An inequality for first-order differences∗

Gord Sinnamon

(Communicated by Victor Burenkov)

2000 Mathematics Subject Classification. 26D10.

Keywords and phrases. Differences, integral inequalities.

Abstract. A question about comparing norms of difference operators that was raised in [1] and presented at the Fourth ISAAC Congress is answered in the affirmative.

Let $1 \leq p < \infty$ and suppose $f : [0, \infty) \to \mathbb{R}$. For $h > 0$, set

$$F(h) \equiv \|\Delta_h f\|_{L^p(0,h)} = \left( \int_0^h \left| f(x+h) - f(x) \right|^p \, dx \right)^{1/p}$$

and

$$G(h) \equiv \|\Delta_h f\|_{L^p(h,3h)} = \left( \int_h^{3h} \left| f(x+h) - f(x) \right|^p \, dx \right)^{1/p}.$$

We wish to compare $F$ and $G$ as functions of $h$. It is easy to see that pointwise comparisons of $F$ and $G$ are impossible for arbitrary $f$ as $F$ depends on values of $f$ on the interval $[0, 2h]$ but $G$ depends on values of

∗Supported by the Natural Sciences and Engineering Research Council of Canada.
An inequality for first-order differences

$f$ on the interval $[h, 4h]$. It is possible to compare the functions $F$ and $G$ in norm, however. A special case of [1, Corollary 2] shows that there is a constant $A$ such that

$$\left( \int_0^\delta F(h)^\theta h^{-\theta} \frac{dh}{h} \right)^{1/\theta} \leq A \left( \int_0^\delta G(h)^\theta h^{-\theta} \frac{dh}{h} \right)^{1/\theta}$$

holds for all measurable $f$. Here $l > 0$, $0 < \delta \leq \infty$, and $1 \leq \theta < \infty$. (The analogous result in the case $\theta = \infty$ is also given there.)

It does not follow from the results of [1] that this inequality still holds if the constant 3 in the definition of $G$ is replaced by a smaller number. That is the substance of the question raised in [1, Remark 3]. Corollary 4, below, shows that the above inequality holds when 3 is replaced by any $\beta > 1$.

The key is the following lemma comparing terms of the form $\|\Delta_t f\|_{L^p(b, c)}$ where as usual

$$\|f\|_{L^p(b, c)} = \begin{cases} \left( \int_b^c |f(x)|^p \, dx \right)^{1/p}, & 0 < p < \infty; \\ \text{ess sup}_{b \leq x \leq c} |f(x)|, & p = \infty. \end{cases}$$

**Lemma 1.** Suppose $1 \leq p \leq \infty$. If $b$, $c$, $t$, $B$, $C$, and $T$ are positive real numbers with $b < c$ and $B < C$ then there exists a finite sequence $s_1, s_2, \ldots, s_N$ of positive real numbers such that

$$\|\Delta_t f\|_{L^p(b, c)} \leq \sum_{n=1}^N \|\Delta_{s_n} T f\|_{L^p(s_n B, s_n C)}$$

for all measurable $f : [0, \infty) \to \mathbb{R}$. The $s_n$ can be chosen so that

$$s_n < \frac{c + t}{B + T}$$

for $n = 1, 2, \ldots, N$.

**Proof.** We begin by supposing that we have a solution to the following problem: Find positive integers $M$ and $K$ and real numbers $t_m$, $b_k$, and $r_{m,k}$ for $m = 1, \ldots, M$ and $k = 1, \ldots, K$ such that

$$0 = t_0 < t_1 < \cdots < t_m = t,$$

$$b_0 < \cdots < b_{K-1} < c \leq b_K,$$

$$\frac{b_k + r_{m,k}}{C} = \frac{\frac{t_m - r_{m,k}}{T}}{\frac{t_m}{T}} < \frac{t_m - r_{m,k}}{T} = \frac{b_k - 1 + r_{m,k}}{B}.$$
An application of Minkowski’s inequality gives

$$\|\Delta_t f\|_{L^p(b,c)} \leq \|\Delta_t f\|_{L^p(b_0,b_K)} \leq \sum_{k=1}^{K} \|\Delta_t f\|_{L^p(b_{k-1},b_k)}$$

and, since

$$\Delta_t f(x) = \sum_{m=1}^{M} f(x + t_m) - f(x + t_{m-1}) = \sum_{m=1}^{M} \Delta_{t_{m-1}}f(x + t_{m-1}),$$

another application yields

$$\|\Delta_t f\|_{L^p(b_{k-1},b_k)} \leq \sum_{m=1}^{M} \|\Delta_{t_{m-1}}f\|_{L^p(b_{k-1}+t_{m-1},b_k+t_{m-1})}$$

