The boundedness of commutators on locally compact Vilenkin groups

Canqin Tang, Qingguo Li and Bolin Ma *

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Abstract. Let G be a locally compact Vilenkin group. In this paper, the authors investigate the boundedness of commutators of singular integral operator on Triebel-Lizorkin spaces on G . Furthermore, the boundedness on the Herz-type Triebel-Lizorkin spaces are also studied.

1. Introduction

The commutators have been studied by many authors for a long time. A well known result which is discovered by Coifman, Rochberg and Weiss ([3], [7], [12]) is that the commutators $[b, T]$ of singular integral operators are bounded on some $L^p(\mathbb{R}^n)$ ($1 < p < \infty$) if and only if $b \in BMO$, where $[b, T]$ is defined by $[b, T]f(x) = b(x)Tf(x) - T(bf)(x)$. Later, Janson in [6] gave that $[b, T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ when $1 < p < q < \infty$ if and only if $b \in \text{Lip}_\beta$ and $\beta = n \left(\frac{1}{p} - \frac{1}{q} \right)$. In 1995, M. Paluszycński

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extended and generalized their results (see [10]). He proved that $[b, T]$ is bounded from $L^p(\mathbb{R}^n)$ to $\dot{F}_p^{\beta, \infty}(\mathbb{R}^n)$ is equivalent to $b \in \text{Lip}_\beta$, where $1 < p < \infty$, $0 < \beta < 1$. Motivated by their works, we consider the cases on Vilenkin groups. Moreover, Xu and Yang ([13]) introduced Herz-type Triebel-Lizorkin spaces on $\mathbb{R}^n$. In this paper, we continue studying the properties of commutator on Herz-type Triebel-Lizorkin spaces on Vilenkin groups.

In order to state our results more precisely we first introduce some notations and definitions.

Throughout this paper, G will denote a bounded locally compact Vilenkin group, that is, G is a locally compact Abelian group containing a strictly decreasing sequence of compact open subgroups $\{G_n\}_{n=-\infty}^\infty$ such that (a) $\cup_{n=-\infty}^\infty G_n = G$ and $\cap_{n=-\infty}^\infty G_n = 0$; (b) $\sup\{\text{order}(G_n/G_{n+1}) : n \in \mathbb{Z}\} = B < \infty$. Choose Haar measure dx on G so that $|G_0| = 1$, where $|A|$ denotes the measure of a measurable subset A of G . Let $|G_n| = (m_n)^{-1}$ for each $n \in \mathbb{Z}$. Since $2m_n \leq m_{n+1} \leq Bm_n$ for each $n \in \mathbb{Z}$, it follows that

$$\sum_{n=k}^{\infty} (m_n)^{-\alpha} \leq c(m_k)^{-\alpha}$$

and

$$\sum_{n=-\infty}^k (m_n)^\alpha \leq c(m_k)^\alpha$$

for any $\alpha > 0, k \in \mathbb{Z}$, where c is a constant independent of k . For each $n \in \mathbb{Z}$ we choose elements $z_{l,n} \in G (l \in \mathbb{Z}_+)$ so that the subsets $G_{l,n} := z_{l,n} + G_n$ of G satisfy $G_{k,n} \cap G_{l,n} = \emptyset$ if $k \neq l$ and $\cup_{l=0}^\infty G_{l,n} = G$; moreover, we choose $z_{0,n}$ such that $G_{0,n} = G_n$. We now define the function $d : G \times G \rightarrow \mathbb{R}$ by $d(x, y) = 0$ if $x - y = 0$ and $d(x, y) = (m_n)^{-1}$ if $x - y \in G_n \setminus G_{n+1}$. Then d is a metric on G and the topology on G generated by this metric is the same as the original topology on G . For $x \in G$, set $|x| = d(x, 0)$. Then $|x| = (m_n)^{-1}$ if and only if $x \in G_n \setminus G_{n+1}$. Let $S(G)$ be the space of test functions and $S'(G)$ be the distribution space on G . Set $\varphi_n = m_n \chi_{G_n} - m_{n+1} \chi_{G_{n+1}}$, where χ_{G_n} is the characteristic function of G_n .

