Modulation spaces $M^{p,q}$ for $0 < p, q \leq \infty$

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Abstract. The purpose of this paper is to construct modulation spaces $M^{p,q}(\mathbf{R}^d)$ for general $0 < p, q \leq \infty$, which coincide with the usual modulation spaces when $1 \leq p, q \leq \infty$, and study their basic properties including their completeness. Given any $g \in \mathcal{S}(\mathbf{R}^d)$ such that $\text{supp } \hat{g} \subset \{\xi \mid |\xi| \leq 1\}$ and $\sum_{k \in \mathbf{Z}^d} \hat{g}(\xi - \alpha k) \equiv 1$, our modulation space consists of all tempered distributions f such that the (quasi)-norm

$$\|f\|_{M_{[g]}^{p,q}} := \left(\int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} |f * (M_\omega g)(x)|^p dx \right)^{\frac{q}{p}} d\omega \right)^{\frac{1}{q}}$$

is finite.

1. Introduction

The purpose of this paper is to construct modulation spaces $M^{p,q}(\mathbf{R}^d)$ for general $0 < p, q \leq \infty$, which coincide with the usual modulation spaces when $1 \leq p, q \leq \infty$, and study their basic properties.

By the usual modulation space $M^{p,q}(\mathbf{R}^d)$, we mean the space of all tempered distributions $f \in \mathcal{S}'(\mathbf{R}^d)$ for which the norm

$$\|f\|_{M^{p,q}} = \left(\int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} |V_g f(x, \omega)|^p dx \right)^{\frac{q}{p}} d\omega \right)^{\frac{1}{q}}$$

is finite, where $V_g f(x, \omega)$ is the Short-Time Fourier Transform of f with respect to a general window $g \in \mathcal{S}(\mathbf{R}^d)$ defined by

$$V_g f(x, \omega) = \langle f, M_\omega T_x g \rangle = \int_{\mathbf{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i \omega \cdot t} dt.$$

Most of the researches on modulation spaces have been restricted to the case $1 \leq p, q \leq \infty$. (See Gröchenig [3].) The reason is that in the case $p < 1$, the estimate of the L^p -norm of the convolution is not easy.

To estimate L^p -norm of convolution of two functions, Y. Galperin and S. Samarah [2] use the fact that if f is a function on $\mathbf{R}^d$ that has polynomial growth, $|f(t)| = O(|t|^N)$, then Bargmann transform $Bf(z)$

$$Bf(z) = 2^{d/4} \int_{\mathbf{R}^d} f(t) e^{2\pi i t \cdot z - \pi t^2 - \frac{\pi}{2} z^2} dt$$

is an entire function on $\mathbf{C}^d$. For such f , they develop a theory of modulation spaces $M^{p,q}(\mathbf{R}^d)$ for the case $0 < p, q \leq \infty$. But for general $f \in \mathcal{S}'(\mathbf{R}^d)$, whether or not $Bf(z)$ is an entire function is non-trivial and is not proved in the paper, and completeness of the space $M^{p,q}(\mathbf{R}^d)$ is not proved. In H. Triebel [6] modulation spaces are studied along the lines of the theory of Besov spaces, but f is restricted to those functions in $\mathcal{S}(\mathbf{R}^d)$ that the support of $\hat{f}$ is compact. The completeness is not proved, either.

In this article we use only a simple, but key, lemma (see Lemma 2.6) to estimate convolutions, and define modulation spaces $M^{p,q}(\mathbf{R}^d)$ for $0 < p, q \leq \infty$ and prove its completeness, in particular.

2. Basic definition

Let $\mathcal{S}(\mathbf{R}^d)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbf{R}^d$ with the topology defined by the semi-norms

$$p_M(\varphi) = \sup_{x \in \mathbf{R}^d} (1 + |x|)^M \sum_{|\alpha| \leq M} |\partial^\alpha \varphi(x)|, \quad M = 1, 2, \dots$$

for $\varphi \in \mathcal{S}(\mathbf{R}^d)$. And let $\mathcal{S}'(\mathbf{R}^d)$ be the topological dual of $\mathcal{S}(\mathbf{R}^d)$. The Fourier transform is $\hat{f}(\omega) = \int f(t)e^{-2\pi i t \cdot \omega} dt$, and the inverse Fourier transform is $\check{f}(t) = \hat{f}(-t)$. We define for $0 < p < \infty$