for each $k$. To reach the conclusion of the lemma it is enough to show that for each $m$ and $k$,

$$(5) \quad \|\Delta_{t_{m-1}}f\|_{L^p(b_{k-1}+t_{m-1},b_k+t_{m-1})}$$

can be dominated by a sum of terms of the form

$$(6) \quad \|\Delta_{sT}f\|_{L^p(sB,sC)}, \text{ for } s > 0.$$

Fix $m$ and $k$ and write $r = r_{m,k}$. Rearranging terms in (4) gives

$$b_{k-1} + r = (t_m + b_{k-1})/(1 + T/B) > 0$$

and we see that $x + r > 0$ for all $x \in (b_{k-1}, b_k)$. Thus we may write

$$\Delta_{t_{m-1}}f(x + t_{m-1}) = f(x + t_m) - f(x + t_{m-1})$$
$$= f(x + t_m) - f(x + r) + f(x + r) - f(x + t_{m-1})$$
$$= \Delta_{t_{m-1}}f(x + r) - \Delta_{t_{m-1}}f(x + r)$$

for such $x$. A third application of Minkowski’s inequality shows that (5) is no greater than

$$\|\Delta_{t_{m-1}}f\|_{L^p(b_{k-1}+r,b_k+r)} + \|\Delta_{t_{m-1}}f\|_{L^p(b_{k-1}+r,b_k+r)}.$$

Using (4), with $s' = (t_m - r)/T$ and $s = (t_{m-1} - r)/T$, we can dominate the last expression by

$$\|\Delta_{s'T}f\|_{L^p(s'B,s'C)} + \|\Delta_{sT}f\|_{L^p(s'B,s'C)},$$
An inequality for first-order differences

which is a sum of terms of the form (6) as required.

To complete the proof we provide a solution to the problem (2)-(4) and verify that $s$ and $s'$ satisfy the upper bound (1). Since $0 < B < C$, both

$$
\mu = \frac{C(B + T)}{B(C + T)} \quad \text{and} \quad \nu = \frac{C + T}{B + T}
$$

are greater than 1. Choose a positive integer $M$ so that $t < b(\mu^M - 1)$ and set $\varepsilon = t/(\mu^M - 1)$. For each $m$ and $k$ define

$$
t_m = \varepsilon(\mu^m - 1) \quad \text{and} \quad b_k = \nu^k(b - \varepsilon) + \varepsilon.
$$

Evidently, (2) is satisfied. The choice of $M$ ensures that $0 < \varepsilon < b$ so the $b_k$'s are increasing and unbounded. Therefore we can choose $K$ to satisfy (3).

The inequality in (4) is automatically satisfied and the two equations in (4) reduce to

$$
r_{m,k} = \frac{Ct_m - Tb_k}{C + T} \quad \text{and} \quad r_{m,k} = \frac{Bt_m - Tb_{k-1}}{B + T}.
$$

A routine calculation shows that these two expressions coincide so that either may be used to define $r_{m,k}$ so that (4) is satisfied. Finally, we estimate $s$ and $s'$ (for fixed $m$ and $k$) by

$$
s < s' = \frac{t_m - r}{T} = \frac{b_{k-1} + t_m}{B + T} < \frac{c + t}{B + T}.
$$

This completes the proof. \(\square\)

Lemma 1 remains valid with the $L^p$-norm replaced by any Banach function space norm since only the triangle inequality is needed. Moreover, it is easily extended to quasinormed function spaces at the expense of an additional constant. In particular, if $0 < p < 1$ one obtains

$$
\| \Delta_tf \|_{L^p(b, c)}^p \leq \sum_{n=1}^{N} \| \Delta_{s_n}Tf \|_{L^p(s_nB, s_nC)}^p.
$$

and hence

$$
\| \Delta_tf \|_{L^p(b, c)} \leq N^{1/p - 1} \sum_{n=1}^{N} \| \Delta_{s_n}Tf \|_{L^p(s_nB, s_nC)}.
$$

Our main result shows that the norms of $F$ and $G$ and other similar expressions are all comparable in great generality. For convenience we
introduce notation for the following weighted Lebesgue norms of functions of the variable $h$.

$$
\| g(h) \|_{L^p(0,\delta)}^\theta = \begin{cases} 
\left( \int_{\delta}^\delta |g(h)|^\theta h^{-\theta} \, dh/h \right)^{1/\theta}, & 1 \leq \theta < \infty; \\
\text{ess sup}_{0 < h < \delta} h^{-1}|g(h)|, & \theta = \infty. 
\end{cases}
$$