Definition 1.1. Let $0 \leq \alpha < \infty$, $0 < p, q < \infty$, the homogeneous Herz spaces $\dot{K}_q^{\alpha, p}(G)$ are defined by

$\dot{K}_q^{\alpha, p}(G) = \{f : f \text{ is a measurable function on } G \text{ with } \|f\|_{\dot{K}_q^{\alpha, p}(G)} < \infty\}$, where

$$\|f\|_{\dot{K}_q^{\alpha, p}(G)} = \left\{ \sum_{j=-\infty}^{\infty} m_l^{-\alpha p} \|f \chi_{G_l \setminus G_{l+1}}\|_{L^q(G)}^p \right\}^{1/p}.$$

Before defining the Herz-type Triebel-Lizorkin spaces on G , we give the notes of a second space of test functions and distributions. We refer to [4]

for details. Let

$$Z(G) = \left\{ \psi \in S(G) : \hat{\psi}(0) = \int_G \psi(t) dt = 0 \right\},$$

and define the convergence in $Z(G)$ to be like in $S(G)$. Let $Z'(G)$ be the space of linear functionals on $Z(G)$ with convergence in $Z'(G)$ defined as in $S'(G)$ and ϱ denotes the set of constant distributions in $S'(G)$.

Definition 1.2. Let $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$. Then

$$\dot{F}_q^{\alpha,p}(G) = \left\{ f \in Z'(G) : \|f\|_{\dot{F}_q^{\alpha,p}(G)} := \left\| \left(\sum_{n=-\infty}^{\infty} (m_n)^{\alpha q} |f * \varphi_n|^q \right)^{1/q} \right\|_p < \infty \right\},$$

with the usual modification if $q = \infty$.

Definition 1.3. Let $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$. Then

$$\begin{aligned} \dot{K}_q^{\alpha,p} \dot{F}_\beta^s(G) &= \left\{ f \in Z'(G) : \|f\|_{\dot{K}_q^{\alpha,p} \dot{F}_\beta^s(G)} \right. \\ &\quad \left. := \left\| \left(\sum_{n=-\infty}^{\infty} ((m_n)^{s\beta} |f * \varphi_n|^\beta) \right)^{1/\beta} \right\|_{\dot{K}_q^{\alpha,p}} < \infty \right\}, \end{aligned}$$

with the usual modification if $\beta = \infty$.

Definition 1.4. Let $0 \leq \alpha < \infty$ and $0 < q < \infty$. A function $b \in L^q(G)$ is said to be a central (α, q) -block if it satisfies

- (i) for some $n \in \mathbb{Z}$, $\text{supp } b \subset G_n$;
- (ii) $\|b\|_{L^q(G)} \leq m_n^\alpha$;

Here $\text{supp } b$ denote the support of function $b(x)$.

By the conclusion in [9], the functions f in $\dot{K}_q^{\alpha,p}(G)$ have the following characterization.

Proposition 1.1 ([9]). *Let $0 < \alpha < \infty$, $1 \leq q < \infty$, $0 < p < \infty$. Then $f \in \dot{K}_q^{\alpha,p}(G)$ if and only if $f = \sum_{j=-\infty}^{\infty} \lambda_j b_j$ in distributional sense, where each b_j is a central (α, q) -block and $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$. Moreover,*

$$\|f\|_{\dot{K}_q^{\alpha,p}(G)} \sim \inf \left\{ \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{\frac{1}{p}} \right\}$$

with the infimum taken over all decompositions of f as above.

In order to study the boundedness of commutators, we also need to define Lipschitz spaces. For $x_0 \in G$, set $I_j = x_0 + G_j$,

Definition 1.5 ([8, 10, 2]). For $0 < \beta < 1$, $1 \leq q \leq \infty$, the Lipschitz spaces $\text{Lip}_\beta(G)$ are defined by

$$\text{Lip}_\beta(G) = \{f \mid \|f\|_\beta < \infty\}$$

where

$$\begin{aligned} \|f\|_\beta &= \sup_{I_j \ni x} \frac{1}{|I_j|^{1+\beta}} \int_{I_j} |f(y) - f_{I_j}| dy \\ &= \sup_{I_j \ni x} \frac{1}{|I_j|^\beta} \left(\frac{1}{|I_j|} \int_{I_j} |f(y) - f_{I_j}|^q dy \right)^{1/q}, \end{aligned}$$

with the f_{I_j} being the average of f over I_j and the usual modification in the case of $q = \infty$.