$$\|f\|_{L^p} = \left(\int_{\mathbf{R}^d} |f(x)|^p dx \right)^{\frac{1}{p}}$$

and $\|f\|_{L^\infty} = \text{ess. sup}_{x \in \mathbf{R}^d} |f(x)|$. We use $\langle f, g \rangle$ to denote the extension to $\mathcal{S}'(\mathbf{R}^d) \times \mathcal{S}(\mathbf{R}^d)$ of the inner product $\langle f, g \rangle = \int f(t)\overline{g(t)} dt$ on $L^2(\mathbf{R}^d)$.

Definition 2.1. If $f \in \mathcal{S}'(\mathbf{R}^d)$ and $g \in \mathcal{S}(\mathbf{R}^d)$, we define the convolution $f * g$ by

$$f * g(x) = \langle f, \overline{g(x - \cdot)} \rangle = \int_{\mathbf{R}^d} f(t)g(x - t)dt, \quad x \in \mathbf{R}^d.$$

We define the translation and the modulation operators by

$$T_x f(t) = f(t - x), \quad \text{and} \quad M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) \quad (x, \omega \in \mathbf{R}^d),$$

respectively. The following Lemmas are useful.

Lemma 2.2. For $g \in \mathcal{S}(\mathbf{R}^d)$, $k \in \mathbf{Z}^d$ and $x \in \mathbf{R}^d$ we have

$$(T_k \hat{g})^\vee(x) = (M_k g)(x).$$

Lemma 2.3. Let Γ be a compact subset of $\mathbf{R}^d$ and $f \in \mathcal{S}'(\mathbf{R}^d)$. If $\text{supp } \hat{f} \subset \Gamma$ then for any $\xi_0 \in \mathbf{R}^d$, we have

$$(1) \quad \text{supp } (M_{\xi_0} f)^\wedge \subset \xi_0 + \Gamma.$$

This is employed to prove that the constant C in the estimate (3) below depends only on Γ .

Definition 2.4. Let $0 < p \leq \infty$, and Γ be a compact subset of $\mathbf{R}^d$. Then L_Γ^p is defined by

$$(2) \quad L_\Gamma^p = \{f \in \mathcal{S}'(\mathbf{R}^d) \mid \exists \xi_0 \in \mathbf{R}^d, \text{supp } \hat{f} \subset \xi_0 + \Gamma, \|f\|_{L^p} < \infty\}.$$

By the Paley-Wiener-Schwartz theorem L_Γ^p consists of entire analytic functions.

Theorem 2.5. Let Γ be a compact subset of $\mathbf{R}^d$ and let $0 < p \leq q \leq \infty$. Then there exists a positive constant C (which depends only on the diameter of Γ and p) such that

$$(3) \quad \|f\|_{L^q} \leq C \|f\|_{L^p}$$

holds for all $f \in L_\Gamma^p$.

Proof. If $\widehat{f} \subset \Gamma$, (3) is just the famous Nikol'skij-Triebel inequality (see [5] Theorem 1.4.1). Let $f \in L^p_\Gamma$. Then we can find $\xi_0 \in \mathbf{R}^d$ such that $\widehat{f} \subset \xi_0 + \Gamma$. By lemma 2.3,

$$\text{supp } (M_{-\xi_0} f)^\wedge \subset \Gamma \quad \text{and} \quad \|f\|_{L^p} = \|M_{-\xi_0} f\|_{L^p}.$$

So we have

$$\|f\|_{L^q} = \|M_{-\xi_0} f\|_{L^q} \leq C \|M_{-\xi_0} f\|_{L^p} = C \|f\|_{L^p}.$$

□

From Theorem 2.5 we have the following key Lemma.

Lemma 2.6. *Let $0 < p \leq 1$ and Γ, Γ' be compact subsets of $\mathbf{R}^d$. Then there exists a positive constant C (which depends only on the diameters of Γ, Γ' and p) such that*

$$(4) \quad \| |f| * |g| \|_{L^p} \leq C \|f\|_{L^p} \|g\|_{L^p}$$

holds for all $f \in L^p_\Gamma$ and all $g \in L^p_{\Gamma'}$.

