Generalized weighted Besov spaces on the Bessel hypergroup

Miloud Assal and Hacen Ben Abdallah

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Abstract. In this paper we study generalized weighted Besov type spaces on the Bessel-Kingman hypergroup. We give different characterizations of these spaces in terms of generalized convolution with a kind of smooth functions and by means of generalized translation operators. Also a discrete norm is given to obtain more general properties on these spaces.

1. Introduction

The Besov spaces has been defined by many ways on $\mathbb{R}^n$ ([8], [4], [14], [15], [12], [13]). This theory has been generalized using Hankel transform and the weight $t^\gamma; \gamma > 0$ ([5], [1]). Our objective is to find weights extending the case $t^\gamma$ to define generalized weighted Besov type spaces on the Bessel-Kingman hypergroup. In the present work we fix $\alpha > -\frac{1}{2}$ and we put $\mathbb{K} = [0, +\infty[$. For $1 \leq p, q \leq \infty$ and an adequate weight $\omega$ (see [2]) we
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define the Besov type spaces \( \Lambda_{p,q}^\omega(X) \) using the harmonic analysis associated with the Bessel operator:

\[
\mathcal{L}_\alpha = \frac{d^2}{dx^2} + \frac{2\alpha + 1}{x} \frac{d}{dx}; \quad x \in ]0, \infty[.
\]

For \( \alpha = \frac{n-1}{2}; \ n \in \mathbb{N} \) and \( n \geq 2 \), the operator \( \mathcal{L}_\alpha \) is the radial part of the Laplacian operator defined on \( \mathbb{R}^n \). We recall that \( x \to j_\alpha(\lambda x) \) \( (\lambda \in [0, \infty[) \) is the unique solution of the following system (see [16])

\[
\begin{cases}
\mathcal{L}_\alpha u = -\lambda^2 u; \\
    u(0) = 1, \quad \frac{du}{dx}(0) = 0
\end{cases}
\]

where \( j_\alpha \) is the normalized Bessel function of first kind and order \( \alpha \). The Fourier Bessel transform for a reasonable function \( f \) is given by ([16]):

\[
(1.1) \quad \mathcal{F}(f)(\lambda) = \int_X j_\alpha(\lambda x)f(x)d\mu_\alpha(x), \quad \text{for all} \ \lambda \in [0, \infty[,
\]

where \( d\mu_\alpha(x) = \frac{x^{2\alpha+1}dx}{2^{\alpha}\Gamma(\alpha+1)} \).

It is well known that the Fourier-Bessel transform is a topological isomorphism from \( S_*(X) \) onto itself (see [16]) where \( S_*(X) \) is the Schwartz subspace consisting of even functions. This paper deals with generalized weighted Besov-Bessel type spaces defined on \( X \) and it is organized as follows. In the first section, we collect some harmonic analysis properties on the Bessel hypergroup which are given in [16]. In the second section we introduce the generalized weighted Besov-Bessel type spaces \( \Lambda_{p,q}^\omega(X) \), we give some properties and inclusion results of these spaces with respect to the parameters \( p, q \) and \( \omega \). Completeness and density results are also given in this section. We introduce an equivalent discrete norm replacing the group \([0, +\infty]\) by the 2-powers group \( \mathbb{D}_2 = \{2^j; j \in \mathbb{Z}\} \) to give a new characterization of the space \( \Lambda_{p,q}^\omega(X) \) and to develop other inclusion properties. In the third section we characterize the space \( \Lambda_{p,q}^\omega(X) = \Lambda_{p,q}^\omega(X) \cap L^p \) under some conditions on weights by equivalent norms using the difference \( \Delta_x f = T_x^{(\alpha)} f - f \) and the generalized modulus of continuity \( m_p(f, x) = \sup_{0 \leq y \leq x} \|\Delta_y f\|_p \) associated with the translation operators \( T_x^{(\alpha)} \).

In proving these results, the main tool used here is the harmonic analysis on the Bessel hypergroup.