Our main results are the following.

Theorem 1.1. *Let $1 < p < \infty$, $0 < \beta < 1$, $b \in \text{Lip}_\beta(G)$. If T is a singular integral operator, then $[b, T]$ is a bounded operator from $L^p(G)$ to $\dot{F}_p^{\beta, \infty}(G)$.*

Theorem 1.2. *Let $1 < q < \infty$, $0 < \beta < 1$, $b \in \text{Lip}_\beta$, $0 < \alpha < 1 - \frac{1}{q}$. If T is a singular integral operator, then $[b, T]$ is a bounded operator from $\dot{K}_q^{\alpha, p}$ to $\dot{K}_q^{\alpha, p} \dot{F}_\beta^\infty$.*

As we will show that

$$\frac{1}{|I_j|^{1+\beta}} \int_{I_j} |[b, T]f - ([b, T]f)_{I_j}| \leq C \|b\|_\beta [M(Tf)(x) + M_r(f)(x) + M(f)(x)],$$

taking the supremum over all I_j such that $x \in I_j$, and then consider the $\dot{K}_q^{\alpha, p}$ norm of both sides, it is obviously to see that Theorem 1.2 is a consequence of the following theorem.

Theorem 1.3. Let $1 \leq r < q < \infty$. If a sublinear operator T satisfies

$$(1.1) \quad |Tf(x)| \leq C \left(\int_{\text{supp } f} \frac{|f(y)|^r}{|x-y|} dy \right)^{1/r} \quad \text{for } x \notin \text{supp } f$$

for all $f \in L^q(G)$ with compact support and T maps $L^q(G)$ into $L^q(G)$, then T is bounded on $\dot{K}_q^{\alpha, p}(G)$, provided $0 < \alpha < \frac{1}{r} - \frac{1}{q}$ and $0 < p \leq \infty$.

In section 2, we'll give the proof of Theorem 1.1. And Theorem 1.3 will be proved in section 3.

2. Boundedness of the commutator on the Triebel-Lizorkin spaces

Let T be a singular integral operator, i. e.

$$Tf(x) = \int_G K(x-y)f(y)dy,$$

where $K(x-y)$ is the kernel and $|K(x-y) - K(x_0-y)| \leq c \frac{|x-x_0|}{|y-x_0|^2}$. Before we start to show our theorems, we first give some equivalent norms of Triebel-Lizorkin spaces. In [1], A. Seeger gave the characterization of Triebel-Lizorkin spaces $\dot{F}_\alpha^{p,q}$ on $\mathbb{R}^n$ via means of maximal functions, which is introduced by Devore and Sharpley in [5] and Christ in [2]. Paralleled to the situation in [1], using the maximal operator, we can obtain the following lemma.

Write

$$D^\beta f(x) = \sup_j m_j^\beta \frac{1}{|\{y|y-x \in I_j\}|} \int_{\{y|y-x \in I_j\}} |f(y) - f_{I_j}| dy,$$

and

$$S^\beta f(x) = \sup_{x \in I_j} m_j^\beta \frac{1}{|I_j|} \int_{I_j} |f(x+h) - f(x)| dh.$$

Lemma 2.1. For $0 < \beta < 1$, $1 < p < \infty$, we have

$$\|D^\beta f\|_p \sim \|f\|_{\dot{F}_p^{\beta,\infty}(G)} \sim \|S^\beta f\|_p.$$

Proof. According to the definition, the average over I_j is dominated as

$$\begin{aligned} \frac{1}{|I_j|} \int_{I_j} |f(x+h) - f(x)| dh &\leq \frac{1}{|I_j|} \int_{I_j} |f(x+h) - f_{I_j}| dh \\ &\quad + \frac{1}{|I_j|} \int_{I_j} |f(x) - f_{I_j}| dh \\ &= A + |f(x) - f_{I_j}|. \end{aligned}$$

Replacing $x+h$ by y , A is easily controlled by

$$(2.1) \quad \frac{1}{|\{y|y-x \in I_j\}|} \int_{\{y|y-x \in I_j\}} |f(y) - f_{I_j}| dy.$$