Proof. Step 1. First we assume $f \in L^p_\Gamma, g \in L^p_{\Gamma'}$ be such that

$$\text{supp } \widehat{f} \subset \Gamma, \quad \text{supp } \widehat{g} \subset \Gamma'.$$

Then for a.e. $x \in \mathbf{R}^d$, $f(y)g(x-y) \in L^p$ (as a function of y) and

$$\text{supp } (f(\cdot)g(x-\cdot))^\wedge \subset \Gamma - \Gamma'.$$

Here we used the notation $-A = \{x \mid -x \in A\}$. By Theorem 2.5

$$\begin{aligned} |f| * |g|(x) &= \left(\int_{\mathbf{R}^d} |f(y)g(x-y)| dy \right) \\ &\leq C \left(\int_{\mathbf{R}^d} |f(y)g(x-y)|^p dy \right)^{\frac{1}{p}}, \end{aligned}$$

where C depends only on Γ, Γ' and p . Taking the L^p -norm of both sides, we have (4).

Step 2. Let $f \in L^p_\Gamma, g \in L^p_{\Gamma'}$. Then there exist $\xi_0, \xi'_0 \in \mathbf{R}^d$ such that $\widehat{f} \subset \xi_0 + \Gamma$ and $\widehat{g} \subset \xi'_0 + \Gamma'$. Applying Step.1 to $M_{-\xi_0} f$ and $M_{-\xi'_0} g$ instead of f and g respectively, we have (4). □

3. Modulation spaces

Definition 3.1. For $\alpha > 0$ we define $\Phi^\alpha(\mathbf{R}^d)$ as follows:

$$\Phi^\alpha(\mathbf{R}^d) := \left\{ g \in \mathcal{S}(\mathbf{R}^d) \mid \text{supp } \widehat{g} \subset \{\xi \mid |\xi| \leq 1\}, \text{ and } \sum_{k \in \mathbf{Z}^d} \widehat{g}(\xi - \alpha k) \equiv 1, \forall \xi \in \mathbf{R}^d \right\}.$$

In the following, we choose a sufficiently small $\alpha > 0$ so that the function space $\Phi^\alpha(\mathbf{R}^d)$ is not empty.

Definition 3.2. Given a $g \in \Phi^\alpha(\mathbf{R}^d)$, and $0 < p, q \leq \infty$, we define the modulation space $M_{[g]}^{p,q}(\mathbf{R}^d)$ to be the space of all tempered distributions $f \in \mathcal{S}'(\mathbf{R}^d)$ such that the (*quasi*)-norm

$$(5) \quad \|f\|_{M_{[g]}^{p,q}} := \left(\int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} |f * (M_\omega g)(x)|^p dx \right)^{\frac{q}{p}} d\omega \right)^{\frac{1}{q}}$$

is finite.

Our definition is close to the original definition of modulation spaces by Feichtinger [1].

Remark. Since $f \in \mathcal{S}'(\mathbf{R}^d)$ and $g \in \mathcal{S}(\mathbf{R}^d)$, we have $V_g f(x, \omega) = e^{-2\pi i x \cdot \omega} f * (M_\omega \tilde{g})(x)$, where $\tilde{g}(x) = \overline{g(-x)}$. Since the usual modulation space $M^{p,q}(\mathbf{R}^d)$ ($1 \leq p, q \leq \infty$) is independent of the choice of a window $g \in \mathcal{S}(\mathbf{R}^d) \setminus \{0\}$ (see [3, Proposition 11.3.2]), our modulation space coincides with the usual one if $1 \leq p, q \leq \infty$.

Theorem 3.3. Let $0 < p \leq \infty$, $0 < q \leq \infty$, then

$$(6) \quad \left(\sum_{k \in \mathbf{Z}^d} \left(\int_{\mathbf{R}^d} |f * (M_{\alpha k} g)(x)|^p dx \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}$$

is an equivalent quasi-norm on $M_{[g]}^{p,q}(\mathbf{R}^d)$, with modifications if p or $q = \infty$.

