A note on two-weight inequalities for multiple Hardy-type operators

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Abstract. Necessary and sufficient conditions on a pair of weights guaranteeing two-weight estimates for the multiple Riemann-Liouville transforms are established provided that the weight on the right-hand side satisfies some additional conditions.

1. Introduction

In 1985 E. Sawyer [15] solved the two-weight problem for the two-dimensional Hardy transform

$$H_2 f(x, y) = \int_0^y \int_0^x f(t, \tau)d\tau dt, \quad x, y > 0.$$ 

Namely he proved the following statement:

**Theorem A.** Let $1 < p \leq q < \infty$. Then for the boundedness of the operator $H_2$ from $L_p^w(R_+^2)$ to $L_q^w(R_+^2)$ it is necessary and sufficient that the following three independent conditions are satisfied:
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(1.1) \( A = \sup_{a,b > 0} (H^\prime_2 v(a,b))^{1/q}(H_2 \sigma(a,b))^{1/p'} < \infty, \)

where \( \sigma = w^{1-p'}, p' = \frac{p}{p-1}; \)

(ii) \( \int_0^a \int_0^b (H_2 \sigma)^{q} v \leq A^q [H_2 \sigma(a,b)]^{p/q} \)

for all \( a, b > 0; \)

(iii) \( \int_a^\infty \int_b^\infty (H^\prime_2 v)^{p'} \sigma \leq A^{p'} [H^\prime_2 v(a,b)]^{p'/q'} \)

for all \( a, b > 0, \) where \( H^\prime_2 f(x,y) = \int_{x}^{\infty} \int_{y}^{\infty} f(t,\tau)dtd\tau, \) \( x, y > 0. \)

In her doctoral thesis A. Wedestig [20] derived a two-weight criterion for the operator \( H_2 \) when the weight on the right–hand side is a product of two functions of separate variables. In particular, she proved

**Theorem B.** Let \( 1 < p \leq q < \infty \) and let \( s_1, s_2 \in (1, p). \) Suppose that the weight function \( w \) on \( \mathbb{R}^2_+ \) has the form \( w(x,y) = w_1(x)w_2(y). \) Then for the boundedness of the operator \( H_2 \) from \( L^p_w(\mathbb{R}^2_+) \) to \( L^q_v(\mathbb{R}^2_+) \) it is necessary and sufficient that

\[
A(s_1, s_2) = \sup_{t_1, t_2 > 0} \left[ W_1(t_1)^{(s_1-1)/p} W_2(t_2)^{(s_2-1)/p} \right]^{1/q} \times \left( \int_{t_1}^{\infty} \int_{t_2}^{\infty} v(x_1, x_2) W_1(x_1)^{2(p-s_2)} dx_1 dx_2 \right)^{1/q} < \infty,
\]

where \( W_1(t_1) = \int_0^{t_1} w_1^{1-p'}(x_1) dx_1 \) and \( W_2(t_2) = \int_0^{t_2} w_2^{1-p'}(x_2) dx_2. \)

Earlier some sufficient conditions for the validity of the two-weight inequality for \( H_2 \) were established in [16] and [19].

Necessary and sufficient conditions on the weight function \( v \) on \( \mathbb{R}^2_+ \) governing the trace inequality

\[
\left( \int_0^{\infty} \int_0^{\infty} |R_{a,b} f(x,y)|^q v(x,y) dx dy \right)^{1/q} \leq c \left( \int_0^{\infty} \int_0^{\infty} |f(x,y)|^p dx dy \right)^{1/p}, \quad 1 < p \leq q < \infty,
\]

\( c \) is a constant.
for the Riemann-Liouville operator with multiple kernels

\[ R_{\alpha,\beta} f(x, y) = \int_0^x \int_0^y \frac{f(t, \tau)}{(x-t)^{1-\alpha}(y-\tau)^{1-\beta}} t dt \tau, \]

where \( \alpha, \beta > 1/p \), have been obtained in [8]. Analogous problem has been solved in [9] for \( 0 < \alpha < 1/p \) and \( \beta > 1/p \).

In this paper we establish boundedness criteria for the operator \( R_{\alpha,\beta} \), \( \alpha, \beta > 1 \), from \( L^p_w(R^2_+) \) to \( L^q_v(R^2_+) \) when the weight \( w \) satisfies the one-dimensional doubling condition uniformly with respect to another variable. As a corollary we conclude that under this restriction the two-weight inequality for the operator \( H_2 \) holds if and only if the condition (1.1) is satisfied. When the weight function \( w \) has the form \( w(x, y) = w_1(x)w_2(y) \) we show that also in this case a two-weight criterion for \( H_2 \) is (1.1).