To deal with the second term, using the differential theorem of integral, we have

$$|f(x) - f_{I_j}| = \left| \lim_{k \rightarrow -\infty, x \in I_k} \frac{1}{|I_k|} \int_{I_k} f(y) dy - f_{I_j} \right|$$

$$\begin{aligned}
&\leq \sum_{k=-\infty}^j |f_{I_{k+1}} - f_{I_k}| \\
&\leq B \sum_{k=-\infty}^j \frac{1}{|I_k|} \int_{I_k} |f(y) - f_{I_k}| dy.
\end{aligned}$$

Since $0 < \beta < 1$, $k \leq j$ and the domination of (2.1), we have

$$\begin{aligned}
&\sup_{x \in I_j} m_j^\beta \frac{1}{|I_j|} \int_{I_j} |f(x+h) - f(x)| dh \\
&\leq C \sup_{x \in I_j} m_j^\beta \sum_{k=-\infty}^j \frac{1}{|\{y|y-x \in I_k\}|} \int_{\{y|y-x \in I_k\}} |f(y) - f_{I_k}| dy \\
&\leq C \sup_{x \in I_j} \frac{1}{|\{y|y-x \in I_j\}|} \int_{\{y|y-x \in I_j\}} |f(y) - f_{I_j}| dy \sum_{k=-\infty}^j m_j^\beta \left(\frac{m_k}{m_j} \right) \\
&\leq C \sup_j m_j^\beta \frac{1}{|\{y|y-x \in I_j\}|} \int_{\{y|y-x \in I_j\}} |f(y) - f_{I_j}| dy.
\end{aligned}$$

Hence $S^\beta f \leq CD^\beta f$.

Next, we have $f = \sum_{k=-\infty}^j \varphi_k * f + \sum_{k=j+1}^{\infty} \varphi_k * f = f_1 + f_2$. It is easy to see

$$\frac{1}{|\{y|y-x \in I_j\}|} \int_{\{y|y-x \in I_j\}} |f_2(y) - f_{2_{I_j}}| dy \leq M f_2,$$

where M is maximal operator. Consequently,

$$\begin{aligned}
&\left\| \sup_j m_j^\beta \frac{1}{|\{y|y-x \in I_j\}|} \int_{\{y|y-x \in I_j\}} |f_2(y) - f_{2_{I_j}}| dy \right\|_p \\
&\leq \left\| \sup_j m_j^\beta M f_2 \right\|_p \\
&\leq C \left\| \sup_j m_j^\beta \sum_{k=j+1}^{\infty} \varphi_k * f \right\|_p \\
&\leq C \left\| \sup_k m_k^\beta |\varphi_k * f| \sum_{k=j+1}^{\infty} m_j^\beta m_k^{-\beta} \right\|_p \\
&\leq C \left\| \sup_k m_k^\beta |\varphi_k * f| \right\|_p.
\end{aligned}$$

$$\text{As } f_1 = \sum_{k=-\infty}^j \varphi_k * f,$$

$$\begin{aligned} & \left\| \sup_j m_j^\beta \frac{1}{|\{y|y-x \in I_j\}|} \int_{\{y|y-x \in I_j\}} |f_1(y) - f_{1_{I_j}}| dy \right\|_p \\ & \leq C \left\| \sup_j m_j^\beta \frac{1}{|\{y|y-x \in I_j\}|} \int_{\{y|y-x \in I_j\}} \left| \sum_{k=-\infty}^j \frac{m_k}{m_j} \sigma_k^* f(y) \right| dy \right\|_p \\ & \leq C \left\| \sup_k m_k \sigma_k^* f \right\|_p \\ & \leq C \left\| \sup_k m_k |\varphi_k * f| \right\|_p \end{aligned}$$

where $\sigma_k^* f(y) = \sup_{z \in G} \frac{|\varphi_k * f(y-z)|}{m_k |z|}$. Thus, $\|D^\beta f\|_p \leq C \|f\|_{\dot{F}_p^{\beta, \infty}}$.

Since $\varphi_k = \triangle_k - \triangle_{k+1}$, $\triangle_k = m_k \chi_{G_k}$, where χ is the characteristic function of a subset of G , we can easily show that $\|f\|_{\dot{F}_p^{\beta, \infty}} \leq C \|S^\beta f\|_p$. Lemma 2.1 is proved. $\square$

From Lemma 2.1, we know that

$$\|f\|_{\dot{F}_p^{\beta, \infty}} \simeq \left\| \sup_{I \ni x} \frac{1}{|I|^{1+\beta}} \int_I |f - f_I| \right\|_p.$$

Now we start to prove Theorem 1.1.

