\infty(\mathbb{R}^{n-1}), u \geq 1 \text{ on } e \}.
\end{align*}
For $l = 0$ one should replace $D_{p,\gamma}$ by $\gamma$. 
Upper estimates for the norm in $M(W^m_p(\mathbb{R}^{n-1}) \to W^l_p(\mathbb{R}^{n-1}))$ are given in the following assertion. By $\text{mes}_{n-1}$ we mean the $(n-1)$-dimensional Lebesgue measure of a compact set $e$.

**Proposition 3 ([3]).** Let $m$ and $l$ be nonintegers, $m \geq l \geq 0$, and let $p \in (1, \infty)$.

(i) If $mp < n - 1$, then

\[
\|\gamma; \mathbb{R}^{n-1}\|_{M(W^m_p \to W^l_p)} \leq \sup_{e \subset \mathbb{R}^{n-1}, \text{diam}(e) \leq 1} \frac{\|D_{p,l} \gamma; e\|_{L_p}}{(\text{mes}_{n-1}(e))^{1/p-m/(n-1)}}
\]

\[
+ \begin{cases} 
\sup_{x \in \mathbb{R}^{n-1}} \|\gamma; B^{n-1}_1(x)\|_{L_1} & \text{for } m > l, \\
\|\gamma; \mathbb{R}^{n-1}\|_{L_\infty} & \text{for } m = l.
\end{cases}
\]

(ii) If $mp = n - 1$, then

\[
\|\gamma; \mathbb{R}^{n-1}\|_{M(W^m_p \to W^l_p)} \leq \sup_{e \subset \mathbb{R}^{n-1}, \text{diam}(e) \leq 1} (\log \frac{2^{n-1}}{\text{mes}_{n-1}(e)})^{1-1/p} \|D_{p,l} \gamma; e\|_{L_p}
\]

\[
+ \begin{cases} 
\sup_{x \in \mathbb{R}^{n-1}} \|\gamma; B^{n-1}_1(x)\|_{L_1} & \text{for } m > l, \\
\|\gamma; \mathbb{R}^{n-1}\|_{L_\infty} & \text{for } m = l.
\end{cases}
\]

Now we list lower estimates for the norm in $M(W^m_p \to W^l_p)$.

**Proposition 4.** [3] Let $m$ and $l$ be nonintegers, $m \geq l \geq 0$, and let $p \in (1, \infty)$.

(i) If $mp < n - 1$, then

\[
\|\gamma; \mathbb{R}^{n-1}\|_{M(W^m_p \to W^l_p)} \geq \sup_{e \subset \mathbb{R}^{n-1}, \tau \in (0,1]} \frac{\|D_{p,l} \gamma; B^{n-1}_1(x)\|_{L_p}}{\tau^{p(n-1)/p-m}}
\]

\[
+ \begin{cases} 
\sup_{x \in \mathbb{R}^{n-1}} \|\gamma; B^{n-1}_1(x)\|_{L_1} & \text{for } m > l, \\
\|\gamma; \mathbb{R}^{n-1}\|_{L_\infty} & \text{for } m = l.
\end{cases}
\]

(ii) If $mp = n - 1$, then

\[
\|\gamma; \mathbb{R}^{n-1}\|_{M(W^m_p \to W^l_p)} \geq \sup_{e \subset \mathbb{R}^{n-1}, \tau \in (0,1]} (\log \frac{2}{\tau})^{1-1/p} \|D_{p,l} \gamma; B^{n-1}_1(x)\|_{L_p}
\]

\[
+ \begin{cases} 
\sup_{x \in \mathbb{R}^{n-1}} \|\gamma; B^{n-1}_1(x)\|_{L_1} & \text{for } m > l, \\
\|\gamma; \mathbb{R}^{n-1}\|_{L_\infty} & \text{for } m = l.
\end{cases}
\]
3. Multipliers in pairs of weighted Sobolev spaces in $\mathbb{R}^n_+$

3.1 Preliminary facts. Let $\mathbb{R}^n_+$ denote the upper half-space $\{z = (x, y) : x \in \mathbb{R}^{n-1}, y > 0\}$. We introduce the weighted Sobolev space $W^{s, \alpha}_p(\mathbb{R}^n_+)$ with the norm

$$
\|(\min\{1, y\})^\alpha \nabla_s U; \mathbb{R}^n_+\|_{L^p} + \|(\min\{1, y\})^\alpha U; \mathbb{R}^n_+\|_{L^p},
$$

where $s$ is nonnegative integer.

It is well known that the fractional Sobolev space $W^s_p(\mathbb{R}^{n-1})$, is the space of traces on $\mathbb{R}^{n-1}$ of functions in the space $W^{s, \alpha}_p(\mathbb{R}^n_+)$, where $s = \lfloor l \rfloor + 1$, $\alpha = 1 - \{l\} - 1/p$, and $p \in (1, \infty)$ (see [5]). We show that a similar result holds for spaces of pointwise multipliers acting in a pair of fractional Sobolev spaces. To be precise, we prove that for all noninteger $m$ and $l$, $m \geq l > 0$, the multiplier space $M(W^m_p(\mathbb{R}^{n-1})) \to W^l_p(\mathbb{R}^{n-1}))$ is the space of traces on $\mathbb{R}^{n-1}$ of functions in $M(W^{m, \beta}_p(\mathbb{R}^n_+)) \to W^{l, \alpha}_p(\mathbb{R}^n_+))$, where $s$ and $\alpha$ are as above and $\beta = 1 - \{m\} - 1/p$, $t = \lceil m \rceil + 1$. Different positive constants depending on $n, p, l, m, s, t$ will be denoted by $c$. We shall omit $\mathbb{R}^n_+$ in notations of norms.

We introduce the notion of $(p, s, \alpha)$-capacity of a compact set $e \subset \mathbb{R}^n_+$:

$$
cap_{p, s, \alpha}(e) = \inf \{\|U; \mathbb{R}^n_+\|_{L^p}^p : U \in C_0^\infty(\mathbb{R}^n_+), U \geq 1 \text{ on } e\}.
$$

The following result is essentially known (see [2], Sections 8.1, 8.2).

**Proposition 5.** Let $k$ be a nonnegative integer, $-1 < \beta p < p - 1$, and let $1 < p < \infty$. Then $\Gamma \in M(W^{k, \beta}_p(\mathbb{R}^n_+)) \to W^{0, \alpha}_p(\mathbb{R}^n_+))$ if and only if

$$
\sup_{e \subset \mathbb{R}^n_+ \atop \dim(e) \leq 1} \frac{\|(\min\{1, y\})^\alpha \Gamma; e\|_{L^p}}{(\cap_{p, k, \beta}(e))^{1/p}} < \infty.
$$

The equivalence relation

$$
\|\Gamma\|_{M(W^{k, \beta}_p(\mathbb{R}^n_+))} \sim \sup_{e \subset \mathbb{R}^n_+ \atop \dim(e) \leq 1} \frac{\|(\min\{1, y\})^\alpha \Gamma; e\|_{L^p}}{(\cap_{p, k, \beta}(e))^{1/p}}
$$

is valid.

We shall use some general properties of multipliers. We start with the inequality

$$
\|\Gamma\|_{M(W^{k-j, \beta}_p(\mathbb{R}^n_+))} \leq c \|\Gamma\|_{M(W^{k, \beta}_p(\mathbb{R}^n_+))}^{(k-j)/s} \|\Gamma\|_{M(W^{s-j, \alpha}_p(\mathbb{R}^n_+))}^{j/s},
$$

where $s$ is nonnegative integer.
where $0 \leq j \leq s$, $-1 < \alpha p < p - 1$, $-1 < \beta p < p - 1$, which follows from the interpolation property of weighted Sobolev spaces (see [4], Section 3.4.2).

The next assertion contains inequalities between multipliers and their mollifiers in variables $x$.