Finally, we mention that \( C \) will be always used to denote a suitable positive constant that is not necessarily the same in each occurrence.
2. Preliminaries

In this section we recall some basic results in harmonic analysis related to the Bessel operators (see [16]). We first begin by some notations.

**Notations.** Let $\alpha > -\frac{1}{2}$ and $X = [0, \infty[$. We denote by

- $\mathbb{R}_+^* = [0, +\infty[$.
- $C_+(X)$ the space of even continuous functions on $\mathbb{R}$.
- $C^\infty(X)$ the space of functions $f : \mathbb{R} \to \mathbb{C}$, even and $C^\infty$ on $\mathbb{R}$.
- $D_+(X)$ the subspace of $C^\infty(X)$ consisting of functions with compact support.
- $C_0(X)$ the space consisting of even continuous functions tending to zero at the infinity.
- $S_+(X)$ the Schwartz subspace consisting of even functions.
- $D'_+(X)$ (resp. $S'_+(X)$) the topological dual of $D_+(X)$ (resp. $S_+(X)$).
- $S_0(X)$ the subset of functions $\psi$ in $S_+(X)$ such that the support of $\mathcal{F}(\psi)$ is a compact subset of $[0, +\infty[$.
- $S_0^1(X)$ the subset of functions $\psi$ in $S_0(X)$ such that
- $\int_0^\infty |\mathcal{F}(r\lambda)|^2 \frac{dr}{r} = 1,$ for all $\lambda \in [0, \infty[$.

These functions are known as generalized wavelets on $X$.

- $L^p(d\mu_\alpha) = L^p(X, d\mu_\alpha)$, $1 \leq p \leq \infty$, the Lebesgue space associated to the measure $d\mu_\alpha$ and $\|\cdot\|_p$ its usual norm given by:
- $\|f\|_p = \left(\int_X |f(x)|^p d\mu_\alpha(x)\right)^{\frac{1}{p}}$, if $p \in [1, \infty[$
- $\|f\|_\infty = \text{esssup}_{x \in X} |f(x)|$.

- $f_r(x) = r^{-(2\alpha+2)} f(x)$; $r > 0$, the dilated of the function $f$ defined on $X$ preserving the integral of $f$ with respect to the measure $d\mu_\alpha$, in the sense that

$$(2.1) \int_X f_r(x) d\mu_\alpha(x) = \int_X f(x) d\mu_\alpha(x), \text{ for all } r > 0 \text{ and } f \in L^1(d\mu_\alpha).$$

**Definition 2.1.** The generalized translation operators $T^{(\alpha)}_x$, $x \geq 0$, associated with the Bessel operators are defined for appropriate functions
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\( f \) by (see [16]):

\[
T_x^{(\alpha)} f(y) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} \int_0^\pi f(\sqrt{x^2 + y^2 + 2xy \cos \theta})(\sin \theta)^{2\alpha} d\theta,
\]

for all \( y \in \mathbb{X} \).

- The generalized convolution product associated with the Bessel operators is defined for a suitable pair of functions \( f \) and \( g \) by (see [16]):

\[
f \ast g(x) = \int_\mathbb{X} T_x^{(\alpha)} f(y) g(y) d\mu_\alpha(y), \quad \text{for all } x \in \mathbb{X}.
\]

We recall that \((\mathbb{X}, \ast, \text{id}_\mathbb{X})\) is an hypergroup in the sense of Jewett ([10], [7]) called Bessel-Kingman hypergroup and we mention that, to make difference between the space variables, the dual variables and the dilation variables, we shall denote them respectively by \( x \in \mathbb{X}, \lambda \in [0, \infty [ \) and \( r > 0 \).