Proof of Theorem 1.1. Let $I_j = x_0 + G_j$ and $x \in I_j$. For a fixed $f \in L^p$, write $f_1 = f \chi_{I_j}$ and $f_2 = f - f_1$. It is easy to see that $[b, T]f = [b - b_{I_j}, T]f$, and so we have

$$\begin{aligned} & \frac{1}{|I_j|^{1+\beta}} \int_{I_j} |[b, T]f(x) - ([b, T]f)_{I_j}| dx \\ & \leq \frac{1}{|I_j|^{1+\beta}} \int_{I_j} |[b - b_{I_j}, T]f(x) - ([b - b_{I_j}, T]f)_{I_j}| dx \\ & \leq C \frac{1}{|I_j|^{1+\beta}} \int_{I_j} |[b - b_{I_j}, T]f(x) - T((b - b_{I_j})f_2)(x_0)| dx \\ & \leq C \frac{1}{|I_j|^{1+\beta}} \int_{I_j} |(b - b_{I_j})Tf(x)| + C \frac{1}{|I_j|^{1+\beta}} \int_{I_j} |T((b - b_{I_j})f_1(x))| \\ & \quad + C \frac{1}{|I_j|^\beta} \sup_{y \in I_j} |T((b - b_{I_j})f_2)(y) - T((b - b_{I_j})f_2)(x_0)| \\ & = L_1 + L_2 + L_3. \end{aligned}$$

We give the estimations for L_1 , L_2 and L_3 respectively. By the definition of the Lipschitz space, the L_1 is easily to be dominated by the maximal

operator, that is

$$\begin{aligned} \frac{1}{|I_j|^{1+\beta}} \int_{I_j} |(b - b_{I_j})Tf| &\leq \frac{1}{|I_j|^\beta} \sup_{y \in I_j} |b(y) - b_{I_j}| \frac{1}{|I_j|} \int_{I_j} |Tf(y)| dy \\ &\leq C \|b\|_\beta M(Tf)(x). \end{aligned}$$

To estimate L_2 , choosing $r > 1$, we have

$$\frac{1}{|I_j|} \|(b - b_{I_j})f_1\|_r^r \leq C \sup_{y \in I_j} |b(y) - b_{I_j}|^r M_r^r(f)(x) \leq C m_j^{-\beta r} \|b\|_\beta^r M_r^r(f)(x).$$

Consequently,

$$\|(b - b_{I_j})f_1\|_r \leq C m_j^{-\beta - \frac{1}{r}} \|b\|_\beta M_r(f)(x).$$

Then, using Hölder inequality and L^r -boundedness of singular operator, we get

$$\begin{aligned} L_2 &= C \frac{1}{|I_j|^{1+\beta}} \int_{I_j} |T((b - b_{I_j})f_1)| \\ &\leq C \frac{1}{|I_j|^{1+\beta}} \|T((b - b_{I_j})f_1)\|_r |I_j|^{1-\frac{1}{r}} \\ &\leq C \frac{1}{|I_j|^{1+\beta}} \|(b - b_{I_j})f_1\|_r |I_j|^{1-\frac{1}{r}} \\ &\leq C \|b\|_\beta M_r f(x). \end{aligned}$$

Now we turn to deal with L_3 . By the hypothesis on the kernel K , we have

$$\begin{aligned} &|T((b - b_{I_j})f_2)(y) - T((b - b_{I_j})f_2)(x_0)| \\ &= \left| \int_G (K(y - z) - K(x_0 - z))(b(z) - b_{I_j})f_2(z) dz \right| \\ &\leq C \int_{\{z | z - x_0 \in G \setminus G_j\}} \frac{|y - x_0|}{|x_0 - z|^2} |b(z) - b_{I_j}| |f(z)| dz \\ &\leq C m_j^{-1} \sum_{k=-\infty}^{j-1} m_k^2 \int_{\{z | z - x_0 \in G_k \setminus G_{k+1}\}} (|b(z) - b_{I_k}| + |b_{I_k} - b_{I_j}|) |f(z)| dz. \end{aligned}$$