**Lemma 1.** Let $\Gamma_\rho$ denote a mollifier of a function $\Gamma$ defined by

$$
\Gamma_\rho(x, y) = \rho^{n+1} \int_{\mathbb{R}^{n-1}} K(\rho^{-1}(x - \xi))\Gamma(\xi, y) d\xi,
$$

where $K \in C^\infty_0(B^1_1)$, $K \geq 0$, and $\|K; \mathbb{R}^{n-1}\|_{L_1} = 1$. Then

$$
(6) \quad \|\Gamma_\rho\|_{M(W^{s,\alpha}_p \rightarrow W^{s,\alpha}_p)} \leq \|\Gamma\|_{M(W^{s,\alpha}_p \rightarrow W^{s,\alpha}_p)} \leq \lim_{\rho \to 0} \inf \|\Gamma_\rho\|_{M(W^{s,\alpha}_p \rightarrow W^{s,\alpha}_p)}.
$$

**Proof.** Let $U \in C^\infty_0$. By Minkowski's inequality

$$
\left( \int_{\mathbb{R}^n} (\min \{1, y\})^{\rho \alpha} |\nabla_{j,z} \int_{\mathbb{R}^{n-1}} \rho^{-n} K(\xi/\rho) \Gamma(x - \xi, y) U(x, y) d\xi|^p d\xi \right)^{1/p}
$$

$$
\leq \int_{\mathbb{R}^{n-1}} \rho^{-n} K(\xi/\rho) \left( \int_{\mathbb{R}^n} (\min \{1, y\})^{\rho \alpha} |\nabla_{j,z} (\Gamma(x, y) U(x + \xi, y))|^p d\xi \right)^{1/p} d\xi,
$$

where $j = 0, s$. Therefore,

$$
\|\Gamma_\rho u\|_{W^{s,\alpha}_p} \leq \|\Gamma\|_{M(W^{s,\alpha}_p \rightarrow W^{s,\alpha}_p)}
$$

$$
\times \int_{\mathbb{R}^{n-1}} \rho^{-n} K(\xi/\rho) \left( \int_{\mathbb{R}^n} (\min \{1, y\})^{\rho \alpha} |\nabla_{j,z} U(x + \xi, y)|^p d\xi \right)^{1/p}
$$

$$
+ \left( \int_{\mathbb{R}^n} (\min \{1, y\})^{\rho \alpha} |U(x + \xi, y)|^p d\xi \right)^{1/p} d\xi.
$$

This gives the left inequality (6). The right inequality (6) follows from

$$
\|\Gamma u\|_{W^{s,\alpha}_p} = \lim \inf_{\rho \to 0} \|\Gamma_\rho U\|_{W^{s,\alpha}_p} \leq \lim \inf_{\rho \to 0} \|\Gamma_\rho\|_{M(W^{s,\alpha}_p \rightarrow W^{s,\alpha}_p)} \|U\|_{W^{s,\alpha}_p}.
$$

The proof is complete. $\square$

**Lemma 2.** Let $\Gamma \in L_{p,loc}$, $p \in (1, \infty)$, $-1 < \beta p < p - 1$, and let $U$ be an arbitrary function in $C^\infty_0(\mathbb{R}^n)$. The best constant in the inequality

$$
(7) \quad \|\min \{1, y\} \Gamma \nabla U\|_{L_p} + \|\min \{1, y\} \Gamma U\|_{L_p} \leq C \|U\|_{W^{s,\alpha}_p}
$$


is equivalent to the norm \( \| \Gamma \|_{M(W_p^{s-1,\sigma} \rightarrow W_p^{0,\sigma})} \).

**Proof.** The estimate \( C \leq c \| \Gamma \|_{M(W_p^{s-1,\sigma} \rightarrow W_p^{0,\sigma})} \) is obvious. To derive the converse estimate, we introduce a function \( x \rightarrow \sigma \) which is positive on \([0, \infty)\) and is equal to \( x \) for \( x > 1 \). For any \( U \in C_0^\infty(\mathbb{R}_+^n) \) there holds

\[
U = (-\Delta)^s(\sigma(-\Delta))^{-[k]-1}u + T(-\Delta)u,
\]

where \( T \) is a function in \( C_0^\infty([0, \infty)) \). Since

\[
(-\Delta)^s = (-1)^s \sum_{|\tau| = s} \frac{s!}{\tau!} D^{2\tau},
\]

it follows from (7) and the theorem on the boundedness of convolution operators in weighted \( L_p \) spaces (see [1]) that

\[
\int_{\mathbb{R}_+^n} \left( \min\{1, y\} \right)^{p\alpha} |\Gamma(z)U(z)|^p dz 
\leq c \left( \| \nabla \sigma(\sigma(-\Delta))^{-s} U \|_{W_p^{s,\sigma}}^p + \| TU \|_{W_p^{s,\sigma}}^p \right) \leq c \| U \|_{W_p^{s,\sigma}}^p.
\]

The proof is complete. \( \Box \)

4. Characterisation of the space \( M(W_p^{t,\beta} \rightarrow W_p^{s,\alpha}) \)

Here we derive necessary and sufficient conditions for a function to belong to the space \( M(W_p^{t,\beta} \rightarrow W_p^{s,\alpha}) \) for \( p \in (1, \infty) \) with \( \alpha \) and \( \beta \) satisfying

(8) \[ -1 < \alpha p < p - 1, \quad -1 < \beta p < p - 1, \quad t \geq s. \]

These inequalities will be assumed everywhere. We start with an assertion on derivatives of multipliers. We shall omit \( \mathbb{R}_+^{n+1} \) in notations of spaces, norms, and integrals.

**Lemma 3.** Suppose

\[
\Gamma \in M(W_p^{t,\beta} \rightarrow W_p^{s,\alpha}) \cap M(W_p^{s-1,\beta} \rightarrow W_p^{0,\alpha}), \quad p \in (1, \infty).
\]

Then \( D^\sigma \Gamma \in M(W_p^{t,\beta} \rightarrow W_p^{s-|\sigma|,\alpha}) \) for any multiindex \( \sigma \) of order \( |\sigma| \leq s \)

and

(9) \[
\| D^\sigma \Gamma \|_{M(W_p^{t,\beta} \rightarrow W_p^{s-|\sigma|,\alpha})} 
\leq \varepsilon \| \Gamma \|_{M(W_p^{s-1,\beta} \rightarrow W_p^{0,\alpha})} + c(\varepsilon) \| \Gamma \|_{M(W_p^{s,\beta} \rightarrow W_p^{s,\alpha})},
\]

where \( \varepsilon \) is an arbitrary positive number.
Proof. Let $U \in W_p^k,\alpha$ and let $\varphi$ be an arbitrary function in $C^\infty_0$. Applying Leibniz formula

$$D^\sigma(\varphi U) = \sum_{\tau_\sigma \geq \tau \geq 0} \frac{\sigma!}{\tau!(\sigma - \tau)!} D^\tau_\sigma \varphi D^{\sigma - \tau} U,$$

we find

$$\int \varphi U(-D)^\sigma \Gamma dz = \int \Gamma D^\sigma(\varphi U) dz$$

$$= \sum_{\tau_\sigma \geq \tau \geq 0} \frac{\sigma!}{\tau!(\sigma - \tau)!} \Gamma D^\tau \varphi D^{\sigma - \tau} U dz$$

$$= \int \varphi \sum_{\tau_\alpha \geq \tau \geq 0} \frac{\sigma!}{\tau!(\sigma - \tau)!} (-D)^\tau (\Gamma D^{\sigma - \tau} U) dz.$$

Therefore,

$$UD^\sigma \Gamma = \sum_{\tau_\sigma \geq \tau \geq 0} \frac{\sigma!}{\tau!(\sigma - \tau)!} (D)^\tau (\Gamma (-D)^{\sigma - \tau} U),$$

which implies the estimate

$$||UD^\sigma \Gamma||_{W_p^{\sigma-1,\alpha}} \leq c \sum_{\tau_\sigma \geq \tau \geq 0} ||\Gamma D^{\sigma - \tau} U||_{W_p^{\sigma-1,\alpha}}.$$

Hence, it suffices to prove (9) for $|\sigma| = 1$. We have

$$||U\nabla\Gamma||_{W_p^{\sigma-1,\alpha}} \leq ||U\Gamma||_{W_p^{\sigma,\alpha}} + ||\nabla U||_{W_p^{\sigma-1,\alpha}}$$

$$\leq (||\Gamma||_{M(W_p^{\sigma,\alpha} \to W_p^{\sigma,\alpha})} + ||\Gamma||_{M(W_p^{\sigma-1,\alpha} \to W_p^{\sigma-1,\alpha})}) ||U||_{W_p^{\sigma,\alpha}}.$$

Estimating the norm $||\Gamma||_{M(W_p^{\sigma-1,\alpha} \to W_p^{\sigma-1,\alpha})}$ by (5) we arrive at (9). \qed

We pass now to two-sided estimates of norms in $M(W_p^{k,\beta} \to W_p^{\sigma,\alpha})$, $p \in (1, \infty)$, given in terms of the spaces $M(W_p^{k,\beta} \to W_p^{\sigma,\alpha})$. We start with lower estimates.