**Properties.** The following results are standard in the sense that they appear in monographs on harmonic analysis like ([16]). For suitable function \( f : \mathbb{X} \to \mathbb{C} \) we have

1. \( T_0^{(\alpha)} f(y) = f(y), \forall \ y \in \mathbb{X} \).
2. \( T_x^{(\alpha)} f(y) = T_y^{(\alpha)} f(x), \forall \ x, y \in \mathbb{X} \).
3. \( T_x^{(\alpha)} f(y) = \int_\mathbb{X} W_\alpha(x, y, z)f(z)d\mu_\alpha(z), \forall \ x, y \in \mathbb{X} \),

where

\[
W_\alpha(x, y, z) = \begin{cases} 
\frac{2^{1-\alpha} \Gamma(2(\alpha+1)) \left( [(x+y)^2-z^2]^{\alpha-\frac{1}{2}} \left( [x^2-(x-y)^2]^{\alpha-\frac{1}{2}} \right) \right)}{(xyz)^{2\alpha}} , & \text{if } |x-y| < z < x+y \\
0, & \text{otherwise.}
\end{cases}
\]

4. Let \( f \) be in \( L^p(d\mu_\alpha), \ 1 \leq p \leq \infty \). Then for all \( x \in \mathbb{X} \), the function \( T_x^{(\alpha)} f \) belongs to \( L^p(d\mu_\alpha) \) and we have

\[
\|T_x^{(\alpha)} f\|_p \leq \|f\|_p.
\]

5. For \( f \) in \( L^{p_1}(d\mu_\alpha) \) and \( g \) in \( L^{p_2}(d\mu_\alpha); \ 1 \leq p_1, p_2 \leq \infty \) the function \( f \ast g \) belongs to \( L^{p_3}(d\mu_\alpha), \ \frac{1}{p_3} + \frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{p_3}, \) and we have

\[
\|f \ast g\|_{p_3} \leq \|f\|_{p_1} \|g\|_{p_2}.
\]

6. Let \( f \) be in \( L^1(d\mu_\alpha) \). Then the function \( \mathcal{F}(f) \) belongs to \( \mathcal{C}_{s,0} \) and we have

\[
\|\mathcal{F}(f)\|_\infty \leq \|f\|_1.
\]
where \( \|F(f)\|_\infty = \sup_{\lambda \in [0, \infty]} |F(f)(\lambda)|. \)

(7) Let \( f \) and \( g \) in \( L^1(d\mu_\alpha) \), then we have
\[
F(f \ast g) = F(f)F(g).
\]

(8) Let \( f \) be in \( L^1(d\mu_\alpha) \). Then, for all \( x \in X \) and \( \lambda \in [0, \infty] \), we have
\[
F(T_\alpha^\alpha x)(\lambda) = j_\alpha(\lambda x)F(f)(\lambda).
\]

**Remark 2.1.** Let \( p \geq 1 \). Then for all \( f \in L^p(d\mu_\alpha) \) and \( \psi \in S_\ast(X) \), \( \langle f, \psi \rangle \) means the value of \( f \in S_\ast(X) \) on \( \psi \) and it is given by:
\[
\langle f, \psi \rangle = \int_0^\infty f(x)\psi(x)d\mu_\alpha(x).
\]

Hence, we identify \( L^p(X) \) with a subspace of \( S_\ast(X) \) via the formula (2.3).

It might be observed that a long list of properties of the classical distributions on \( \mathbb{R}^n \) remains valid also in our context. On the other hand it is not difficult to verify that the Fourier-Bessel transform and the Bessel operators are symmetric in the following sense
\[
\int_0^\infty fF(g)d\mu_\alpha = \int_0^\infty F(f)gd\mu_\alpha
\]
and
\[
\int_0^\infty L_\alpha fgd\mu_\alpha = \int_0^\infty L_\alpha gf d\mu_\alpha, \text{ for all } f, g \in S_\ast(X).
\]

Therefore, we extend naturally the Fourier-Bessel transform and the operator \( L_\alpha \) on \( S_\ast(X) \) as follows:

For all \( T \in S_\ast(X) \) and \( \psi \in S_\ast(X) \) we put
\[
\langle F(T), \psi \rangle = \langle T, F(\psi) \rangle \quad \text{and} \quad \langle L_\alpha T, \psi \rangle = \langle T, L_\alpha \psi \rangle.
\]

It has been mentioned in [16], p.108 that for all \( f \in D_\ast'(X) \) (resp. \( S_\ast'(X) \)) and \( \psi \in D_\ast(X) \) (resp. \( S_\ast(X) \)), the generalized convolution product associated with Bessel operators of \( f \) and \( \psi \) is the function in \( C_\ast^\infty(X) \) given by \( \psi \ast f(x) = \langle f, T_\alpha^\alpha \psi \rangle \).