It is easy to see

$$\begin{aligned} &\int_{\{z | z - x_0 \in G_k \setminus G_{k+1}\}} |b(z) - b_{I_k}| |f(z)| dz \\ &\leq \sup_{z \in \{z | z - x_0 \in G_k \setminus G_{k+1}\}} |b(z) - b_{I_k}| \int_{\{z | z - x_0 \in G_k \setminus G_{k+1}\}} |f(z)| dz \\ &\leq \|b\|_\beta m_k^{-\beta-1} (Mf)(x) \end{aligned}$$

and

$$\begin{aligned} \int_{\{z|z-x_0 \in G_k \setminus G_{k+1}\}} |b_{I_k} - b_{I_j}| |f(z)| dz &\leq |b_{I_k} - b_{I_j}| m_k^{-1} (Mf)(x) \\ &\leq \|b\|_\beta m_k^{-\beta-1} (Mf)(x), \end{aligned}$$

where $|b_{I_k} - b_{I_j}|$'s have the following estimates

$$\begin{aligned} |b_{I_k} - b_{I_j}| &= \left| \frac{1}{|I_j|} \int_{I_j} b(z) dz - b_{I_k} \right| \\ &\leq \frac{1}{|I_j|} \int_{I_j} |b(z) - b_{I_k}| dz \\ &\leq \sup_{z \in I_j} |b(z) - b_{I_k}| \\ &\leq \sup_{z \in I_k} |b(z) - b_{I_k}| \\ &\leq \|b\|_\beta m_k^{-\beta} \end{aligned}$$

for $k \leq j-1$ and $I_k \supset I_j$. Thus, we obtain

$$\begin{aligned} &|T((b - b_{I_j})f_2)(y) - T((b - b_{I_j})f_2)(x_0)| \\ &\leq C \|b\|_\beta m_j^{-1} \sum_{k=-\infty}^{j-1} m_k^2 m_k^{-\beta-1} Mf(x) \\ &\leq C \|b\|_\beta m_j^{-\beta} Mf(x) \end{aligned}$$

and we can deduce that $L_3 \leq C \|b\|_\beta Mf(x)$.

Taking account of all the estimates for L_1 , L_2 and L_3 together, we obtain

$$\frac{1}{|I_j|^{1+\beta}} \int_{I_j} |[b, T]f - ([b, T]f)_{I_j}| \leq C \|b\|_\beta (M(Tf)(x) + M_r(f)(x) + M(f)(x)).$$

Taking the supremum over all I_j such that $x \in I_j$, and the L_p -norm on both sides, and using Lemma 2.1 we conclude that

$$\|[b, T]f\|_{\dot{F}_p^{\beta, \infty}} \leq C \|b\|_\beta \|f\|_p.$$

Thus, Theorem 1.1 is proved. $\square$

3. Boundedness of some sublinear operators on Herz space

In this section, we will consider the boundedness of some sublinear operators on Herz space.

Proof of Theorem 1.3. We first prove the case when $r = 1$. Let $f \in \dot{K}_q^{\alpha, p}(G)$. Then by Proposition 1.1, we have $f(x) = \sum_{j=-\infty}^{\infty} \lambda_j b_j(x)$,

where $\left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\right)^{\frac{1}{p}} \leq C\|f\|_{\dot{K}_q^{\alpha,p}}$, and b_j 's are (α, q) -block supported in G_j , $j \in \mathbb{Z}$. Let $\chi_n = \chi_{G_n \setminus G_{n+1}}$. Since T is a sublinear operator, we have

$$\begin{aligned} \|Tf\|_{\dot{K}_q^{\alpha,p}} &= \left\{ \sum_{n=-\infty}^{\infty} m_n^{-\alpha p} \|(Tf)\chi_n\|_q^p \right\}^{1/p} \\ &\leq \left\{ \sum_{n=-\infty}^{\infty} m_n^{-\alpha p} \left(\sum_{j=-\infty}^n |\lambda_j| \|(Tb_j)\chi_n\|_q \right)^p \right. \\ &\quad \left. + \sum_{n=-\infty}^{\infty} m_n^{-\alpha p} \left(\sum_{j=n+1}^{\infty} |\lambda_j| \|(Tb_j)\chi_n\|_q \right)^p \right\}^{1/p} \\ &= (L_1 + L_2)^{1/p}. \end{aligned}$$