**Lemma 4.** Let $\Gamma \in M(W_p^{l,\beta} \to W_p^{s,\alpha})$. Then

$$||\nabla s\Gamma||_{M(W_p^{l,\beta} \to W_p^{s,\alpha})} + ||\Gamma||_{M(W_p^{l-1,\beta} \to W_p^{s,\alpha})} \leq c ||\Gamma||_{M(W_p^{l,\beta} \to W_p^{s,\alpha})}.$$

Proof. Suppose first that $\Gamma \in M(W_p^{l-1,\beta} \to W_p^{s,\alpha})$. We have

$$||\nabla sU||_{W_p^{l,\alpha}} \leq ||\Gamma||_{M(W_p^{l,\beta} \to W_p^{s,\alpha})} ||U||_{W_p^{l,\beta}} + c \sum_{\tau_\sigma \geq \tau \geq 0} ||D^\tau UD^\sigma\Gamma||_{W_p^{\sigma,\alpha}}$$

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\begin{align}
\left( \| \Gamma \|_{M(W^{s-\alpha}_{p} \rightarrow W^{0}_{p})} + c \sum_{j=1}^{s} \| \nabla_{j} \Gamma \|_{M(W^{s-\alpha_{j}}_{p} \rightarrow W^{0}_{p})} \right) \| U \|_{W^{s}_{p}},
\end{align}

By Lemma 3,
\begin{align}
\| \nabla_{j} \Gamma \|_{M(W^{s-\alpha_{j}}_{p} \rightarrow W^{0}_{p})}
\end{align}
\begin{align}
\leq \varepsilon \| \Gamma \|_{M(W^{s-\alpha_{j}}_{p} \rightarrow W^{0}_{p})} + c(\varepsilon) \| \Gamma \|_{M(W^{s-\alpha_{j}}_{p} \rightarrow W^{0}_{p})}.
\end{align}

Estimating the last norm by (5) we obtain
\begin{align}
\| \nabla_{j} \Gamma \|_{M(W^{s-\alpha_{j}}_{p} \rightarrow W^{0}_{p})}
\end{align}
\begin{align}
\leq \varepsilon \| \Gamma \|_{M(W^{s-\alpha_{j}}_{p} \rightarrow W^{0}_{p})} + c(\varepsilon) \| \Gamma \|_{M(W^{s}_{p} \rightarrow W^{0}_{p})}.
\end{align}

Substitution of this into (11) gives
\begin{align}
\| \nabla_{s} U \|_{W^{0}_{p}}
\end{align}
\begin{align}
\leq \left( \varepsilon \| \Gamma \|_{M(W^{s-\alpha_{j}}_{p} \rightarrow W^{0}_{p})} + c(\varepsilon) \| \Gamma \|_{M(W^{s}_{p} \rightarrow W^{0}_{p})} \right) \| U \|_{W^{s}_{p}}.
\end{align}

Besides,
\begin{align}
\| \Gamma U \|_{W^{0}_{p}} \leq \| \Gamma \|_{M(W^{s}_{p} \rightarrow W^{0}_{p})} \| U \|_{W^{s}_{p}}.
\end{align}

Summing up two last estimates and applying Lemma 2 we arrive at
\begin{align}
\| \Gamma \|_{M(W^{s-\alpha_{j}}_{p} \rightarrow W^{0}_{p})} \leq \varepsilon \| \Gamma \|_{M(W^{s-\alpha_{j}}_{p} \rightarrow W^{0}_{p})} + c(\varepsilon) \| \Gamma \|_{M(W^{s}_{p} \rightarrow W^{0}_{p})}.
\end{align}

Hence,
\begin{align}
\| \Gamma \|_{M(W^{s-\alpha_{j}}_{p} \rightarrow W^{0}_{p})} \leq c \| \Gamma \|_{M(W^{s}_{p} \rightarrow W^{0}_{p})}.
\end{align}

Now, we are going to remove the assumption \( \Gamma \in M(W^{s-\alpha_{j}}_{p} \rightarrow W^{0}_{p}) \).

Since \( \Gamma \in M(W^{s}_{p} \rightarrow W^{0}_{p}) \), then
\begin{align}
\| \Gamma \|_{W^{0}_{p}} \leq c \| \eta \|_{W^{s}_{p}},
\end{align}
where \( \eta \in C^{\infty}_{0}(B^{2}_{2}(z)) \), \( \eta = 1 \) on \( B^{1}_{1}(z) \), and \( z \) is an arbitrary point in \( \mathbb{R}^{n}_{+} \).

Hence \( \Gamma \in W^{s}_{p,\text{unif}}(\mathbb{R}^{n}_{+}) \) which implies that for any \((n-1)\)-dimensional multiindex \( \tau \) the derivative \( D_{\tau}^{\gamma} \Gamma \) belongs to \( W^{s}_{p,\text{unif}}(\mathbb{R}^{n}_{+}) \). Therefore, \( \Gamma_{\rho} \in L^{\infty}(\mathbb{R}^{n}_{+}) \) which in its turn guarantees that \( \Gamma_{\rho} \in M(W^{s-\alpha_{j}}_{p} \rightarrow W^{0}_{p}) \).
Thus, we may put \( \Gamma_p \) into (15) in order to obtain
\[
\| \Gamma_p \|_{M(W_p^{r_1,\alpha} \to W_p^{0,\alpha})} \leq c \| \Gamma_p \|_{M(W_p^{r_1,\alpha} \to W_p^{s,\alpha})}.
\]

Letting \( \rho \to 0 \) and using Lemma 1 we arrive at (15) for all \( \Gamma \in M(W_p^{1,\beta} \to W_p^{0,\alpha}) \).

To estimate the first term in the right-hand side of (10), we combine (15) with (12) for \( j = s \).

The estimate converse to (10) is contained in the following lemma.

**Lemma 5.** Let \( \Gamma \in M(W_p^{1,\beta} \to W_p^{0,\alpha}) \) and let \( \nabla_y \Gamma \in M(W_p^{1,\beta} \to W_p^{0,\alpha}) \). Then \( \Gamma \in M(W_p^{q,\alpha} \to W_p^{0,\alpha}) \) and the estimate
\[
(16) \quad \| \Gamma \|_{M(W_p^{q,\alpha} \to W_p^{s,\alpha})} \leq c \left( \| \nabla_y \Gamma \|_{M(W_p^{q,\alpha} \to W_p^{s,\alpha})} + \| \Gamma \|_{M(W_p^{q,\alpha} \to W_p^{s,\alpha})} \right)
\]
is valid.

**Proof.** By Lemma 4 and (5) we have
\[
\| \nabla_y \Gamma \|_{M(W_p^{q,\alpha} \to W_p^{s,\alpha})} \leq c \| \Gamma \|_{M(W_p^{q,\alpha} \to W_p^{s,\alpha})}
\]

(17)
\[
\leq c \| \| \Gamma \|_{M(W_p^{q,\alpha} \to W_p^{s,\alpha})} \| \Gamma \|_{M(W_p^{q,\alpha} \to W_p^{s,\alpha})},
\]

where \( j = 1, \ldots, s \). For any \( U \in C_0^\infty \),
\[
\| (\min \{1, y \}) a \nabla_s (\Gamma U) \|_{L_p} \leq c \sum_{j=0}^s \| (\min \{1, y \}) a \| \nabla_j \Gamma \| \nabla_{s-j} U \|_{L_p}
\]
\[
\leq c \left( \| \nabla_s \Gamma \|_{M(W_p^{q,\alpha} \to W_p^{s,\alpha})} + \| \Gamma \|_{M(W_p^{q,\alpha} \to W_p^{s,\alpha})} \right)
\]
\[
+ \sum_{j=1}^{s-1} \| \nabla_j \Gamma \|_{M(W_p^{q,\alpha} \to W_p^{s,\alpha})} \| U \|_{W_p^{q,\alpha}}.
\]

This and (17) imply
\[
\| (\min \{1, y \}) a \nabla_s (\Gamma U) \|_{L_p}
\]
\[
\leq c \left( \| \nabla_s \Gamma \|_{M(W_p^{q,\alpha} \to W_p^{s,\alpha})} + \| \Gamma \|_{M(W_p^{q,\alpha} \to W_p^{s,\alpha})} \right) \| U \|_{W_p^{q,\alpha}}.
\]