Here $g(h)$ is understood to be formula involving the variable $h$ rather than a function name. In addition to homogeneity, the triangle inequality, and the Fatou property of these norms we will need the following dilation property, easily proved by a change of variable. For any $s > 0$,

$$
(7) \quad \| g(h) \|_{L^p(0,\delta)} = s^\theta \| g(h/s) \|_{L^p(0,s\delta)}.
$$

Note also that $\| g(h) \|_{L^p(0,\delta_1)} \leq \| g(h) \|_{L^p(0,\delta_2)}$ whenever $\delta_1 \leq \delta_2$.

**Theorem 2.** Suppose $1 \leq p \leq \infty$, $1 \leq \theta \leq \infty$, $l > 0$, and $0 < \delta \leq \infty$. If $t > 0$, $0 \leq b < c$, $T > 0$, and $0 \leq B < C$ then, for all measurable $f : [a, \infty) \to \mathbb{R}$,

$$
(8) \quad \| \Delta_thf \|_{L^p(a+bh,ah+ch)} \|_{L^p(0,\delta)} \leq A \| \Delta_Thf \|_{L^p(a+Bh,ah+Ch)} \|_{L^p(0,\delta)}.
$$

The constant $A$ above depends only on $t$, $b$, $c$, $T$, $B$, $C$ and $l$.

**Proof.** By translating the function $f$ the theorem is easily reduced to the case $a = 0$ so we assume $a = 0$ henceforth.

We first prove the theorem in the case that $b$ and $B$ are positive. Lemma 1 yields $s_1, s_2, \ldots, s_N$ such that $s_n < (c + t)/(B + T)$ for all $n$ and

$$
(9) \quad \| \Delta_tf \|_{L^p(b,c)} \leq \sum_{n=1}^N \| \Delta_sTf \|_{L^p(s_B,s_C)}
$$

for all $f$. For each $h$, define the dilation $f_h$ by $f_h(x) = f(hx)$ and observe that $\Delta_tf_h(x) = \Delta_thf(hx)$. Changes of variable show that

$$
\| \Delta_tf_h \|_{L^p(b,c)} = h^{-1/p} \| \Delta_thf \|_{L^p(bh,ch)}
$$

and

$$
\| \Delta_sTf_h \|_{L^p(s_B,s_C)} = h^{-1/p} \| \Delta_sTf \|_{L^p(s_Bh,s_Ch)}
$$

so inequality (9), applied to $f_h$, implies

$$
\| \Delta_thf \|_{L^p(bh,ch)} \leq \sum_{n=1}^N \| \Delta_sTf \|_{L^p(s_Bh,s_Ch)}.
$$
It is important to point out that \( N \) and \( s_1, s_2, \ldots, s_N \) do not depend on \( h \). We use the triangle inequality and property (7) to get

\[
\left\| \Delta_{ih} f \right\|_{L^p(bh, ch)} \leq \sum_{n=1}^{N} \left\| \Delta_{sn} T_{h} f \right\|_{L^p(bh, ch)} \left\| L^p_{\nu}(0, \delta) \right\|
\]

\[
= \sum_{n=1}^{N} s_n \left\| \Delta_{T_{h}} f \right\|_{L^p(ch, ch)} \left\| L^p_{\nu}(0, s_n \delta) \right\|
\]

\[
\leq \left( \sum_{n=1}^{N} s_n \right) \left\| \Delta_{T_{h}} f \right\|_{L^p(ch, ch)} \left\| L^p_{\nu}(0, \frac{s_n \delta}{s_n \delta}) \right\|
\]

This proves the theorem in the case that \( b \) and \( B \) are positive and it is a simple matter to show that it remains valid when \( B = 0 \) since replacing \( B \) by zero only makes the right hand side of (8) larger.

The case \( b = 0 \) requires some additional argument. For each integer \( j \geq 0 \), apply (7) with \( s = 1/2 \) to get

\[
N_j = \left\| \Delta_{ih} f \right\|_{L^p(2^{-j}ch, ch)} \left\| L^p_{\nu}(0, \delta) \right\|
\]

\[
= 2^{-j} \left\| \Delta_{2ih} f \right\|_{L^p(2^{-j+1}ch, ch)} \left\| L^p_{\nu}(0, \delta/2) \right\|
\]