2. Preliminaries

Let \( \rho \) be an almost everywhere positive function on a subset \( E \) of \( \mathbb{R}^n \). We denote by \( L^p_\rho(E) \), \( 1 < p < \infty \), the set of all measurable functions \( f : E \rightarrow \mathbb{R}^1 \) for which the norm

\[ \|f\|_{L^p_\rho(E)} = \left( \int_E |f(x)|^p \rho(x) dx \right)^{1/p} \]

is finite.

Let us recall some well-known results for one-dimensional Hardy-type transforms.

A solution of the two-weight problem for the one-dimensional Hardy transform

\[ H f(x) = \int_0^x f(t) dt \]

has been given by B. Muckenhoupt [13] for \( p = q \); by V. Kokilashvili [6], J. Bradley [2] and V. Maz’ya [11, Chapter 1] for \( p \leq q \). Namely the following statement holds.

**Theorem C.** Let \( 1 < p \leq q < \infty \). Then the inequality

\[ \left( \int_0^\infty \left( \int_0^x f(y) dy \right)^{q/p} dx \right)^{1/q} \leq c \left( \int_0^\infty |f(x)|^p dx \right)^{1/p} \]

with the positive constant \( c \) independent of \( f \) holds if and only if

\[ A \equiv \sup_{t>0} \left( \int_t^\infty v(x) dx \right)^{1/q} \left( \int_0^t w^{1-p'}(x) dx \right)^{1/p'} < \infty. \]
Moreover, if \( c \) is the best constant in (2.1), then there exists a positive constant \( b \) depending only on \( p \) and \( q \) such that the inequality \( A \leq c \leq bA \) holds.

Later on F. J. Martín-Reyes and E. Sawyer [10] and V. Stepanov [17] proved the next statement, which gives two-weight criteria for the Riemann-Liouville transform

\[
R_\alpha f(x) = \int_0^x \frac{f(y)}{(x-y)^{1-\alpha}} dy,
\]

where \( \alpha > 1 \).

**Theorem D.** Let \( 1 < p \leq q < \infty \), \( \alpha > 1 \). Then the operator \( R_\alpha \) acts boundedly from \( L^p(R_+) \) to \( L^q(R_+) \) if and only if the following two conditions

\[
A_1 =: \sup_{t>0} \left( \int_t^\infty \frac{v(x)}{(x-t)^{1-\alpha}} dx \right)^{1/q} \left( \int_0^t w^{1-p'}(x) dx \right)^{1/p'} < \infty;
\]

\[
A_2 =: \sup_{t>0} \left( \int_t^\infty v(x) dx \right)^{1/q} \left( \int_0^t \frac{w^{1-p'}(x)}{(t-x)^{1-\alpha}} dx \right)^{1/p'} < \infty
\]

hold. Moreover, there exist positive constants \( c_1 \) and \( c_2 \) depending only on \( \alpha \), \( p \) and \( q \) such that \( c_1 \max\{A_1, A_2\} \leq \|R_\alpha\| \leq c_2 \max\{A_1, A_2\} \).

Criteria for the boundedness of \( R_\alpha \) from \( L^p(R_+) \) to \( L^q(R_+) \) when \( 1 < p \leq q < \infty \) and \( \alpha > 1/p \) have been obtained in [12] (see also [14]), while the similar result has been derived in [7], [3, Chapter 2], for \( 1 < p \leq q < \infty \) and \( 0 < \alpha < 1/p \). When \( 1 < p < q < \infty \) a solution of the two-weight problem for potential operators has been given in [5].

The next statement concerning the discrete Hardy operator defined on \( \mathbb{Z} \) perhaps is already known, but we give the proof of the theorem for the completeness (see also [1], [4] for two-weight criteria for the Hardy transform on \( \mathbb{Z}_+ \)):