In the case of $0 < p \leq 1$, using L^q -boundedness of T , we get

$$\begin{aligned} L_1 &= \sum_{n=-\infty}^{\infty} m_n^{-\alpha p} \left(\sum_{j=-\infty}^n |\lambda_j| \|(Tb_j)\chi_n\|_q \right)^p \\ &\leq C \sum_{n=-\infty}^{\infty} m_n^{-\alpha p} \left(\sum_{j=-\infty}^n |\lambda_j|^p \|b_j\|_q^p \right) \\ &\leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^p m_j^{\alpha p} \left(\sum_{n=j}^{\infty} m_n^{-\alpha p} \right) \\ &\leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty. \end{aligned}$$

In the case of $1 < p \leq \infty$, by Hölder's inequality and the L^q boundedness of T , we obtain

$$\begin{aligned} L_1 &\leq C \sum_{n=-\infty}^{\infty} m_n^{-\alpha p} \left(\sum_{j=-\infty}^n |\lambda_j|^p \|b_j\|_q^p \right) \\ &\leq C \sum_{n=-\infty}^{\infty} \left\{ \left(\sum_{j=-\infty}^n |\lambda_j|^p \left(\frac{m_j}{m_n} \right)^{\alpha p/2} \right) \left(\sum_{j=-\infty}^n \left(\frac{m_j}{m_n} \right)^{\alpha p'/2} \right)^{\frac{p}{p'}} \right\} \\ &\leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty, \end{aligned}$$

where p' is the conjugate index of p .

Now we turn to estimate L_2 . For $j \geq n+1$, $G_j \subset G_{n+1} \subset G_n$. If $x \in G_n \setminus G_{n+1}$ and $y \in G_j$, then $x - y \in G_n \setminus G_{n+1}$. By the condition of T ,

we have

$$\begin{aligned}
\|(Tb_j)\chi_n\|_q &= \left(\int_{G_n \setminus G_{n+1}} |Tb_j|^q dx \right)^{1/q} \\
&\leq C \left(\int_{G_n \setminus G_{n+1}} \left(\int_{G_j} \frac{|b_j(y)|}{|x-y|} dy \right)^q dx \right)^{1/q} \\
&\leq C m_n^{1-\frac{1}{q}} \|b_j\|_q |G_j|^{\frac{1}{q'}} \\
&\leq C m_n^{1-\frac{1}{q}} m_j^\alpha m_j^{-(1-\frac{1}{q})}.
\end{aligned}$$

Similar to the estimation of L_1 , we also consider two cases.

Case A: $0 < p \leq 1$. In this case,

$$\begin{aligned}
L_2 &= \sum_{n=-\infty}^{\infty} m_n^{-\alpha p} \left(\sum_{j=n+1}^{\infty} |\lambda_j| \|(Tb_j)\chi_n\|_q \right)^p \\
&\leq C \sum_{n=-\infty}^{\infty} m_n^{-\alpha p} \left(\sum_{j=n+1}^{\infty} |\lambda_j|^p \left(\frac{m_n}{m_j} \right)^{(1-\frac{1}{q})p} m_j^{\alpha p} \right) \\
&\leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{n=-\infty}^{j-1} \left(\frac{m_n}{m_j} \right)^{(1-\frac{1}{q}-\alpha)p} \\
&\leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty
\end{aligned}$$

since $0 < \alpha < 1 - 1/q$.

Case B: $1 < p \leq \infty$. Let $\frac{1}{p} + \frac{1}{p'} = 1$. Using the Hölder's inequality,

$$\begin{aligned}
L_2 &\leq C \sum_{n=-\infty}^{\infty} m_n^{-\alpha p} \left(\sum_{j=n+1}^{\infty} |\lambda_j| \left(\frac{m_n}{m_j} \right)^{1-\frac{1}{q}} m_j^\alpha \right)^p \\
&\leq C \sum_{n=-\infty}^{\infty} \left(\sum_{j=n+1}^{\infty} |\lambda_j| \left(\frac{m_n}{m_j} \right)^{(1-\frac{1}{q}-\alpha)} \right)^p \\
&\leq C \sum_{n=-\infty}^{\infty} \left(\sum_{j=n+1}^{\infty} |\lambda_j|^p \left(\frac{m_n}{m_j} \right)^{(1-\frac{1}{q}-\alpha)p/2} \right) \\
&\quad \times \left(\sum_{j=n+1}^{\infty} \left(\frac{m_n}{m_j} \right)^{(1-\frac{1}{q}-\alpha)p'/2} \right)^{\frac{p}{p'}} \\
&\leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty.
\end{aligned}$$