Note that \( N_0 = 0 \). To estimate \( N_j \) for \( j \geq 1 \) we break up \( \Delta_{2ih} f \) as

\[
\Delta_{2ih} f(x) = \Delta_{ih} f(x + ih) + \Delta_{ih} f(x)\chi_{(ch, \infty)}(x) + \Delta_{ih} f(x)\chi_{(0, ch)}(x)
\]

and use Minkowski’s inequality to obtain

\[
\left\| \Delta_{2ih} f \right\|_{L^p(2^{-j+1}ch, ch)} \leq \left\| \Delta_{ih} f \right\|_{L^p(2^{-j+2}ch, ch)} \left\| \Delta_{ih} f \right\|_{L^p(2^{-j+1}ch, ch)} \left\| \chi_{(0, ch)} \right\|
\]

Using the first part of the proof we get constants \( A_1 \) and \( A_2 \) such that

\[
\left\| \Delta_{ih} f \right\|_{L^p(ch, ch)} \leq A_1 \left\| \Delta_{ih} f \right\|_{L^p(Bh, Bh)} \left\| L^p_{\nu}(0, \delta/2) \right\|
\]

\[
\left\| \Delta_{ih} f \right\|_{L^p((ch, ch))} \leq A_2 \left\| \Delta_{ih} f \right\|_{L^p(Bh, Bh)} \left\| L^p_{\nu}(0, \delta/2) \right\|
\]

\[
\left\| \Delta_{ih} f \right\|_{L^p((ch, ch))} \leq A_2 \left\| \Delta_{ih} f \right\|_{L^p(Bh, Bh)} \left\| L^p_{\nu}(0, \delta/2) \right\|
\]
With $K = (A_1 + A_2) \|\Delta_{Th}f\|_{L^p(Bh,Ch)} \|_{L^q(0,\frac{t}{n+1},\delta)}$ we use the estimate (11) in (10) to get

$$N_j \leq 2^{-l} \left(K + \|\Delta_{th}f\|_{L^p(2^{-j},ch,ch)} \|_{L^q(0,\delta/2)} \right) \leq 2^{-l}(K + N_{j-1}).$$

A simple induction, starting with $N_0 = 0$, shows that $N_j \leq K/(2^{l} - 1)$ for all $j \geq 0$ and by the Fatou property we have

$$\|\Delta_{th}f\|_{L^p(0,\delta)} \|_{L^q(0,\delta)} = \lim_{j \to \infty} N_j \leq \frac{A_1 + A_2}{2^l - 1} \|\Delta_{Th}f\|_{L^p(Bh,Ch)} \|_{L^q(0,\frac{t}{n+1},\delta)}.$$

This completes the proof. □

If $\delta = \infty$ Theorem 2 becomes a general equivalence.

**Corollary 3.** Suppose $1 \leq p \leq \infty$, $1 \leq \theta \leq \infty$, $l > 0$, and $a \in \mathbb{R}$. If $t > 0$, $0 \leq b < c$, $T > 0$, and $0 \leq B < C$ then

$$\|\Delta_{th}f\|_{L^p(a+bh,a+ch)} \|_{L^q(0,\infty)} \approx \|\Delta_{Th}f\|_{L^p(a+Bh,a+Cch)} \|_{L^q(0,\infty)}$$

for all measurable $f : [a, \infty) \to \mathbb{R}$. The constants in this equivalence depend only on $t$, $b$, $c$, $T$, $B$, $C$ and $l$.

The answer to the question raised in [1, Remark 3] is the special case $b = 0$, $c = t = B = T = 1$, $C = \beta$ of Theorem 2.

**Corollary 4.** Suppose $1 \leq p \leq \infty$, $1 \leq \theta \leq \infty$, $l > 0$, $a \in \mathbb{R}$, and $0 < \delta \leq \infty$. If $\beta > 1$ then there exists a constant $A$ depending only on $l$ and $\beta$ such that

$$\|\Delta_{h}f\|_{L^p(a,a+h)} \|_{L^q(0,\delta)} \leq A \|\Delta_{h}f\|_{L^p(a+h,a+\beta h)} \|_{L^q(0,\delta)}$$

for all measurable $f : [a, \infty) \to \mathbb{R}$.

The techniques used in this paper do not seem to extend to higher order differences. It is natural to ask whether or not Theorem 2 holds with $\Delta$ replaced by $\Delta^k$ for $k > 1$.

**Acknowledgements.** The author would like to thank Professor Victor Burenkov for comments that greatly improved the proof of Theorem 2.

**References**

Department of Mathematics
University of Western Ontario
London, Ontario
N6A 5B7 Canada
(E-mail: sinnamon@uwo.ca)

(Received: November 2003)
Submit your manuscripts at
http://www.hindawi.com