Proof. Step 1. We first prove that quasi-norm (5) can be estimated from above by the quasi-norm (6). For this purpose fix an $\omega \in \mathbf{R}^d$. Then there exists $k = (k_1, \dots, k_d) \in \mathbf{Z}^d$ such that

$$(7) \quad \omega \in [\alpha k_1, \alpha(k_1 + 1)] \times \dots \times [\alpha k_d, \alpha(k_d + 1)] = [\alpha k, \alpha(k + 1)].$$

Then there exists a positive constant N which depends only on the size of $\text{supp } \widehat{g}$, $\alpha > 0$ and the dimension d , such that

$$\begin{aligned}
 f * (M_\omega g) &= f * (T_\omega \widehat{g})^\vee \\
 &= f * \left(\sum_{|r| \leq N} T_{\alpha(k+r)} \widehat{g} \cdot T_\omega \widehat{g} \right)^\vee \\
 &= f * \left(\sum_{|r| \leq N} T_{\alpha(k+r)} \widehat{g} \right)^\vee * (T_\omega \widehat{g})^\vee \\
 &= \sum_{|r| \leq N} \left(f * (M_{\alpha(k+r)} g) \right) * (M_\omega g).
 \end{aligned}$$

Taking the L^p -norm on both sides and p -th power, we obtain

$$\begin{aligned}
 \|f * (M_\omega g)\|_{L^p}^p &= \int_{\mathbf{R}^d} \left| \sum_{|r| \leq N} (f * (M_{\alpha(k+r)} g)) * (M_\omega g)(x) \right|^p dx \\
 (8) \quad &\leq C \sum_{|r| \leq N} \| (f * (M_{\alpha(k+r)} g)) * (M_\omega g) \|_{L^p}^p \\
 &\leq C' \sum_{|r| \leq N} \|f * (M_{\alpha(k+r)} g)\|_{L^p}^p.
 \end{aligned}$$

Here, for the last inequality we used Young's theorem for $p \geq 1$

$$\|F * G\|_{L^p} \leq \|F\|_{L^p} \|G\|_{L^1}$$

and Lemma 2.6 for $p \leq 1$. We take the q/p -th power of (8), integrate over $[\alpha k, \alpha(k+1)]$, and sum over $k \in \mathbf{Z}^d$, and obtain

$$\begin{aligned}
 \int_{\mathbf{R}^d} \|f * (M_\omega g)\|_{L^p}^q d\omega &\leq C' \sum_{k \in \mathbf{Z}^d} \int_{[\alpha k, \alpha(k+1)]} \left(\sum_{|r| \leq N} \|f * (M_{\alpha(k+r)} g)\|_{L^p}^p \right)^{\frac{q}{p}} d\omega \\
 &\leq C'' \sum_{k \in \mathbf{Z}^d} \|f * (M_{\alpha k} g)\|_{L^p}^q.
 \end{aligned}$$

Step 2. Fix a $k \in \mathbf{Z}^d$. Then for all $\omega \in [\alpha k, \alpha(k+1)]$ we have

$$\begin{aligned}
 f * (M_{\alpha k} g) &= f * \left(\sum_{|r| \leq N} T_{\omega + \alpha r} \widehat{g} \cdot T_{\alpha k} \widehat{g} \right)^\vee \\
 &= \sum_{|r| \leq N} f * (M_{\omega + \alpha r} g) * (M_{\alpha k} g).
 \end{aligned}$$

Taking p -th power and integrating over $\mathbf{R}^d$, we obtain

$$\|f * (M_{\alpha k} g)\|_{L^p}^p \leq C \sum_{|r| \leq N} \|f * (M_{\omega + \alpha r} g)\|_{L^p}^p.$$

Similarly we have

$$\begin{aligned} \|f * (M_{\alpha k} g)\|_{L^p}^q &\leq C \left(\sum_{|r| \leq N} \|f * (M_{\omega + \alpha r} g)\|_{L^p}^p \right)^{\frac{q}{p}} \\ &\leq C' \int_{[\alpha k, \alpha(k+1)]} \left(\sum_{|r| \leq N} \|f * (M_{\omega + \alpha r} g)\|_{L^p}^p \right)^{\frac{q}{p}} d\omega \\ &\leq C'' \sum_{|r| \leq N} \int_{[\alpha k, \alpha(k+1)]} \|f * (M_{\omega + \alpha r} g)\|_{L^p}^q d\omega. \end{aligned}$$

Taking sum over $k \in \mathbf{Z}^d$, we get

$$\sum_{k \in \mathbf{Z}^d} \|f * (M_{\alpha k} g)\|_{L^p}^q \leq C' \int_{\mathbf{R}^d} \|f * (M_{\omega} g)\|_{L^p}^q d\omega.$$

So the proof is complete. $\square$

In the sequel, we shall not distinguish between equivalent quasi-norms of a given quasi-normed space.