**Theorem E.** Let \( 1 < p \leq q < \infty \) and let \( \{a_n\}, \{b_n\} \) be positive sequences. The inequality

\[
(2.2) \quad \left( \sum_{n=-\infty}^{\infty} \left| \sum_{k=-\infty}^{n} g_k a_n^q \right|^{q/p'} \right)^{1/p} \leq c \left( \sum_{n=-\infty}^{\infty} |g_n|^{p'b_n} \right)^{1/p}
\]
with the positive constant $c$ independent of $\{g_k\}$ ($g_k \in l_{b_k}^p(\mathbb{Z})$), holds if and only if

$$B := \sup_{n \in \mathbb{Z}} \left( \sum_{k=n}^{\infty} a_k^q \right)^{1/q} \left( \sum_{k=-\infty}^{n} b_k^{-p'} \right)^{1/p'} < \infty.$$ 

Moreover, if $c$ is the best constant in (2.2), then

$$B \leq c \leq Bq^{1/q} \left( \frac{q}{q-1} \right)^{(p-1)/p}.$$ 

Proof (Sufficiency). Let $\alpha_n = \left( \sum_{k=-\infty}^{n} b_k^{-p'} \right)^{1/p'}$. Due to Hölder’s inequality we have

$$\left( \sum_{n=-\infty}^{\infty} a_n^q \left| \sum_{k=-\infty}^{n} f_k \right|^p \right)^{p/q} \leq \left( \sum_{n=-\infty}^{\infty} a_n^q \left( \sum_{k=-\infty}^{n} |f_k \alpha_kb_k|^p \right)^{a_n^q} \left( \sum_{k=-\infty}^{n} (\alpha_kb_k)^{-p'} \right)^{\frac{p}{p'}} \right)^{p/q} =: \left( \sum_{n=-\infty}^{\infty} A_n \right)^{\frac{p}{q}}.$$

By Minkowsky’s inequality ($\frac{q}{p} \geq 1$) we obtain

$$\left( \sum_{n=-\infty}^{\infty} A_n \right)^{p/q} = \left( \sum_{n=-\infty}^{\infty} \left( \sum_{k=-\infty}^{n} |f_k \alpha_kb_k|^p a_n^q \left( \sum_{k=-\infty}^{n} (\alpha_kb_k)^{-p'} \right)^{p} \right)^{\frac{q}{p'}} \right)^{\frac{p}{q}} \leq \sum_{k=-\infty}^{\infty} |f_k \alpha_kb_k|^p \left( \sum_{n=k}^{\infty} a_n^q \left( \sum_{k=-\infty}^{n} (\alpha_kb_k)^{-p'} \right)^{\frac{p}{p'}} \right)^{\frac{q}{p}} =: \sum_{k} D_k.$$

For the intrinsic sum we have

$$\sum_{n=k}^{\infty} a_n^q \left( \sum_{k=-\infty}^{n} (\alpha_kb_k)^{-p'} \right)^{\frac{p}{p'}} = \sum_{n=k}^{\infty} a_n^q \left( \sum_{k=-\infty}^{n} b_k^{-p'} \right)^{\frac{p}{p'}}.$$

Moreover, the next easily verifiable inequality
\[ \sum_{k=-\infty}^{n} b_k^{-p'} \left( \sum_{i=-\infty}^{k} b_i^{-p'} \right)^{-\frac{q}{p'}} \leq q' \left( \sum_{k=-\infty}^{n} b_k^{-p'} \right)^{1/q'} \]
gives
\[ \sum_{n=k}^{\infty} a_n^q \left( \sum_{i=-\infty}^{n} (\alpha_ib_i)^{-p'} \right)^{\frac{q}{q'}} = \left( q' \right)^{\frac{q}{q'}} \sum_{n=k}^{\infty} a_n^q \left( \sum_{i=-\infty}^{n} b_i^{-p'} \right)^{\frac{q}{q'}}. \]

Further, the latter sum does not exceed
\[ \lambda = B^{q'} (q')^{\frac{q}{q'}} \sum_{n=k}^{\infty} a_n^q \left( \sum_{i=-\infty}^{n} a_i^q \right)^{-\frac{q}{q'}}. \]
The inequality
\[ \sum_{n=k}^{\infty} a_n^q \left( \sum_{i=-\infty}^{n} a_i^q \right)^{-\frac{q}{q'}} \leq q \left( \sum_{i=-\infty}^{n} a_i^q \right)^{1/q} \]
can also be easily verified. Therefore we obtain
\[ \lambda \leq qB^{q'} (q')^{\frac{q}{q'}} \left( \sum_{i=-\infty}^{k} a_i \right)^{1/q} \leq qB \cdot B^{q'} (q')^{\frac{q}{q'}} \left( \sum_{i=0}^{k} \right)^{-\frac{q}{q'}} = qB^{q'} (q')^{\frac{q}{q'}} a_k^{-p}. \]