It remains to note that
\[
\| (\min \{1, y \}) a \Gamma U \|_{L_p} \leq \| \Gamma \|_{M(W_p^{r_1,\alpha} \to W_p^{0,\alpha})} \| U \|_{W_p^{r_1,\alpha}}.
\]
The proof is complete. \( \square \)

Using Lemmas 4 and 5 we arrive at the following description of the space 
\[ M\left( W_p^{t,\beta}(\mathbb{R}_+^n) \rightarrow W_p^{s,\alpha}(\mathbb{R}_+^n) \right). \]

**Theorem 1.** A function \( \Gamma \) belongs to the space 
\[ M\left( W_p^{t,\beta} \rightarrow W_p^{s,\alpha} \right) \]  
if and only if \( \Gamma \in W_{p,\text{loc}}^{s,\alpha} \), \( \nabla_s \Gamma \in M(W_p^{t,\beta} \rightarrow W_p^{0,\alpha}) \).  
Moreover,
\[ ||\Gamma||_{M(W_p^{t,\beta} \rightarrow W_p^{s,\alpha})} \sim ||\nabla_s \Gamma||_{M(W_p^{t,\beta} \rightarrow W_p^{0,\alpha})} + ||\Gamma||_{M(W_p^{t,\beta} \rightarrow W_p^{0,\alpha})}. \]

The equivalence relation (4) enables one to reformulate Theorem 1.

**Theorem 2.** A function \( \Gamma \) belongs to the space 
\[ M\left( W_p^{t,\beta} \rightarrow W_p^{s,\alpha} \right) \]  
if and only if \( \Gamma \in W_{p,\text{loc}}^{s,\alpha} \) and for any compact set \( e \subset \mathbb{R}_+^n \)
\[ ||(\min\{1, y\})^a \nabla_s \Gamma; e||_{L_p} \leq c \text{ cap}_{p,t,\beta}(e), \]
\[ ||(\min\{1, y\})^a \Gamma; e||_{L_p} \leq c \text{ cap}_{p,t,\beta}(e). \]

Moreover,
\[ ||\Gamma||_{M(W_p^{t,\beta} \rightarrow W_p^{s,\alpha})} \]
\[ \sim \sup_{\substack{e \subset \mathbb{R}_+^n \cap \text{diam}(e) \leq 1 \\text{diam}(e) \leq 1}} \left( \frac{||(\min\{1, y\})^a \nabla_s \Gamma; e||_{L_p}}{(\text{cap}_{p,t,\beta}(e))^{1/p}} + \frac{||(\min\{1, y\})^a \Gamma; e||_{L_p}}{(\text{cap}_{p,t,\beta}(e))^{1/p}} \right). \]

An important particular case of Theorem 2 is \( t = s \).

**Corollary 1.** A function \( \Gamma \) belongs to the space 
\[ MW_p^{s,\alpha} \]  
if and only if \( \Gamma \in W_{p,\text{loc}}^{s,\alpha} \) and for any compact set \( e \subset \mathbb{R}_+^n \)
\[ ||(\min\{1, y\})^a \nabla_s \Gamma; e||_{L_p} \leq c \text{ cap}_{p,s,\alpha}(e). \]

Moreover,
\[ ||\Gamma||_{MW_p^{s,\alpha}} \sim \sup_{\substack{e \subset \mathbb{R}_+^n \cap \text{diam}(e) \leq 1 \\text{diam}(e) \leq 1}} \left( \frac{||(\min\{1, y\})^a \nabla_s \Gamma; e||_{L_p}}{(\text{cap}_{p,s,\alpha}(e))^{1/p}} + ||\Gamma||_{L_{\infty}} \right). \]

5. Trace theorems for multipliers in weighted Sobolev spaces

We start with the following simple fact concerning traces of multipliers.
Theorem 3. Let $m$ and $l$ be positive noninteger, $m \geq l$ and let

$$
\Gamma \in M(W^{l,\beta}(\mathbb{R}^n_+) \to W^{s,\alpha}(\mathbb{R}^n_+))
$$

where $t = [m] + 1$, $s = [l] + 1$, $\beta = 1 - \{m\} - 1/p$, and $\alpha = 1 - \{l\} - 1/p$. If $\gamma$ is the trace of $\Gamma$ on $\mathbb{R}^{n-1}$, then

$$
\gamma \in M(W^m(\mathbb{R}^n-1) \to W^l_p(\mathbb{R}^{n-1}))
$$

and the estimate

$$(20) \quad \|\gamma; \mathbb{R}^{n-1}\|_{M(W^m \to W^l_p)} \leq c \|\Gamma; \mathbb{R}^n_+\|_{M(W^{s,\alpha} \to W^{l,\beta})}
$$

holds.

Proof. Let $U \in W^{l,\beta}(\mathbb{R}^n_+)$ and let $u$ be the trace of $U$ on $\mathbb{R}^{n-1}$. By setting $\Gamma U$ and $\gamma u$ instead of $U$ and $u$, respectively, in the inequality

$$
\|u; \mathbb{R}^{n-1}\|_{W^l_p} \leq c \|U; \mathbb{R}_+^n\|_{W^{s,\alpha}}
$$

we arrive at the estimate

$$
\|\gamma u; \mathbb{R}^{n-1}\|_{W^l_p} \leq c \|\Gamma; \mathbb{R}_+^n\|_{M(W^{s,\alpha} \to W^{l,\beta})} \|U; \mathbb{R}_+^n\|_{W^{s,\alpha}}.
$$

Minimizing the right-hand side over all extensions $U$ of $u$ we obtain

$$
\|\gamma u; \mathbb{R}^{n-1}\|_{W^l_p} \leq c \|\Gamma; \mathbb{R}_+^n\|_{M(W^{s,\alpha} \to W^{l,\beta})} \|u; \mathbb{R}^{n-1}\|_{W^l_p}
$$

which gives (20). \hfill \Box

We state an extension theorem for functions in $M(W^m(\mathbb{R}^n-1) \to W^l_p(\mathbb{R}^{n-1}))$ to be proved in Section 7.

Theorem 4. Let $m$ and $l$ be positive noninteger, $m \geq l$, $p \in (1, \infty)$, and let

$$
\gamma \in M(W^m_p(\mathbb{R}^n-1) \to W^l_p(\mathbb{R}^{n-1})).
$$

Then the Dirichlet problem

$$(21) \quad \Delta \Gamma = 0 \text{ on } \mathbb{R}^n_+, \quad \Gamma|_{\mathbb{R}^{n-1}} = \gamma
$$

has a unique solution in $M(W^{l,\beta}(\mathbb{R}^n_+) \to W^{s,\alpha}(\mathbb{R}^n_+))$, where $t = [m] + 1$, $s = [l] + 1$, $\beta = 1 - \{m\} - 1/p$, and $\alpha = 1 - \{l\} - 1/p$. There holds the estimate

$$(22) \quad \|\Gamma; \mathbb{R}^n_+\|_{M(W^{s,\alpha} \to W^{l,\beta})} \leq c \|\gamma; \mathbb{R}^{n-1}\|_{M(W^m \to W^l)}.
$$
6. Auxiliary estimates for an extension operator

6.1 Pointwise estimate for $T\gamma$ and $\nabla T\gamma$. For functions $\gamma \in L_{1,m,loc}(\mathbb{R}^{n-1})$, we introduce the operator

$$ (T\gamma)(x,y) = y^{1-n} \int_{\mathbb{R}^{n-1}} \zeta \left( \frac{x - \xi}{y} \right) \gamma(\xi) d\xi, \quad (x,y) \in \mathbb{R}^n, $$

where $\zeta$ is a continuously differentiable function defined on $\mathbb{R}^n_+$ outside the origin. We assume that

$$ (|z| + 1)|\nabla \zeta(z)| + |\zeta(z)| \leq C (|z| + 1)^{-n}. \tag{24} $$

**Lemma 6.** Let $\gamma \in M(W_p^{m-l}(\mathbb{R}^{n-1}) \to L_p(\mathbb{R}^{n-1}))$, where $m \geq l$ and $1 < p < \infty$. Then

$$ |T\gamma(z)| + y|\nabla(T\gamma(z))| \leq c(1 + y^{m-1}) ||\gamma||_{M(W_p^{m-l} \to L_p)}. \tag{25} $$

**Proof.** In view of (24)

$$ |T\gamma(z)| + y|\nabla(T\gamma(z))| \leq cy^{1-n} \left( \int_{E_y^{-1}(x)} |\gamma(\xi)| d\xi + y^n \int_{\mathbb{R}^{n-1} \setminus E_y^{-1}(x)} \frac{|\gamma(\xi)| d\xi}{|\xi - z|^n} \right). $$