**Definition 2.2.** A weight will be a measurable function \( \omega : \mathbb{R}_+^* \to \mathbb{R}_+^* \) satisfying
\[
\exists \ C > 0; \ \forall \ r, \rho > 0 \ (\frac{1}{2} \leq \frac{r}{\rho} \leq 2 \Rightarrow w(r) \leq Cw(\rho)).
\]
An elementary example of weight is the function defined on $\mathbb{R}^*_+$ by $w_{\gamma,\beta}(r) = r^\gamma (1 + |\log r|)^\beta$; $\alpha, \beta \in \mathbb{R}$.

3. Generalized Weighted Besov-Bessel Type Spaces

In what follows we endow the group $\mathbb{R}^*_+$ with the invariant measure $\frac{dr}{r}$.

We introduce the generalized weighted Besov-Bessel type space $\Lambda_{p,q}^{\omega,\psi}(X)$ and we give some properties.

**Definition 3.1.** Let $1 \leq p, q \leq \infty$, $\psi \in S^1_{*,0}(X)$ and $\omega$ a weight. We define the generalized weighted Besov-Bessel type space $\Lambda_{p,q}^{\omega,\psi}(X)$ as the set of all $f \in S_*(X)$ such that

\[(3.1) \quad f = \int_0^\infty f_{\#}\psi_r \psi_r \frac{dr}{r}\]

and $A_{p,q}^{\omega,\psi}(f) < \infty$, where

\[A_{p,q}^{\omega,\psi}(f) = \begin{cases} \left( \int_0^\infty \left( \frac{\|f_{\#}\psi_r\|_p}{\omega(r)} \right)^q \frac{dr}{r} \right)^{\frac{1}{q}}, & \text{if } 1 \leq q < \infty, \\ \operatorname{esssup}_{r>0} \left( \frac{\|f_{\#}\psi_r\|_p}{\omega(r)} \right), & \text{if } q = \infty. \end{cases} \]

**Remark 3.1.** (1) We begin by mentioning that the definition of the generalized weighted Besov-Bessel type space $\Lambda_{p,q}^{\omega,\psi}(X)$ given here is the same than that introduced by Chemin in the classical case for $w(r) = r^\gamma$ (see [9]) which is generalized by Bahouri, Gérard and Xu on the Heisenberg group (see [6]) and by M. Assal and H. Ben Abdallah on the Laguerre hypergroup (see [3]). We do not choose the classical definition introduced by Poetze (see [12]) in which $\Lambda_{p,q}^{\omega,\psi}(X)$ is defined as a set of distributions modulo polynomials.

(2) We note here that the expression (3.1) is independent, in $S_*(X)$, of the choice of $\psi$ in $S^1_{*,0}$.

(3) If $f$ belongs to $L^2(X)$, then (3.1) holds in $L^2(X)$, which is a consequence of Plancherel’s formula (see [16]).

(4) The expression (3.1) is not true in $S_*(X)$ if $f$ is a polynomial function on $X$. Indeed in this case, for all $r > 0$, we have $f_{\#}\psi_r = 0$. Thus, let $P_*[X]$ the subspace of even polynomials, then $P_*[X]$ is invariant under the generalized translation operators $T_x^{\omega,\psi}$; $x \in X$ and we have $f_{\#}\psi_r = 0$ for all $\psi \in S^1_{*,0}$. To prove the above result, it suffice to take $P(y) = y^{2k}; k \in \mathbb{N}$.
Then it holds
\[
T_x^{(\alpha)} P(y) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} \int_0^{\pi} (x^2 + y^2 + 2xy \cos \theta)^k (\sin \theta)^{2\alpha} d\theta \\
= \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} \sum_{j=0}^{k} 2^j C_j^k (x^2 + y^2)^{k-j} (xy)^j \\
\times \int_0^{\pi} (\cos \theta)^j (\sin \theta)^{2\alpha} d\theta.
\]

The result follows using the fact that \(\int_0^{\pi} (\cos \theta)^j (\sin \theta)^{2\alpha} d\theta = 0\) for all odd integer \(j\). Now, using the fact that \(\mathcal{F}(\psi)\) belongs to \(S_{\omega,0}[X]\) and the invariance of \(P^*\) under \(T_x^{(\alpha)}\) we obtain \(f \circ \psi_r = 0\).