Finally we have
\[ \sum_{k \in \mathbb{Z}} D_k \leq q^{\frac{q}{q'}} B^{p} (q')^{\frac{q}{q'}} \sum_{k=-\infty}^{\infty} |f_k\alpha_kb_k| p a_k^{-p} = q^{\frac{q}{q'}} B^{p} (q')^{\frac{q}{q'}} \sum_{k=-\infty}^{\infty} |f_k| p \beta_k^p. \]

In order to prove necessity we take the sequence
\[ g_k = \begin{cases} \beta_k^{-p'}, & k \leq n, \\ 0, & k > n. \end{cases} \]
Then we have
\[
\left( \sum_{i \in \mathbb{Z}} \left( \sum_{k = -\infty}^{i} g_k \right)^q a_i^q \right)^{1/q} \geq \left( \sum_{i = n}^{\infty} a_i^q \right)^{1/q} \left( \sum_{k = -\infty}^{n} \beta_k^{-p'} \right)^{1/p'}.
\]

On the other hand,
\[
\left( \sum_{k = -\infty}^{\infty} |g_k|^p \beta_k^p \right)^{1/p} = \left( \sum_{k = -\infty}^{n} \beta_k^{-p'} \right)^{1/p'}
\]
and finally
\[B < \infty. \]

Analogously it follows

**Theorem F.** Let $1 < p \leq q < \infty$ and let $m$ be an integer. Suppose that $\{a_n\}_{n=-\infty}^{m}$, $\{b_n\}_{n=-\infty}^{m}$ are positive sequences. Then the two-weight inequality
\[
(2.3) \quad \left( \sum_{n=-\infty}^{m} \left| g_n \right|^p b_n^p \right)^{1/p} \leq c \left( \sum_{n=-\infty}^{m} \left| g_n \right|^p b_n^p \right)^{1/p'}
\]
holds if and only if
\[
B_m =: \sup_{-\infty < n \leq m} \left( \sum_{k = n}^{m} a_k^q \right)^{1/q} \left( \sum_{k = -\infty}^{n} b_k^{-p'} \right)^{1/p'} < \infty.
\]
Moreover, if $c$ is the best constant in (2.3), then
\[
B_m \leq c \leq B_m q^2 \left( \frac{q}{q - 1} \right)^{(p-1)/p}.
\]

### 3. The Main Results

In order to formulate the main results of this paper we need the following definition:

**Definition.** A nonnegative function $\rho : R^d_+ \rightarrow R^1$ is said to be a weight function with doubling condition uniformly with respect to $x \in R_+$ if there exists a positive constant $c$ such that for arbitrary $t > 0$ and almost all $x > 0$ the inequality
\[
\rho(x) \leq c \rho(x + t).
\]
\[ \int_0^{2t} \rho(x,y)dy \leq c \int_0^t \rho(x,y)dy \]

holds. In this case we write \( \rho \in DC(y) \). Analogously we define the class \( DC(x) \).

Note that if the weight \( \rho \) is integrable on \([0,a]^2\), \( a > 0 \), then \( \rho \in DC(y) \) is equivalent to the condition: there exists a constant \( c > 0 \) such that for all intervals of finite length \( I \subset R_+ \) and all \( t > 0 \) the inequality

\[ \int_I \int_0^{2t} \rho(x,y)dxdy \leq c \int_I \int_0^t \rho(x,y)dxdy \]

holds.

**Theorem 3.1.** Let \( 1 < p \leq q < \infty \) and let \( \alpha, \beta \geq 1 \). Suppose that \( w^{1-p'} \in DC(y) \). Then the operator \( R_{\alpha,\beta} \) is bounded from \( L^p_w(R_+^2) \) to \( L^q_v(R_+^2) \) if and only if

\begin{enumerate}
\item[(i)]
\[ A_1 := \sup_{a,b>0} \left( \int_0^a \int_0^b \frac{w^{1-p'}(x,y)}{(a-x)^{(1-\alpha)q}} dxdy \right)^{1/p'} \times \left( \int_a^\infty \int_b^\infty \frac{v(x,y)}{y^{(1-\beta)q}} dxdy \right)^{1/q} < \infty; \]
\item[(ii)]
\[ A_2 := \sup_{a,b>0} \left( \int_0^a \int_0^b w^{1-p'}(x,y)dxdy \right)^{1/p'} \times \left( \int_a^\infty \int_b^\infty \frac{v(x,y)}{(x-a)^{(1-\alpha)q}(y^{1-\beta)q}} dxdy \right)^{1/q} < \infty. \]
\end{enumerate}

Moreover, \( \| R_{\alpha,\beta} \| \approx \max\{A_1, A_2\} \).