Corollary 3.4. *Let $0 < p_0 \leq p_1 \leq \infty$ and $0 < q_0 \leq q_1 \leq \infty$. Then*

$$M_{[g]}^{p_0, q_0}(\mathbf{R}^d) \subset M_{[g]}^{p_1, q_1}(\mathbf{R}^d).$$

Proof. This follows from the monotonicity of the l^p -spaces and Theorem 2.5. $\square$

Lemma 3.5. *If $0 < p \leq \infty$, $0 < q \leq \infty$ and $g_1, g_2 \in \Phi^\alpha(\mathbf{R}^d)$, then $\|f\|_{M_{[g_1]}^{p, q}}$ and $\|f\|_{M_{[g_2]}^{p, q}}$ are equivalent quasi-norms on $M^{p, q}(\mathbf{R}^d)$.*

Proof. It is easy to see that both $\|f\|_{M_{[g_1]}^{p, q}}$ and $\|f\|_{M_{[g_2]}^{p, q}}$ are quasi-norms. In order to prove the equivalence of these two quasi-norms we apply Young's inequality (or Lemma 2.6 if $p \leq 1$). Since $T_{\alpha k} \hat{g}_1 = \sum_{|r| \leq N} T_{\alpha(k+r)} \hat{g}_2 \cdot T_{\alpha k} \hat{g}_1$ we have

$$f * (M_{\alpha k} g_1) = \left(\hat{f} \cdot T_{\alpha k} \hat{g}_1 \cdot \sum_{|r| \leq N} T_{\alpha(k+r)} \hat{g}_2 \right)^\vee$$

$$= f * \left(\sum_{|r| \leq N} M_{\alpha(k+r)} g_2 \right) * (M_{\alpha k} g_1).$$

Hence there exists a positive constant c such that

$$\|f * (M_{\alpha k} g_1)\|_{L^p} \leq c \left\| f * \left(\sum_{|r| \leq N} M_{\alpha(k+r)} g_2 \right) \right\|_{L^p}.$$

This proves that $\|f\|_{M_{[g_1]}^{p,q}}$ can be estimated from above by $c\|f\|_{M_{[g_2]}^{p,q}}$. Hence these two quasi-norms are equivalent to each other. $\square$

Theorem 3.6. *We have the continuous imbeddings*

$$(9) \quad \mathcal{S}(\mathbf{R}^d) \subset M^{p,q}(\mathbf{R}^d) \subset \mathcal{S}'(\mathbf{R}^d)$$

for $0 < p \leq \infty$ and $0 < q \leq \infty$.

Proof. To prove the left-hand side of (9), let $f \in \mathcal{S}(\mathbf{R}^d)$ and denote $\Delta = \sum_{j=1}^d (\partial^2 / \partial_j^2)$. Then for any positive integer L we have

$$\begin{aligned} & (1 + 4\pi^2 |\omega|^2)^L |f * (M_\omega g)(x)| \\ &= \left| \int_{\mathbf{R}^d} f(x-t) g(t) (1 + 4\pi^2 |\omega|^2)^L e^{2\pi i \omega \cdot t} dt \right| \\ &= \left| \int_{\mathbf{R}^d} f(x-t) g(t) (1 - \Delta)^L e^{2\pi i \omega \cdot t} dt \right| \\ &= \left| \int_{\mathbf{R}^d} \sum_{|\alpha+\beta| \leq 2L} C_{\alpha,\beta} \partial^\alpha f(x-t) \partial^\beta g(t) e^{2\pi i \omega \cdot t} dt \right|. \end{aligned}$$

Therefore

$$|f * (M_\omega g)(x)| (1 + |x|^2)^M (1 + 4\pi^2 |\omega|^2)^L \leq C p_{2(L+M)}(f).$$

Taking L, M sufficiently large, we have

$$\|f\|_{M^{p,q}} \leq C p_{2(L+M)}(f).$$

Next we prove the right-hand side of (9) with $q = \infty$. First note that there exists a constant N (depending only on the size of $\text{supp } \hat{g}$, $\alpha > 0$ and dimension d) such that $T_{\alpha k} \hat{g} = \sum_{|r| \leq N} T_{\alpha(k+r)} \hat{g} \cdot T_{\alpha k} \hat{g}$ for all $k \in \mathbf{Z}^d$. If