The following proposition gives the independence of the space \(\Lambda_{\omega,\psi}^{\omega,\psi}(X)\) on the choice of \(\psi\) in \(S_{\omega,0}[X]\).

**Proposition 3.1.** Let \(1 \leq p, q \leq \infty\) and \(\omega\) a weight. Then the space \(\Lambda_{\omega,\psi}^{\omega,\psi}(X)\) is independent of the choice of the function \(\psi\) in \(S_{\omega,0}[X]\).

Not that the function \(w(r) = r^\gamma\) satisfies the condition (2.4) for all \(\gamma \in \mathbb{R}\).

The main idea in the proof of the above proposition is based on the following lemma.

**Lemma 3.1.** Let \(w\) a weight. Then the following assertions are equivalent
(i) \(\exists C > 0; \forall r, \rho > 0 \left( \frac{1}{2} \leq \frac{r}{\rho} \leq 2 \Rightarrow w(r) \leq Cw(\rho) \right)\).
(ii) For all \(\alpha, \beta\) such that \(0 < \alpha < \beta\), \(\exists C > 0; \forall r, \rho > 0\), \(\alpha \leq \frac{r}{\rho} \leq \beta \Rightarrow w(r) \leq Cw(\rho)\).

**Proof of Proposition 3.1.** Assume \(\psi\) and \(\phi\) be a pair of functions belonging to \(S_{\omega,0}[X]\). To get the desired result it suffices to prove that \(\Lambda_{\omega,\psi}^{\omega,\psi}(X) \subseteq \Lambda_{\omega,\psi}^{\omega,\psi}(X)\). Since \(\mathcal{F}\phi\) and \(\mathcal{F}\psi\) are supported by some compact subsets of \([0, \infty]\), then there exist \(\alpha, \beta\) such that \(\phi_{\rho} \circ \psi_r = 0\) if \(\frac{r}{\rho} \notin [\alpha, \beta]\). Moreover we have \(\|\phi_{\rho} \circ \psi_r\|_1 \leq \|\phi\|_1\|\psi\|_1\) and we obtain for all \(f \in \Lambda_{\omega,\psi}^{\omega,\psi}(X)\)
\[
f \circ \phi_r = \int_{\alpha\rho}^{\beta\rho} (f \circ \psi_r) \circ \psi_r \circ \phi_r \frac{dr}{r}.
\]

Using the fact that \(w\) satisfies (2.4) and Lemma 3.1 we get
\[
\frac{\|f \circ \phi_r\|_p}{w(\rho)} \leq C \int_{\alpha}^{\beta} \frac{\|f \circ \psi_r\|_p dr}{w(\rho r)}.
\]
And so, Minkowski’s inequality leads to
\[
\left( \int_0^\infty \left( \frac{\|f \# \phi \|_p}{w(\rho)} \right)^q \frac{d\rho}{\rho} \right)^{1/q} \leq C \int_0^\infty \left( \int_0^\infty \left( \frac{\|f \# \psi \|_p}{w(r\rho)} \right)^q \frac{d\rho}{\rho} \right)^{1/q} \frac{dr}{r}.
\]
This completes the proof of the proposition. □

**Remark 3.2.** In view of their independence of the function \( \psi \) in \( S_{1,0}^1 \), the spaces \( \Lambda_{p,q}^\omega (X) \); \( 1 \leq p, q \leq \infty \) will be denoted indifferently with or without \( \psi \) which will be chosen adequately.

The following proposition deals with an elementary example of weights transmitting its homogeneity to the space \( \Lambda_{p,q}^\omega (X) \).