**Theorem 3.2.** Let \( 1 < p \leq q < \infty \) and let \( \alpha, \beta \geq 1 \). Suppose that \( w^{1-p'} \in DC(x) \). Then the operator \( R_{\alpha,\beta} \) is bounded from \( L^p_w(R_+^2) \) to \( L^q_v(R_+^2) \) if and only if

\begin{enumerate}
\item[(i)]
\[ A_1 := \sup_{a,b>0} \left( \int_0^a \int_0^b \frac{w^{1-p'}(x,y)}{(a-x)^{(1-\alpha)q}} dxdy \right)^{1/p'} \times \left( \int_a^\infty \int_b^\infty \frac{v(x,y)}{y^{(1-\beta)q}} dxdy \right)^{1/q} < \infty; \]
\item[(ii)]
\[ A_2 := \sup_{a,b>0} \left( \int_0^a \int_0^b w^{1-p'}(x,y)dxdy \right)^{1/p'} \times \left( \int_a^\infty \int_b^\infty \frac{v(x,y)}{(x-a)^{(1-\alpha)q}(y^{1-\beta)q}} dxdy \right)^{1/q} < \infty. \]
\end{enumerate}

Moreover, \( \| R_{\alpha,\beta} \| \approx \max\{A_1, A_2\} \).

**Theorem 3.2.** Let \( 1 < p \leq q < \infty \) and let \( \alpha, \beta \geq 1 \). Suppose that \( w^{1-p'} \in DC(x) \). Then the operator \( R_{\alpha,\beta} \) is bounded from \( L^p_w(R_+^2) \) to \( L^q_v(R_+^2) \) if and only if

\begin{enumerate}
\item[(i)]
\[ A_1 := \sup_{a,b>0} \left( \int_0^a \int_0^b \frac{w^{1-p'}(x,y)}{(a-x)^{(1-\alpha)q}} dxdy \right)^{1/p'} \times \left( \int_a^\infty \int_b^\infty \frac{v(x,y)}{y^{(1-\beta)q}} dxdy \right)^{1/q} < \infty; \]
\item[(ii)]
\[ A_2 := \sup_{a,b>0} \left( \int_0^a \int_0^b w^{1-p'}(x,y)dxdy \right)^{1/p'} \times \left( \int_a^\infty \int_b^\infty \frac{v(x,y)}{(x-a)^{(1-\alpha)q}(y^{1-\beta)q}} dxdy \right)^{1/q} < \infty. \]
\end{enumerate}

Moreover, \( \| R_{\alpha,\beta} \| \approx \max\{A_1, A_2\} \).

**Theorem 3.2.** Let \( 1 < p \leq q < \infty \) and let \( \alpha, \beta \geq 1 \). Suppose that \( w^{1-p'} \in DC(x) \). Then the operator \( R_{\alpha,\beta} \) is bounded from \( L^p_w(R_+^2) \) to \( L^q_v(R_+^2) \) if and only if

\begin{enumerate}
\item[(i)]
\[ A_1 := \sup_{a,b>0} \left( \int_0^a \int_0^b \frac{w^{1-p'}(x,y)}{(a-x)^{(1-\alpha)q}} dxdy \right)^{1/p'} \times \left( \int_a^\infty \int_b^\infty \frac{v(x,y)}{y^{(1-\beta)q}} dxdy \right)^{1/q} < \infty; \]
\item[(ii)]
\[ A_2 := \sup_{a,b>0} \left( \int_0^a \int_0^b w^{1-p'}(x,y)dxdy \right)^{1/p'} \times \left( \int_a^\infty \int_b^\infty \frac{v(x,y)}{(x-a)^{(1-\alpha)q}(y^{1-\beta)q}} dxdy \right)^{1/q} < \infty. \]
\end{enumerate}

Moreover, \( \| R_{\alpha,\beta} \| \approx \max\{A_1, A_2\} \).
(i)

\[ B_1 =: \sup_{a,b>0} \left( \int_0^a \int_0^b w^{1-p'}(x,y) \frac{dy}{y^{(1-\beta)q}} dx \right)^{1/p'} \times \left( \int_a^\infty \int_b^\infty \frac{v(x,y)}{x^{(1-\alpha)q}} dxdy \right)^{1/q} < \infty; \]