By Hölder’s inequality,

$$ \int_{E_y^{-1}(x)} |\gamma(\xi)| d\xi \leq cy^{1-n} (\int_{E_y^{-1}(x)} |\gamma|^{(n-1)(p-1)/p} d\xi)^{1/(n-1)}. \tag{26} $$

Let $y \in (0, 1)$. The right-hand side in (26) does not exceed

$$ cy^{-m+l+n-1} \sup_{x \in \mathbb{R}^{n-1}} (1 + r^{m-l} r^{-n}) ||\gamma||_{L_p} \tag{27} $$

This and Proposition 2 show that for $y \in (0, 1)$

$$ \int_{E_y^{-1}(x)} |\gamma(\xi)| d\xi \leq cy^{-m+l+n-1} ||\gamma||_{M(W_p^{m-l} \to L_p)}. \tag{28} $$

Suppose $y > 1$. Since

$$ \text{cap}_{p,m-l}(B_r^{n-1}(x)) \sim r^{n-1} \text{ for } r > 1, \tag{29} $$
it follows that the right-hand side of (26) is dominated by
\[ cy^{n-1} \left( \text{cap}_{p,m-1}(B_{y}^{n-1}(x)) \right)^{-1/p} \left\| \gamma; B_{y}^{n-1}(x) \right\|_{L_{p}}. \]

Combining this with (27) and Proposition 2 we conclude that
\[ \int_{E_{y}^{n-1}(x)} |\gamma(\xi)| d\xi \leq cy^{n-1}(1 + y^{l-m}) \left\| \gamma; B_{y}^{n-1} \right\|_{M(W_{p}^{m-1} \rightarrow L_{p})}. \]

We now estimate the second integral in the right-hand side of (25). Clearly,
\[ \int_{R^{n-1}\setminus E_{y}^{n-1}(x)} \frac{|\gamma(\xi)| d\xi}{|\xi - x|^{n}} \leq n \int_{y}^{\infty} \frac{d\rho}{\rho^{n+1}} \int_{E_{\rho}^{n-1}(x)} |\gamma(\xi)| d\xi. \]

By Hölder’s inequality the right-hand side of (31) admits the majorant
\[ c \int_{y}^{\infty} \rho^{-2-\frac{n-1}{p}} \left\| \gamma; B_{\rho}^{n-1}(x) \right\|_{L_{p}} d\rho. \]

Using (29) we see that the function (32), for \( y > 1 \), does not exceed
\[ cy^{-1} \sup_{r \in (0,1]} \left( \text{cap}_{p,m-1}(B_{r}^{n-1}(x)) \right)^{-1/p} \left\| \gamma; B_{r}^{n-1}(x) \right\|_{L_{p}} \]

which in view of Proposition 2 is dominated by
\[ cy^{-1} \left\| \gamma; B_{y}^{n-1} \right\|_{M(W_{p}^{m-1} \rightarrow L_{p})}. \]

Let \( y < 1 \). Then
\[ \int_{y}^{1} \rho^{-2-\frac{n-1}{p}} \left\| \gamma; B_{\rho}^{n-1}(x) \right\|_{L_{p}} d\rho \]
\[ \leq cy^{-m+1-1} \sup_{r \in (0,1]} \left( 1 + r^{m-1-\frac{n-1}{p}} \right) \left\| \gamma; B_{r}^{n-1}(x) \right\|_{L_{p}} \]
\[ \leq cy^{-m+1-1} \sup_{r \in (0,1]} \left( \text{cap}_{p,m-1}(B_{r}^{n-1}(x)) \right)^{-1/p} \left\| \gamma; B_{r}^{n-1}(x) \right\|_{L_{p}}. \]

Furthermore, by (29)
\[ \int_1^\infty \rho^{-2 - \frac{n-k}{p}} \| \gamma; B_\rho^n(x) \|_{L_p} \, d\rho \]

\[ \leq c \int_1^\infty \rho^{-2} (\text{cap}_{p,m-l}(B_\rho^n(x)))^{-1/p} \| \gamma; B_\rho^n(x) \|_{L_p} \, d\rho \]

\[ \leq c \sup_{r \in \mathbb{N}^{-1}} (\text{cap}_{p,m-l}(B_r^n(x)))^{-1/p} \| \gamma; B_r^n(x) \|_{L_p}. \]

Summing up this inequality and (34), and using Proposition 2 we conclude that the integral (32) is majorized, for \( y < 1 \), by

\[ cy^{-m+l-1} \| \gamma; \mathbb{R}^{n-1} \|_{M(W^{m-1}_p \to L_p)}. \]

This, together with (33), imply that for all \( y > 0 \) the integral (32) does not exceed

\[ cy^{-1} (1 + y^{l-m}) \| \gamma; \mathbb{R}^{n-1} \|_{M(W^{m-1}_p \to L_p)}. \]

Hence, the result follows from (30), (31), and (25). \( \square \)

6.2 Weighted \( L_p \)-estimates for \( T\gamma \) and \( \nabla T\gamma \).

Lemma 7. Let the extension operator \( T \) be defined by (23) and suppose that \( \gamma \in M(W^{m-1}(\mathbb{R}^{n-1}); L_p(\mathbb{R}^{n-1})) \), where \( l \in (0, 1) \), \( |m| \geq 1 \), \( 1 < p < \infty \). Then, for \( k = 1, \ldots, |m| \),

\[ \left( \int_0^1 y^{p(k-l)-1} (|T\gamma(z)| + y|\nabla(T\gamma)(z)|)^p \, dy \right)^{1/p} \]

\[ \leq c \| \gamma; \mathbb{R}^{n-1} \|_{M(W^{m-1}_p \to L_p)} \left[ |\mathcal{M}\gamma(x)| \right] \frac{n-k}{n-1}, \]

where \( \mathcal{M} \) is the Hardy-Littlewood maximal operator in \( \mathbb{R}^{n-1} \).

Proof. Let \( \delta \) be a number in \((0,1]\) to be chosen later. We set

\[ \int_0^1 y^{p(k-l)-1} (|T\gamma(z)| + y|\nabla(T\gamma)(z)|)^p \, dy = \int_0^\delta \ldots \, dy + \int_\delta^1 \ldots \, dy. \]

In view of (25)

\[ \int_0^\delta \ldots \, dy \leq c \int_0^\delta y^{p(k+1-l-n)-1} \left( \int_{B_\delta^n(x)} |\gamma(\xi)| \, d\xi \right)^p \, dy \]
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\[ + c \int_0^\delta y^{p(k+1-l)-1} \left( \int_{\mathbb{R}^{n-1}/\mathbb{R}^{n-1}(x)} \frac{|\gamma(\xi)| d\xi}{|\xi - x|^\alpha} \right)^p dy. \]

By the definition of \( \mathcal{M} \),

\[ \int_0^\delta y^{p(k+1-l)-1} \left( \int_{\mathbb{R}^{n-1}/\mathbb{R}^{n-1}(x)} |\gamma(\xi)| d\xi \right)^p dy \leq c[(\mathcal{M}_\gamma)(x)]^p \delta^{p(k-l)}. \]

Using (31) we obtain

\[ \int_0^\delta y^{p(k+1-l)-1} \left( \int_{\mathbb{R}^{n-1}/\mathbb{R}^{n-1}(x)} \frac{|\gamma(\xi)| d\xi}{|\xi - x|^\alpha} \right)^p dy \leq c[(\mathcal{M}_\gamma)(x)]^p \delta^{p(k-l)}. \]

Combining (36) and (37) we conclude that

\[ \int_0^\delta \ldots dy \leq c[(\mathcal{M}_\gamma)(x)]^p \delta^{p(k-l)}. \]

By Lemma 6,

\[ \int_\delta^1 y^{p(k-l)-1} (|T\gamma| + y|\nabla(T\gamma)(\xi)|)^p dy \]

\[ \leq c \|\gamma; \mathbb{R}^{m-1}\|_{M(W_p^{m-1} \to L_p)}^p \delta^{p(k-m)}. \]

Summing up (38) and (39) we find

\[ \int_0^1 y^{p(k-l)-1} (|T\gamma| + y|\nabla(T\gamma)(\xi)|)^p dy \]

\[ \leq c[(\mathcal{M}_\gamma)(x)]^p \delta^{p(k-l)} + \|\gamma; \mathbb{R}^{m-1}\|_{M(W_p^{m-1} \to L_p)}^p \delta^{p(k-m)}). \]