$f \in M^{p,\infty}(\mathbf{R}^d)$ and $\psi \in \mathcal{S}(\mathbf{R}^d)$, then

$$\begin{aligned} |\langle f, \psi \rangle| &= \left| \sum_{k \in \mathbf{Z}^d} \langle (T_{\alpha k} \widehat{g} \cdot \widehat{f})^\vee, \sum_{|r| \leq N} (M_{\alpha(k+r)} g) * \psi \rangle \right| \\ &\leq \sum_{k \in \mathbf{Z}^d} \|f * (M_{\alpha k} g)\|_{L^\infty} \left\| \sum_{|r| \leq N} (M_{\alpha(k+r)} g) * \psi \right\|_{L^1}. \end{aligned}$$

Recall that both $f * (M_{\alpha k} g)$ and $\sum_{|r| \leq N} (M_{\alpha(k+r)} g) * \psi$ are analytic functions.

Hence, the last estimate makes sense. By Theorem 2.5 we have

$$\|f * (M_{\alpha k} g)\|_{L^\infty} \leq C \|f * (M_{\alpha k} g)\|_{L^p}.$$

Hence,

$$|\langle f, \psi \rangle| \leq C' \|f\|_{M^{p,\infty}} \sum_{k \in \mathbf{Z}^d} \left\| \sum_{|r| \leq N} (M_{\alpha(k+r)} g) * \psi \right\|_{L^1}.$$

The last sum can be estimated from above by $C \|\psi\|_{M^{1,1}}$. Consequently, if M is a sufficiently large natural number, we have

$$|\langle f, \psi \rangle| \leq C \|f\|_{M^{p,\infty}} p_M(\psi)$$

for any $\psi \in \mathcal{S}(\mathbf{R}^d)$. This proves that $M^{p,\infty}(\mathbf{R}^d)$ is continuously embedded in $\mathcal{S}'(\mathbf{R}^d)$. $\square$

Theorem 3.7. $M^{p,q}(\mathbf{R}^d)$ is a quasi-Banach space if $0 < p \leq \infty$ and $0 < q \leq \infty$ (Banach space if $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$).

To prove the theorem we prepare the following lemma.

Definition 3.8. For $0 < p, q \leq \infty$ we define $L^{p,q}(\mathbf{R}^d \times \mathbf{Z}^d)$ as follows:

$$\begin{aligned} &L^{p,q}(\mathbf{R}^d \times \mathbf{Z}^d) \\ &:= \left\{ f : \mathbf{R}^d \times \mathbf{Z}^d \rightarrow \mathbf{C} \text{ is measurable and } \|f\|_{L^{p,q}} := \left\| \|f\|_{L^p} \right\|_{l^q} < \infty \right\}. \end{aligned}$$

This is a quasi-normed space and hence there is an $0 < r < \infty$ such that

$$(10) \quad \left\| \sum_{l=1}^n f_l \right\|_{L^{p,q}}^r \leq 2 \sum_{l=1}^n \|f_l\|_{L^{p,q}}^r.$$

(See Köthe [4] p.162.)

Lemma 3.9. *Let $0 < p, q \leq \infty$. If $\{f_n(x, k)\}_{n=1}^\infty$ is a sequence in $L^{p,q}(\mathbf{R}^d \times \mathbf{Z}^d)$ such that*

$$(11) \quad \sum_{n \in \mathbf{N}} \|f_n\|_{L^{p,q}}^r < \infty.$$

Then the series

$$(12) \quad f(x, k) = \sum_{n \in \mathbf{N}} f_n(x, k)$$

converges absolutely for a.e. $x \in \mathbf{R}^d$ and for all $k \in \mathbf{Z}^d$ and is in $L^{p,q}(\mathbf{R}^d \times \mathbf{Z}^d)$ and

$$(13) \quad \|f_1 + \cdots + f_n - f\|_{L^{p,q}} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

In particular, $L^{p,q}(\mathbf{R}^d \times \mathbf{Z}^d)$ is complete.