(ii)

\[ B_2 =: \sup_{a,b>0} \left( \int_0^a \int_0^b w^{1-p'}(x,y) dxdy \right)^{1/p'} \times \left( \int_a^\infty \int_b^\infty \frac{v(x,y)}{(y-b)^{(1-\beta)q}x^{(1-\alpha)q}} dxdy \right)^{1/q} < \infty. \]

Moreover, \( \|R_{\alpha,\beta}\| \approx \max\{B_1, B_2\}. \)

**Corollary 3.1.** Let \( 1 < p \leq q < \infty. \) Suppose that \( w^{1-p'} \in DC(x) \) or \( w^{1-p'} \in DC(y). \) Then the operator \( H_2 \) is bounded from \( L^p_w(R^2_+) \) to \( L^q_v(R^2_+) \) if and only if (1.1) holds.

More general form of this corollary is the next statement:

**Theorem 3.3.** Let \( 1 < p \leq q < \infty. \) Assume that the weight function \( w^{1-p} \) satisfies the condition

\[
\sup_{x \in \mathbb{R}} \left( \sum_{|k| \leq b} \left( \int_0^{2^k} w^{1-p'}(x,y) dy \right)^{1-p} \left( \int_0^{2^{k+1}} w^{1-p'}(x,y) dx \right)^{p-1} \right) < \infty.
\]

Then the boundedness of \( H_2 \) from \( L^p_w(R^2_+) \) to \( L^q_v(R^2_+) \) is equivalent to (1.1).

The following theorem states that if the weight function \( w \) has the form \( w(x,y) = w_1(x)w_2(y), \) then the boundedness of the operator \( H_2 \) from \( L^p_w(R^2_+) \) to \( L^q_v(R^2_+) \) is equivalent to the first condition in the E. Sawyer’s theorem.

**Theorem 3.4.** Let \( 1 < p \leq q < \infty \) and \( w(x,y) = w_1(x)w_2(y). \) Then the operator \( H_2 \) is bounded from \( L^p_w(R^2_+) \) to \( L^q_v(R^2_+) \) if and only if

\[
D =: \sup_{a,b>0} \left( \int_0^a w_1^{1-p'}(x) dx \right)^{1/p'} \left( \int_0^b w_2^{1-p'}(y) dy \right)^{1/p'} \times \left( \int_a^\infty \int_b^\infty v(x,y) dxdy \right)^{1/q} < \infty.
\]
4. Proof of the Main Results

In this section we present the proofs of the results formulated in the previous section.

Proof of Theorem 3.1. Sufficiency. First of all note that (see e.g., [18]) the condition \( w^{1-p'} \in DC(y) \) implies the reverse doubling condition for \( w^{1-p'} \) uniformly with respect to \( x \), i.e., there exists the constants \( \eta_1, \eta_2 > 1 \) such that for all \( t > 0 \) and a.e. \( x \in R_+ \) the inequality

\[
\int_0^{\eta_1 t} w^{1-p'}(x,y) dy \geq \eta_2 \int_0^t w^{1-p'}(x,y) dy
\]

holds.

In the sequel we shall use the notation:

\[
v_1(x,y) = \frac{v(x,y)}{y^{1-p}q}; \quad \tilde{v}_{1,j}(x) = \int_{\eta_1^j}^{\eta_1^{j+1}} v_1(x,y) dy;
\]

\[
F_j(t) = \int_0^{\eta_1^{j+1}} f(t,\tau) d\tau; \quad A = \max\{A_1, A_2\}.
\]

Let \( f \geq 0 \). Then taking into account the fact \( \alpha \geq 1 \) and using Theorem D we find that

\[
I = \int_0^\infty \int_0^\infty v(x,y)(R_{a,\beta}f)^q(x,y) dxdy
\]

\[
\leq \int_0^\infty \int_0^\infty v_1(x,y) \left( \int_0^x (x-t)^{\alpha-1} \left( \int_0^y f(t,\tau) d\tau \right) dt \right)^q dxdy
\]

\[
\leq \sum_{j \in Z} \int_0^\infty \tilde{v}_{1,j}(x) \left( \int_0^x (x-t)^{\alpha-1} F_j(t) dt \right)^q dx
\]

\[
\leq cA^q \sum_{j \in Z} \left[ \int_0^\infty \left( \int_0^{\eta_1^j} w^{1-p'}(x,y) dy \right)^{1-p} F_j^p(x) dx \right]^{q/p}
\]

\[
\leq cA^q \left[ \int_0^\infty \sum_{j \in Z} \left( \int_0^{\eta_1^j} w^{1-p'}(x,y) dy \right)^{1-p} F_j^p(x) dx \right]^{q/p}.
\]