Using all the estimations above, we obtain

$$\|Tf\|_{\dot{K}_q^{\alpha,p}} \leq C \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \leq C \|f\|_{\dot{K}_q^{\alpha,p}}.$$

When $r > 1$, similar to the proof of $r = 1$, we can decompose $\|Tf\|_{\dot{K}_q^{\alpha,p}}$ into two parts which are dominated by $(L_1 + L_2)^{1/p}$. The estimate for L_1 is easily obtained as in the proof of $r = 1$. We only consider to estimate L_2 now. By the hypothesis (1.1), we have

$$\begin{aligned} \|(Tb_j)\chi_n\|_q &\leq C \left(\int_{G_n \setminus G_{n+1}} \left(\int_{G_j} \frac{|b_j(y)|^r}{|x-y|} dy \right)^{\frac{q}{r}} dx \right)^{1/q} \\ &\leq C m_n^{\frac{1}{r}-\frac{1}{q}} \left(\int_{G_j} |b_j(y)|^r dy \right)^{1/r} \\ &\leq C m_n^{\frac{1}{r}-\frac{1}{q}} m_j^{\alpha+\frac{1}{q}-\frac{1}{r}}. \end{aligned}$$

So, if $0 < p \leq 1$, then

$$\begin{aligned} L_2 &\leq C \sum_{n=-\infty}^{\infty} m_n^{-\alpha p} \left(\sum_{j=n+1}^{\infty} |\lambda_j|^p \left(\frac{m_n}{m_j} \right)^{(\frac{1}{r}-\frac{1}{q})p} m_j^{\alpha p} \right) \\ &\leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{n=-\infty}^{j-1} \left(\frac{m_n}{m_j} \right)^{(\frac{1}{r}-\frac{1}{q}-\alpha)p} \right) \\ &\leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty, \end{aligned}$$

where $0 < \alpha < \frac{1}{r} - \frac{1}{q}$.

If $1 < p \leq \infty$, then

$$\begin{aligned} L_2 &\leq C \sum_{n=-\infty}^{\infty} m_n^{-\alpha p} \left(\sum_{j=n+1}^{\infty} |\lambda_j| \left(\frac{m_n}{m_j} \right)^{(\frac{1}{r}-\frac{1}{q})} m_j^{\alpha} \right)^p \\ &\leq C \sum_{n=-\infty}^{\infty} \left(\sum_{j=n+1}^{\infty} |\lambda_j| \left(\frac{m_n}{m_j} \right)^{\frac{1}{r}-\frac{1}{q}-\alpha} \right)^p \\ &\leq C \sum_{n=-\infty}^{\infty} \left(\sum_{j=n+1}^{\infty} |\lambda_j|^p \left(\frac{m_n}{m_j} \right)^{(\frac{1}{r}-\frac{1}{q}-\alpha)p/2} \right) \\ &\quad \times \left(\sum_{j=n+1}^{\infty} \left(\frac{m_n}{m_j} \right)^{(\frac{1}{r}-\frac{1}{q}-\alpha)p'/2} \right)^{\frac{p}{p'}} \end{aligned}$$

$$\leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty.$$

Combining the estimations of L_1 with L_2 , we can deduce that

$$\|Tf\|_{\dot{K}_q^{\alpha,p}} \leq C \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \leq C \|f\|_{\dot{K}_q^{\alpha,p}}.$$

This finishes the proof of Theorem 1.3. $\square$

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Department of Mathematics

Hunan University

Changsha, Hunan

P. R. China

and

Department of Mathematics

Hunan University of Arts and Science

Changde, Hunan

P. R. China

(*E-mail* : *tangcq2000@yahoo.com.cn*)

Department of Mathematics

Hunan University

Changsha, Hunan

P. R. China

(*E-mail* : *liqingguoli@yahoo.com.cn*)

Department of Mathematics

Hunan University

Changsha, Hunan

P. R. China

(*E-mail* : *blma@hnu.net.cn*)

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