The right-hand side in this inequality attains its minimum value for

\[ \delta = \left( \frac{\|\gamma; \mathbb{R}^{m-1}\|_{M(W_p^{m-1} \to L_p)}^p}{(\mathcal{M}_\gamma)(x)^{1/(m-l)}} \right)^{1/(m-l)}. \]

The proof is complete. \( \square \)

**Lemma 8.** Let the operators \( T \) and \( D_{p,i} \) be defined by (23) and (2). Then

\[ \int_0^1 y^{p(1-l)-1} |\nabla(T\gamma)(\xi)|^p dy \leq c (D_{p,i}\gamma)(x)^p. \]
Proof. Let $R(\xi, x) = \gamma(\xi) - \gamma(x)$. Using the identity
\[
y^{-n+1} \int_{\mathbb{R}^{n-1}} \zeta \left( \frac{\xi - x}{y} \right) d\xi = \text{const}
\]
we have
\[
(40) \quad \frac{\partial T_\gamma}{\partial y}(x, y) = \frac{\partial}{\partial y} \left( y^{-n+1} \int_{\mathbb{R}^{n-1}} \zeta \left( \frac{\xi - x}{y} \right) R(\xi, x) d\xi \right).
\]
Furthermore, it is clear that
\[
\frac{\partial T_\gamma}{\partial x_j}(x, y) = y^{-n+1} \int_{\mathbb{R}^{n-1}} R(\xi, x) \frac{\partial}{\partial x_j} \left( \frac{\xi - x}{y} \right) d\xi.
\]
Therefore,
\[
|\nabla (T\gamma)(x, y)| \leq cy^{-n} \sum_{k=0}^{1} \int_{\mathbb{R}^{n-1}} |\nabla \zeta \left( \frac{\xi - x}{y} \right) | \left( 1 + \frac{|\xi - x|}{y} \right)^k |R(\xi, x)| d\xi.
\]
This estimate and (24) imply
\[
|\nabla (T\gamma)(x, y)| \leq cy^{-n} \int_{\mathbb{R}^{n-1}} \left( 1 + \frac{|\xi - x|}{y} \right)^{-n} |R(\xi, x)| d\xi
\]
\[
= cy^{-1/p} \int_{\mathbb{R}^{n-1}} \left( \frac{\xi - x}{y} \right)^{n-1/p} \left( 1 + \frac{|\xi - x|}{y} \right)^{-n} \frac{|R(\xi, x)|}{|\xi - x|^{n-1/p}} d\xi.
\]
Consequently,
\[
\int_0^1 y^{p(1-\ell)-1} |\nabla (T\gamma)(x, y)|^p dy
\]
\[
\leq c \int_0^1 \left( \int_{\mathbb{R}^{n-1}} f\left( \frac{|\xi - x|}{y} \right) \frac{|R(\xi, x)|}{|\xi - x|^{n-1/p}} d\xi \right)^p y^{p(1-\ell)-1} dy,
\]
where $f(\eta) = \eta^{n-1/p}(1 + \eta)^{-n}$. We write the last integral over $(0, 1)$ as
\[
\int_0^1 \left( \int_0^\infty f\left( \frac{t}{y} \right) g(t, x) \frac{dt}{t} \right)^p y^{p(1-\ell)-1} dy
\]
\[
= \int_0^1 \left( \int_0^\infty f(s) g(sy, x) \frac{ds}{s} \right)^p y^{p(1-\ell)-1} \frac{dy}{y},
\]
(41)
with
\[ g(t, x) = t^{1/p - 1} \int_{\partial \mathbb{R}^{n-1}} |R(t\theta + x)| d\theta. \]

By Minkowski’s inequality, the right-hand side of (41) does not exceed
\[
\left( \int_0^\infty \left( \int_0^1 \left( f(s) \left( \int_0^s (g(\tau, x))^{p\tau^{l-1}} d\tau \right)^{1/p} \frac{ds}{s} \right)^p \frac{ds}{s} \right)^p \right)
\leq \left( \int_0^\infty f(s) \left( \int_0^s (g(\tau, x))^{p\tau^{l-1}} d\tau \right)^{1/p} \frac{ds}{s} \right)^p \int_0^\infty (g(\tau, x))^{p\tau^{l-1}} d\tau.
\]

Therefore,
\[
\int_0^1 (g(\tau, x))^{p\tau^{l-1}} |\nabla (T\gamma)(x, y)|^p dy \leq c \int_0^\infty (g(\tau, x))^{p\tau^{l-1}} d\tau.
\]

It remains to note that
\[
\int_0^\infty (g(\tau, x))^{p\tau^{l-1}} d\tau \leq \int_0^\infty \tau^{-p} \left( \int_{\partial \mathbb{R}^{n-1}} |\gamma(\tau\theta + x) - \gamma(x)| d\theta \right)^p \frac{d\tau}{\tau} \\
\leq c \int_0^\infty \int_{\partial \mathbb{R}^{n-1}} |\gamma(\tau\theta + x) - \gamma(x)|^p d\theta \frac{d\tau}{\tau^{p+1}} \\
\leq c \int_{\partial \mathbb{R}^{n-1}} |\gamma(x + h) - \gamma(x)|^p \\ h^{p+n-1} dh \\
= c \left( (D_p \gamma)(x) \right)^p.
\]

The result follows. \( \square \)

### 7. Proof of Theorem 4

#### 7.1 The case \( l < 1 \)

Our aim now is to prove that for \( l < 1 \) and \( s = 1 \) the operator \( T \) defined by (23) maps \( M(W^{m}(\mathbb{R}^{n-1}) \to W^l(\mathbb{R}^{n-1})) \) into \( M(W^{m+1, \alpha}_{p} (\mathbb{R}^1) \to W^1, \alpha_{p} (\mathbb{R}^1)) \) with \( \alpha = 1 - l - 1/p, \beta = 1 - \{m\} - 1/p \) and there holds the estimate
\[
\|T\gamma; \mathbb{R}^{n-1}\|_{M(W^{m+1, \alpha}_{p} \to W^1, \alpha_{p})} \leq c \|\gamma; \mathbb{R}^{n-1}\|_{M(W^{m}_{p} \to W^l_{p})},
\]

where \( C \) is the constant in (24).
We have
\[
\|(\min\{1, y\})^a \nabla (UT\gamma); \mathbb{R}^n_+\|_{L^p}^p \leq \int_0^1 y^{\alpha p} \int_{\mathbb{R}^{n-1}} (|\nabla (T\gamma)|^p |U|^p + |T\gamma|^p |\nabla U|^p) dz + c \int_{1}^{\infty} \int_{\mathbb{R}^{n-1}} (|\nabla (T\gamma)|^p |U|^p + |T\gamma|^p |\nabla U|^p) dz
\]
\[= c \int_{0 < y < 1} \ldots dz + c \int_{y > 1} \ldots dz. \tag{44}\]

By Lemma 1, for \( y > 1 \)
\[
y |\nabla (T\gamma)(z)| + ||(T\gamma)(z)|| \leq c \||\gamma|; \mathbb{R}^{n-1}\|_{M(W^{\infty -1}_p, L_p)}.
\]

Hence,
\[
\int_{y > 1} \ldots dz \leq c \||\gamma|; \mathbb{R}^{n-1}\|_{W^{\infty -1}_p, L_p} \|U; \mathbb{R}^n_+\|_{W^{1+a}_p}. \tag{45}\]

It remains to refer to the estimate
\[
\|U; \mathbb{R}^n_+\|_{W^{1+a}_p} \leq c \|U; \mathbb{R}^n_+\|_{W^{1+a}_p}
\]
which follows from the one dimensional Hardy inequality.

Introducing the notation
\[
\mathcal{R}_0 U(z) = U(z) - \sum_{k=0}^{[m]} \frac{\partial^k}{\partial y^k} U(x, 0) \frac{y^k}{k!},
\]
\[
\mathcal{R}_1 U(z) = \begin{cases} \nabla U(z) - \sum_{k=0}^{[m]-1} \frac{\partial^k}{\partial y^k} \nabla U(x, 0) \frac{y^k}{k!} & \text{for } m > 1 \\ \nabla U(z) & \text{for } m < 1 \end{cases}
\]
we have
\[
\int_{0 < y < 1} \ldots dz \leq c \int_{0 < y < 1} y^{p(1 - l)} \sum_{j=0}^{1} |\nabla_j (T\gamma)|^p \mathcal{R}_{1-j} U(z) dz
\]
\[+ c \int_{0 < y < 1} y^{-p l - 1} (|T\gamma(z)| + y |\nabla (T\gamma)(z)|)^p \sum_{k=1}^{[m]} |\nabla_k U(x, 0)|^k y^{pk} dz
\]
\[+ c \int_{0 < y < 1} y^{p(1 - l) - 1} |\nabla T\gamma(z)|^p |U(x, 0)|^p dz. \tag{46}\]
for \( m > 1 \). In case \( m < 1 \) the second integral in the right hand side of (46) should be omitted.