On the other hand, we have

\[
\sup_{x > 0} J(x,j) = \sup_{x > 0} \sum_{j \in Z} \left( \int_0^{\eta_1^j} w^{1-p'}(x,y) dy \right)^{1-p}
\]
Indeed, (4.1) and the condition $w^{1-p'} \in DC(y)$ lead to the inequality:

$$J(x, j) = \sum_{j=0}^{\infty} \left( \int_0^{\eta_1^j} w^{1-p'}(x, y)dy \right)^{1-p} \left( \int_0^{\eta_1^{k+1}} w^{1-p'}(x, y)dy \right)^{p-1}$$

$$\leq \left( \sum_{j=0}^{\infty} \eta_1^{(j-k)(1-p)} \right) \left( \int_0^{\eta_1^{k+1}} w^{1-p'}(x, y)dy \right)^{1-p}$$

$$\times \left( \int_0^{\eta_1^{k+1}} w^{1-p'}(x, y)dy \right)^{p-1} \leq c.$$

Consequently, by virtue of Theorem E and Hölder’s inequality we find that

$$\left[ \int_0^{\infty} \sum_{j \in \mathbb{Z}} \left( \int_0^{\eta_1^j} w^{1-p'}(x, y)dy \right)^{1-p} F_j^p(x)dx \right]^{q/p}$$

$$\leq c \left[ \int_0^{\infty} \sum_{j \in \mathbb{Z}} \left( \int_0^{\eta_1^j} w^{1-p'}(x, y)dy \right)^{1-p} \left( \sum_{k=0}^{j} \int_0^{\eta_1^k} f(x, t)dt \right)^{p} dx \right]^{q/p}$$

$$\leq c \left[ \int_0^{\infty} \sum_{k \in \mathbb{Z}} \left( \int_0^{\eta_1^k} w^{1-p'}(x, y)dy \right)^{1-p} \left( \int_0^{\eta_1^{k+1}} f(x, t)dt \right)^{p} dx \right]^{q/p}$$

$$\leq c \left[ \int_0^{\infty} \sum_{j \in \mathbb{Z}} \left( \int_0^{\eta_1^j} w(x, t)f^p(x, t)dt \right) \left( \int_0^{\eta_1^{j+1}} w^{1-p'}(x, t)dt \right)^{p-1} dx \right]^{q/p}$$

$$\leq c ||f||_{L^p(R_2^+)}.$$

**Necessity.** Let $f \geq 0$ and let $a, b > 0$. It is easy to see that

$$I \geq \int_a^{\infty} \int_0^{\infty} v_1(x, y) \left( \int_0^{x^{b/2}} f(t, \tau) \frac{f(t, \tau)}{(x-t)^{1-\alpha} (y-\tau)^{1-\beta}} dt \right)^{q} dy$$

$$\geq c \left( \int_a^{\infty} \int_0^{\infty} v_1(x, y) dx dy \right) \left( \int_0^{a^{b/2}} f(t, \tau) \frac{f(t, \tau)}{(a-t)^{1-\alpha}} dt \right)^{q}.$$
a, b > 0, we find that
\[
\left( \int_0^a \int_0^{b/2} \frac{w_1^{1-p'}(x, y)}{(a-x)^{(1-\alpha)p}} dxdy \right)^q \int_a^\infty \int_b^\infty v_1(x, y)dxdy \leq I \leq c \left( \int_0^a \int_0^{b/2} \frac{w_1^{1-p'}(x, y)}{(a-x)^{(1-\alpha)p}} dxdy \right)^{q/p} < \infty.
\]
Hence this inequality and the condition \( w \in CD(y) \) give us the condition \( A_1 < \infty \).

Taking into account the arguments used above and the fact that the operator \( R_{\alpha, \beta} \) is bounded from \( L^p(w_2) \) to \( L^q(v'_2) \) if and only if its dual operator
\[
W_{\alpha, \beta}f(x, y) = \int_x^\infty \int_y^\infty (t-x)^{\alpha-1}(\tau-y)^{\beta-1}f(t, \tau)dtd\tau
\]
is bounded from \( L^{q'}(v'_1) \) to \( L^{p'}(w'_1) \), we obtain that \( A_2 < \infty \).