Proof. When either p or $q = \infty$, the following inequalities give the proof with $r = \min\{q, 1\}$ or $\min\{p, 1\}$.

$$\begin{aligned} \left\| \left\| \sum_{n \in \mathbf{N}} |f_n(x, k)| \right\|_{L^\infty} \right\|_{l^q}^r &\leq \left\| \sum_{n \in \mathbf{N}} \|f_n(x, k)\|_{L^\infty} \right\|_{l^q}^r \\ &\leq \sum_{n \in \mathbf{N}} \|f_n(x, k)\|_{L^{p,q}}^r, \quad \text{if } q \leq p = \infty, \\ \left\| \sum_{n \in \mathbf{N}} |f_n(x, k)| \right\|_{L^p}^r &\leq \sum_{n \in \mathbf{N}} \|f_n(x, k)\|_{L^p}^r \\ &\leq \sum_{n \in \mathbf{N}} \left(\sup_k \|f_n(x, k)\|_{L^p} \right)^r, \quad \text{if } p < q = \infty. \end{aligned}$$

Let $0 < p, q < \infty$ and define

$$g_n(x, k) := \sum_{l=1}^n |f_l(x, k)| \quad \text{and} \quad g(x, k) := \lim_{n \rightarrow \infty} g_n(x, k) = \sum_{n=1}^\infty |f_n(x, k)|.$$

By (11) and by the Beppo-Levi theorem

$$\|g\|_{L^{p,q}}^r = \lim_{n \rightarrow \infty} \|g_n\|_{L^{p,q}}^r \leq \lim_{n \rightarrow \infty} 2 \sum_{l=1}^n \|f_l\|_{L^{p,q}}^r = 2 \sum_{n=1}^\infty \|f_n\|_{L^{p,q}}^r < \infty.$$

This proves that $g \in L^{p,q}$. Consequently, for a.e. x and for all k , the series (12) converges absolutely. Since $|f(x, k)|^p \leq g(x, k)^p \in L^1(\mathbf{R}^d)$ and $\|f(x, k)\|_{L^p}^q \leq \|g(x, k)\|_{L^p}^q \in l^1(\mathbf{Z}^d)$, we conclude that $f \in L^{p,q}(\mathbf{R}^d \times \mathbf{Z}^d)$.

Similarly, since $|f_1 + \cdots + f_n - f| \leq \sum_{l=n+1}^{\infty} |f_l| \leq g \in L^{p,q}$, we have (13) by a repeated application of the Lebesgue dominant convergence theorem.

To prove the completeness of $L^{p,q}$ let $\{h_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $L^{p,q}$. Then we can choose a subsequence $\{h_{n_k}\}_{k=1}^{\infty}$ so that

$$\|h_{n_{k+1}} - h_{n_k}\|_{L^{p,q}}^r \leq 2^{-(k+1)} \quad \text{for all } k \in \mathbf{N}.$$

From above argument, $\{h_{n_k}\}_{k=1}^{\infty}$ converges to $f = \lim_{k \rightarrow \infty} h_{n_k}$ in $L^{p,q}$. Since $\{h_n\}_{n=1}^{\infty}$ is a Cauchy sequence, $\{h_{n_k}\}$ also converges to f in $L^{p,q}$. $\square$

Proof of Theorem 3.7. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $M^{p,q}(\mathbf{R}^d)$ such that

$$\sum_{n \in \mathbf{N}} \|f_n\|_{M^{p,q}}^r < \infty.$$

Then by Lemma 3.9, the series

$$\sum_{n \in \mathbf{N}} (f_n * (M_{\alpha k} g)(x))$$

converges absolutely for a.e. $x \in \mathbf{R}^d$ and for all $k \in \mathbf{Z}^d$ and is in $L^{p,q}(\mathbf{R}^d \times \mathbf{Z}^d)$. And by Theorem 3.6, we have $f = \sum_{n \in \mathbf{N}} f_n$ converges in $\mathcal{S}'(\mathbf{R}^d)$. Since for all $x \in \mathbf{R}^d$ and $k \in \mathbf{Z}^d$

$$\begin{aligned} f * (M_{\alpha k} g)(x) &= \left\langle \sum_{n \in \mathbf{N}} f_n, \overline{M_{\alpha k} g(x - \cdot)} \right\rangle \\ &= \sum_{n \in \mathbf{N}} \langle f_n, \overline{M_{\alpha k} g(x - \cdot)} \rangle \\ &= \sum_{n \in \mathbf{N}} (f_n * (M_{\alpha k} g)(x)), \end{aligned}$$

we have $f \in M^{p,q}(\mathbf{R}^d)$. So $M^{p,q}(\mathbf{R}^d)$ is complete. $\square$