By Lemma 6, for \( 0 < y < 1 \)

\[
|T\gamma(z) + y|\nabla(T\gamma)(z)| \leq c y^{1-m}\|\gamma; \mathbb{R}^{n-1}\|_{M(W^{m-1}_p \rightarrow L_p)}.
\]

Since for \( j = 0, 1 \)

\[
|\mathcal{R}_{1-j}U(z)| \leq \frac{y^{j-j}}{([m] + j - 1)!} \left( \int_0^\infty |\nabla U(x, t)| dt \right)^p dz,
\]

we have

\[
\int_{0<y<1} y^{p(1-(m))-1} |\mathcal{R}_{1-j}U(z)|^p dz
\]

\[
\leq c \int_{0<y<1} y^{-p[m]-1} \left( \int_0^y \left| \nabla [\nabla U(x, t)] \right| \left| \nabla U(x, 0) \right|^p dx \right)^p dz.
\]

By Hardy’s inequality the right-hand side does not exceed \( c \|U; \mathbb{R}^n\|_{W^{1+1/\alpha}_p}^p \).

Combining this with (47) we obtain that the first integral in the right-hand side of (46) does not exceed

\[
\|\gamma; \mathbb{R}^{n-1}\|_{M(W^{m-1}_p \rightarrow L_p)}^p \|U; \mathbb{R}^n\|_{W^{1+1/\alpha}_p}^p.
\]

We now pass to the estimate of the second integral in the right-hand side of (46) for \( k = 1, \ldots, [m], m > 1 \). Applying Lemma 7, we find

\[
\int_{0<y<1} y^{p(k-j)-1} (|T\gamma(z)| + y|\nabla(T\gamma)(z)|)^p |\nabla_k U(x, 0)|^p dx \leq c \|\gamma; \mathbb{R}^{n-1}\|_{M(W^{m-1}_p \rightarrow L_p)}^p \mathcal{M}(\gamma(x))^{\frac{p}{p-\alpha}} |\nabla_k U(x, 0)|^p dx.
\]

The last integral is not greater than

\[
\|\mathcal{M}(\gamma)|^{\frac{p}{p-\alpha}}; \mathbb{R}^{n-1}\|_{M(W^{m-1}_p \rightarrow L_p)}^p |\nabla_k U(x, 0)|^p W^{-\alpha}_p.
\]

Using Proposition 2 with \( \lambda = m - k, \mu = m - l \) and Verbitsky’s theorem on the boundedness of the maximal operator \( \mathcal{M} \) in the space \( M(W^{m-\lambda}_p(\mathbb{R}^{n-1}) \rightarrow L_p(\mathbb{R}^{n-1})) \) (see [3], Ch.2), we find that (51) is dominated by

\[
c \|\gamma; \mathbb{R}^{n-1}\|_{M(W^{m-1}_p \rightarrow L_p)}^{p(m-k)} |U(x, 0)|^p W^{-\alpha}_p,
\]
Hence and by (50)
\[
\int_{0 < y < 1} y^{p(k-\ell)-1} |(T\gamma(z) + y|\nabla(T\gamma)(z))|^p |\nabla_k U(x, 0)|^p dz
\leq c \|\gamma; \mathbb{R}^n\|^{p}_{M(W_p^{n-1} \to L_p)} \|U; \mathbb{R}^n\|^{p}_{W_p^{m+1, \alpha}}.
\]  
(52)

By Lemma 8, the integral
\[
\int_{0 < y < 1} y^{p(1-\ell)-1} |\nabla(T\gamma)(z)|^p |U(x, 0)|^p dz
\]

does not exceed
\[
c \int_{\mathbb{R}^{n-1}} (D_{p,1}\gamma(x))^p |U(x, 0)|^p dx
\leq c \|D_{p,1}\gamma; \mathbb{R}^n\|^{p}_{M(W_p^{n-1} \to L_p)} \|U(., 0); \mathbb{R}^n\|^{p}_{W_p^m}
\]

(54)

Thus we arrive at the inequality
\[
\int_{0 < y < 1} y^{p\alpha} |
\nabla(UT\gamma)(z)|^p dz \leq c \|\gamma; \mathbb{R}^n\|^{p}_{M(W_p^{n-1} \to W_p^m)} \|U; \mathbb{R}^n\|^{p}_{W_p^{m+1, \alpha}}.
\]

It remains to estimate the integral
\[
\int_{0 < y < 1} y^{p(1-\ell)-1} |(T\gamma)(z)|^p |U(z)|^p dz.
\]

Clearly,
\[
\int_{0 < y < 1} y^{p(1-\ell)-1} |(T\gamma)(z)|^p |U(z)|^p dz \leq \int_{0 < y < 1} y^{p(1-\ell)-1} |(T\gamma)(z)|^p |\mathcal{R}_0 U(z)|^p dz
\]

\[
+ \sum_{k=0}^{[m]} \int_{0 < y < 1} y^{p(k-\ell)-1} |(T\gamma)(z)|^p |\nabla_k U(x, 0)|^p dz.
\]  
(55)
By (47) and (48) with \( j = 1 \) we have

\[
\int_{0 < y < 1} y^{p(1-l) - 1} |(T \gamma)(z)|^p \| \mathcal{R}_0 U(z) \|^p \, dz
\]

\[
\leq c \| \gamma \|_{\mathbb{R}^{n-1}}^p \|_{M(W^{m-1}_p \rightarrow L_p)}^p \int_{0 < y < 1} y^{p(1-(m-1)) - 1} \left( \int_0^y |\nabla_{[m] + 1} U(x,t)| \, dt \right)^p \, dz
\]

which by Hardy’s inequality is dominated by (49). In view of (52)

\[
\int_{0 < y < 1} y^{p(k-l) - 1} |(T \gamma)(z)|^p |\nabla_k U(x,0)|^p \, dz
\]

\[
\leq c \| \gamma \|_{\mathbb{R}^{n-1}}^p \|_{M(W^{m-1}_p \rightarrow L_p)}^p \| U; \mathbb{R}^n \|^p \| W^{m+\sigma}_{W_p} \|^p_\mathbb{R}^{n+1}\sigma.
\]

Thus we arrive at the estimate

\[
\int_{0 < y < 1} \ldots \, dz \leq c \| \gamma \|_{\mathbb{R}^{n-1}}^p \|_{M(W^{m-1}_p \rightarrow W^1_p)}^p \| U; \mathbb{R}^n \|^p \| W^{m+\sigma}_{W_p} \|^p_\mathbb{R}^{n+1}\sigma.
\]

Since the Poisson kernel satisfies condition (24), Theorem 4 with \( l < 1 \) follows.