**Proof of Theorem 3.2.** The proof is similar to that of Theorem 3.1. \( \square \)

**Proof of Theorem 3.3.** Necessity is obvious. In order to prove sufficiency, it is enough to take the sequence \( 2^k \) instead of \( \eta_k \) in the proof of Theorem 3.1. \( \square \)

**Proof of Theorem 3.4.** First suppose that \( S := \int_0^\infty w_2^{1-p'}(y)dy = \infty \). Let \( \{x_k \}_{k=-\infty}^{+\infty} \) be a sequence of positive numbers for which the equality
\[
(4.3) \quad 2^k = \int_0^{x_k} w_2^{1-p'}(y)dy
\]
holds for all \( k \in Z \). It is clear that \( \{x_k \} \) is increasing and \( R_+ = \cup_{k \in Z} [x_k, x_{k+1}] \). Besides, it is easy to verify that
\[
2^k = \int_{x_k}^{x_{k+1}} w_2^{1-p'}(y)dy.
\]
Let \( f \geq 0 \). We have
\[
I =: \int_0^\infty \int_0^\infty v(x, y)(H_2(x, y))^2dxdy
\]
\[
= \sum_{k \in Z} \int_0^\infty \int_{x_k}^{x_{k+1}} v(x, y) \left( \int_0^x \int_0^y f(t, \tau)dtd\tau \right)^q dxdy
\]
\[
\leq \sum_{k \in Z} \int_0^\infty \left( \int_{x_k}^{x_{k+1}} v(x, y)dy \right) \left( \int_0^2 \left( \int_0^{x_{k+1}} f(t, \tau)d\tau \right)^q dt \right) dx.
\]
\[
\sum_{k \in \mathbb{Z}} \int_{0}^{\infty} V_k(x) \left( \int_{0}^{x} F_k(t) dt \right)^q dx,
\]

where

\[
V_k(x) := \int_{x_k}^{x_{k+1}} v(x, y) dy; \quad F_k(t) := \int_{0}^{t} f(t, \tau) d\tau.
\]

Further, it is obvious that

\[
D^q \geq \sup_{a > 0} \left( \int_{a}^{\infty} \int_{x_j}^{x_{j+1}} v(x, y) dx dy \right) \left( \int_{0}^{a} \int_{x_j}^{x_{j+1}} w_{1}^{1-p'}(x, y) dx dy \right)^{q/p'}.
\]

Therefore by Theorem C we obtain

\[
I \leq cD^q \sum_{j \in \mathbb{Z}} \left[ \int_{0}^{\infty} w_1(x) \left( \int_{0}^{x_j} w_2^{1-p'}(y) dy \right)^{1-p} \left( \int_{j}^{x_{j+1}} w_2^{1-p'}(y) dy \right)^{p-1} \right]^{q/p}
\]

On the other hand, (4.3) yields

\[
\sum_{k=n}^{+\infty} \left( \int_{0}^{x_k} w_2^{1-p'}(y) dy \right)^{1-p} \left( \sum_{k=-\infty}^{n} \int_{x_k}^{x_{k+1}} w_2^{1-p'}(y) dy \right)^{p-1}
\]

for all \( n \in \mathbb{Z} \). Hence by Theorem E and Hölder’s inequality we have

\[
I \leq cD^q \left[ \int_{0}^{\infty} w_1(x) \sum_{j \in \mathbb{Z}} \left( \int_{j}^{x_{j+1}} w_2^{1-p'}(y) dy \right)^{1-p} \right]^{q/p} \times \left( \int_{x_k}^{x_{k+1}} f(x, \tau) d\tau \right)^{p} dx \left[ \int_{0}^{\infty} w_1(x) \sum_{j \in \mathbb{Z}} \left( \int_{j}^{x_{j+1}} w_2^{1-p'}(y) dy \right)^{p-1} \right]^{q/p}
\]

\[
= cD^q \left\| f \right\|^q_{L^p(R^2)}.
\]
If $S < \infty$, then without loss of generality we can assume that $S = 1$. In this case we choose the sequence $\{x_k\}_{k=-\infty}^{0}$ for which (4.3) holds for all $k \leq 0$. Arguing as in the case $S = \infty$ and using Theorem F instead of Theorem E, we finally obtain the desired result. □

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