Theorem 3.10. *If $0 < p < \infty$ and $0 < q < \infty$, then $\mathcal{S}(\mathbf{R}^d)$ is dense in $M^{p,q}(\mathbf{R}^d)$.*

Proof. Let $f \in M^{p,q}(\mathbf{R}^d)$ and set

$$f_n = \sum_{|k| \leq n} f * (M_{\alpha k} g), \quad n \in \mathbf{N}.$$

Of course, $f_n \in M^{p,q}(\mathbf{R}^d)$. Then, it follows that

$$\begin{aligned} (f - f_n) * (M_{\alpha l}g)(x) &= \sum_{|k| > n} f * (M_{\alpha k}g) * (M_{\alpha l}g)(x) \\ &= \sum_{r \in \Lambda_{l,n}} f * (M_{\alpha(l+r)}g) * (M_{\alpha l}g)(x) \end{aligned}$$

where

$$\Lambda_{l,n} := \left\{ r \in \mathbf{Z}^d \mid |r + l| > n \text{ and } \text{supp}(T_{\alpha(l+r)}\widehat{g}) \cap \text{supp}(T_{\alpha l}\widehat{g}) \neq \emptyset \right\}.$$

Note that

$$|\Lambda_{l,n}| \leq \exists N, \quad (\forall l \in \mathbf{Z}^d, \forall n \in \mathbf{N}).$$

Moreover

$$\begin{aligned} \left\| \sum_{r \in \Lambda_{l,n}} f * (M_{\alpha(l+r)}g) * (M_{\alpha l}g)(x) \right\|_{L^p} \\ \leq C \left\| \sum_{r \in \Lambda_{l,n}} f * (M_{\alpha(l+r)}g) \right\|_{L^p} \|M_{\alpha l}g\|_{L^{\min\{1,p\}}} \end{aligned}$$

and thus

$$\begin{aligned} \|f - f_n\|_{M^{p,q}}^q &= \sum_{l \in \mathbf{Z}^d} \left\| \sum_{r \in \Lambda_{l,n}} f * (M_{\alpha(l+r)}g) \right\|_{L^p}^q \\ &\leq C' \sum_{|r| > n-N} \left\| f * (M_{\alpha r}g) \right\|_{L^p}^q \rightarrow 0, \quad (n \rightarrow \infty). \end{aligned}$$

Hence f_n approximates f in $M^{p,q}(\mathbf{R}^d)$. Let $\varphi \in \mathcal{S}(\mathbf{R}^d)$ with $\varphi(0) = 1$ and $\text{supp } \widehat{\varphi} \subset \{\xi \mid |\xi| \leq 1\}$. Let $(f_n)_\delta(x) = \varphi(\delta x)f_n(x)$ with $0 < \delta < 1$ then $(f_n)_\delta \in \mathcal{S}(\mathbf{R}^d)$ and this is an approximation of f_n in $M^{p,q}(\mathbf{R}^d)$. This proves that $\mathcal{S}(\mathbf{R}^d)$ is dense in $M^{p,q}(\mathbf{R}^d)$. $\square$

Lemma 3.11. *Let Γ be a compact subset of $\mathbf{R}^d$. If $\beta > 0$ is sufficiently small (depending on Γ), then we have equivalency*

$$\left(\sum_{k \in \mathbf{Z}^d} |f(\beta k)|^p \right)^{\frac{1}{p}} \sim \|f\|_{L^p}, \quad f \in L_\Gamma^p, \quad 0 < p \leq \infty.$$

Proof. See for instance [5] Theorem 1.4.1. $\square$

Combining Theorem 3.3 and Lemma 3.11, we have the following Corollary.

Corollary 3.12. *Let $0 < p \leq \infty$, $0 < q \leq \infty$. If $\beta > 0$ is sufficiently small then*

$$(14) \quad \left(\sum_{k \in \mathbf{Z}^d} \left(\sum_{l \in \mathbf{Z}^d} |f * (M_{\alpha k} g)(\beta l)|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}$$

is an equivalent quasi-norm on $M^{p,q}(\mathbf{R}^d)$.

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