### 7.2 The case \( l > 1 \).

**Lemma 9.** Let \( m \) and \( l \) be nonintegers, \( m \geq l > 0 \), and let \( T \) be the extension operator (23). Suppose that \( \gamma \in M(W^{m-1}_p(\mathbb{R}^{n-1}) \rightarrow L_p(\mathbb{R}^{n-1})) \). Then

\[
T \gamma \in M(W^{m-\lfloor l \rfloor}_p \rightarrow \mathbb{R}^{n+1}_+), \mathbb{R}^{n,\alpha}(\mathbb{R}^{n+1}_+)
\]

and

\[
\| T \gamma; \mathbb{R}^{n,\alpha}_+ \|_{M(W^{m-\lfloor l \rfloor,\alpha}_p \rightarrow \mathbb{R}^{n,\alpha}_+)} \leq c \| \gamma; \mathbb{R}^{n-1} \|_{M(W^{m-1}_p \rightarrow L_p)}.
\]

**Proof.** To begin with, let \( [m] = [l] \). Then by (47)

\[
\int_{0 < y < 1} y^{p(l-1) - 1} |(T \gamma)(z)|^p \, dz
\]

\[
\leq c \| \gamma; \mathbb{R}^{n-1} \|_{M(W^{m-1}_p \rightarrow L_p)} \int_{0 < y < 1} y^{p(l-1) - 1} |U(z)|^p \, dz
\]

which gives the result.
Suppose $[m] \geq [l] + 1$. We introduce the function

$$\mathcal{R}U = U(z) - \sum_{j=0}^{[m]-[l]-1} \frac{\partial^j U}{\partial y^j}(x, 0) \frac{y^j}{j!}$$

which, clearly, satisfies

$$|\mathcal{R}U(z)| \leq \frac{y^{[m]-[l]-1}}{([m] - [l] - 1)!} \int_0^y |\nabla_{[m]-[l]} U(x, t)| dt.$$ 

This and (47) imply

$$\int_{0 < y < 1} y^{p(1-(l)) - 1} |T\gamma(z)|^p |\mathcal{R}U(z)|^p dz$$

$$\leq c||\gamma; \mathbb{R}^n||^p_{M(W_p^{m-1-\varepsilon} \to L_p)} \int_{0 < y < 1} y^{-p(m) - 1} \left( \int_0^y |\nabla_{[m]-[l]} U(x, t)| dt \right)^p dz.$$ 

By Hardy’s inequality the right-hand side is dominated by

$$c||\gamma; \mathbb{R}^n||^p_{M(W_p^{m-1-\varepsilon} \to L_p)} ||U; \mathbb{R}^n||^p_{W_p^{m-1-\varepsilon}}.$$ 

Furthermore, by Lemma 7 with $m$ replaced by $m - [l]$, $l$ replaced by $\{l\}$ and $k = j + 1$ we have for $j = 0, \ldots, [m] - [l] - 1$

$$\int_{0 < y < 1} y^{p(j+1-(l)) - 1} |T\gamma(z)|^p |\nabla_j U(x, 0)|^p dz$$

(57) $\leq c||\gamma; \mathbb{R}^n||^p_{M(W_p^{m-1-\varepsilon} \to L_p)} \int_{\mathbb{R}^{m-1}} (\mathcal{M}\gamma(x))^{p \frac{[m]-[l]-1}{m-1-\varepsilon}} |\nabla_j U(x, 0)|^p dx.$

The last integral is dominated by

$$||\mathcal{M}\gamma; \mathbb{R}^n||^p_{M(W_p^{m-1-\varepsilon} \to L_p)} ||U; \mathbb{R}^n||^p_{W_p^{m-1-\varepsilon}}$$

which by Proposition 3 does not exceed

$$||\mathcal{M}\gamma; \mathbb{R}^n||^p_{M(W_p^{m-1-\varepsilon} \to L_p)} ||U; \mathbb{R}^n||^p_{W_p^{m-1-\varepsilon}}.$$ 

Hence and by (57)

$$\int_{0 < y < 1} y^{p(j+1-(l)) - 1} |T\gamma(z)|^p |\nabla_j U(x, 0)|^p dz$$
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\[ \leq c_{\gamma} \| \gamma \|_{W^{m-1}}^{p} \| u \|_{L_p(U^n)}^{p} \| W^{m-1}_{W^{\alpha}} \|_{L_p(U^n)}^{p}. \]

The result follows. \qed

*Proof of Theorem 4* for \( l > 1 \). Suppose Theorem has been proved for \( [l] = 1, \ldots, \mathcal{L} - 1 \), where \( \mathcal{L} \geq 2 \). Let \( [l] = \mathcal{L} \) and let

\[ \gamma \in M(W^{m}(\mathbb{R}^{n-1}) \to W^{d}_{p}(\mathbb{R}^{n-1})) \] for \( m \geq \mathcal{L} \).

Let \( T \gamma \) denote the Poisson integral. Since by Proposition 1 one has

\[ \gamma \in M(W^{m-1}(\mathbb{R}^{n-1}) \to L_p(\mathbb{R}^{n-1})), \]

it follows from Lemma 9 that

\[ T \gamma \in M(W^{m-1}_{p}(\mathbb{R}^{n-1}) \to W^{0,\alpha}_{p}(\mathbb{R}^{n})). \]

and (56) holds. Next we show that

\[ \nabla_{\mathcal{L}+1}(T \gamma) \in M(W^{m+1}_{p}(\mathbb{R}^{n} \to W^{0,\alpha}_{p}(\mathbb{R}^{n})). \]

Using Proposition 1, we obtain

\[ \frac{\partial \gamma}{\partial x_k} \in M(W^{m}_{p}(\mathbb{R}^{n-1}) \to W^{d-1}_{p}(\mathbb{R}^{n-1}), \quad k = 1, \ldots, n - 1. \]

Then, by the induction hypothesis applied to \( \partial \gamma/\partial x_k \),

\[ \frac{\partial}{\partial x_k}(T \gamma) = T \frac{\partial \gamma}{\partial x_k} \in M(W^{m+1}_{p}(\mathbb{R}^{n} \to W^{1,\alpha}_{p}(\mathbb{R}^{n})). \]

By Lemma 3,

\[ \nabla_{\mathcal{L}} \frac{\partial}{\partial x_k}(T \gamma) \in M(W^{m+1}_{p}(\mathbb{R}^{n} \to W^{0,\alpha}_{p}(\mathbb{R}^{n})). \]

Using the harmonicity of \( T \gamma \) and (61) we find

\[ \frac{\partial^{\mathcal{L}+1}(T \gamma)}{\partial y^{\mathcal{L}+1}} = -\frac{\partial^{\mathcal{L}-1}(\Delta_{\gamma}(T \gamma))}{\partial y^{\mathcal{L}-1}} \in M(W^{m+1}_{p}(\mathbb{R}^{n} \to W^{0,\alpha}_{p}(\mathbb{R}^{n})). \]

which together with (61) implies the inclusion (59). Combining this with (56) we find that \( T \gamma \in M(W^{m+1}_{p}(\mathbb{R}^{n} \to W^{[d+1,\alpha]}_{p}(\mathbb{R}^{n})). \) It remains to note that all above inclusions are accompanied by the corresponding estimates. The result follows. \qed
8. Extension of multipliers defined on $\partial \Omega$

We return to the assertion stated in Introduction.

**Theorem 5.** Let $\gamma \in M(W^m_p(\partial \Omega) \to W^l_p(\partial \Omega))$, where $m$ and $l$ are nonintegers, $m \geq l > 0$, $p \in (1, \infty)$. There exists a linear extension operator

$$\gamma \to \Gamma \in M(W^{t,\beta}_p(\Omega) \to W^{s,\alpha}_p(\Omega)),$$

where $t = [m] + 1$, $s = [l] + 1$, $\beta = 1 - \{m\} - 1/p$, and $\alpha = 1 - \{l\} - 1/p$.

**Proof.** It suffices to construct an extension $\Gamma$ only for $\gamma$ with sufficiently small support. To be precise, we assume that $\gamma = 0$ outside the ball $B^n_\rho$ centered at $0 \in \partial \Omega$, where $\rho$ is small enough. We introduce a cut off function $\varphi \in C_c^\infty(B^n_{3\rho})$, equal to one on $B^n_{2\rho}$. Let us define cartesian coordinates $\zeta = (\xi, \eta)$ with the origin $0$, where $\xi \in \mathbb{R}^{n-1}$ and $\eta \in \mathbb{R}^1$. Let $\Omega \cap B^n_{3\rho} = \{\zeta : \xi \in B_{3\rho}^{n-1}, \eta > f(\xi)\}$, where $f$ is a smooth function. We make the standard change of variables $\kappa: \zeta \to (x, y)$, where $x = \xi$, $y = \eta - f(\xi)$. The diffeomorphism $\kappa$ maps $\Omega \cap B^n_{3\rho}$ into the half space $\mathbb{R}^n_+ = \{(x, y) : x \in \mathbb{R}^{n-1}, y > 0\}$. Clearly, the function $\tilde{\gamma} = \gamma \circ \kappa^{-1}$ belongs to $M(W^m_p(\mathbb{R}^{n-1}) \to W^l_p(\mathbb{R}^{n-1}))$. Its harmonic extension to $\mathbb{R}^n_+$, denoted by $\tilde{\Gamma}$, is in $M(W^{t,\beta}_p(\mathbb{R}^n_+) \to W^{s,\alpha}_p(\mathbb{R}^n_+))$ and satisfies the estimate (22) according to Theorem 4. Hence the function $\gamma = (\tilde{\Gamma} \circ \kappa)\varphi$ is a desired extension. The proof is complete. $\square$

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