**Proposition 3.2.** Let \( 1 \leq p, q \leq \infty \), and \( \omega(\rho) = \rho^\gamma \); \( (\gamma \in \mathbb{R}) \). Then the Besov-Bessel type space \( \Lambda_{p,q}^\omega (X) \) is homogeneous of degree \( d(p, \gamma) = \frac{2\alpha + 2}{p} - \gamma \) in the sense that, for all \( f \in \Lambda_{p,q}^\omega (X) \)
\[
A_{p,q}^\omega (d_r f) = r^{\frac{2\alpha + 2}{p} - \gamma} A_{p,q}^\omega (f), \text{ for all } r > 0
\]
where \( d_r f \) is the distribution given by \( \langle d_r f, \psi \rangle = \langle f, \psi_{1/r} \rangle \) for all \( \psi \in S_{1,0}^1 \).

**Proof.** Let \( 1 \leq p, q \leq \infty \) and \( \gamma \in \mathbb{R} \). Assume \( f \) in \( \Lambda_{p,q}^\omega (X) \), then
\[
A_{p,q}^\omega (d_r f) = \left\| \frac{\| (d_r f) \# \psi \|_p}{\rho^\gamma} \right\|_{L^q(\frac{d\rho}{\rho})} = \frac{\| f \# \psi \|_p}{\rho^\gamma} \left\| \frac{\| (d_r f) \# \psi \|_p}{\rho^\gamma} \right\|_{L^q(\frac{d\rho}{\rho})} = \frac{\| f \# \psi \|_p}{\rho^\gamma} \left\| \frac{\| (d_r f) \# \psi \|_p}{\rho^\gamma} \right\|_{L^q(\frac{d\rho}{\rho})} = r^{\frac{2\alpha + 2}{p} - \gamma} A_{p,q}^\omega (f).
\]

**Definition 3.2.** Let \( 1 \leq p, q, q' \leq \infty \); \( \frac{1}{q} + \frac{1}{q'} = 1 \). We denote by \( W^{2\alpha + 2}_{p, p'} \) the class of weights satisfying
\[
(i) \quad w(r)r^{-\frac{(2\alpha + 2)}{p}} \text{ belongs to } L^{q'}((1, +\infty], \frac{dr}{r}).
\]
\[
(ii) \quad \text{There exists } \varepsilon > 0 \text{ such that } w(r)r^\varepsilon \text{ belongs to } L^{q'}([0, 1], \frac{dr}{r}).
\]

The following proposition holds.
Proposition 3.3. Let $1 \leq p, q \leq \infty$ and $\omega \in W^{\frac{2\alpha+2}{p}}$. Then $\Lambda_{p,q}^\omega(X)$ is a Banach space.

Remark that the weight $w(r) = r^\gamma$ belongs to $W^{\frac{2\alpha+2}{p}}$ for $\gamma < \frac{2\alpha+2}{p}$ which coincides with the classical case. To prove the above proposition we need the following lemmas.

Lemma 3.2. Let $h \in S_*(X)$ and $\psi \in S^1_{*,0}(X)$. Then, for all $k \in \mathbb{N}$, there exists $\psi[k] \in S^1_{*,0}(X)$ such that

\[(3.2) \quad h \ast \psi_r = r^{2k} (L^k h) \ast (\psi[k])_r \]

where $L_\alpha$ is the differential operator given in the introduction part. Furthermore there exists $C > 0$ such that

\[(3.3) \quad \|h \ast \psi_r\|_1 \leq Cr^{2k} \quad \text{for all } 0 \leq r \leq 1 \]

\[(3.4) \quad \|h \ast \psi_r\|_1 \leq C \quad \text{for all } r \geq 1. \]

Proof. We obtain (3.2) by induction on $k$. The inequalities (3.3) and (3.4) follow from (3.2). \hfill \Box

Lemma 3.3. Let $1 \leq p, q \leq \infty$ and $\omega \in W^{\frac{2\alpha+2}{p}}$. For $\phi$ in $S^1_{*,0}$, put

\[(3.5) \quad \Phi(g) = \int_0^\infty w(r)g(r) \ast \phi_r \frac{dr}{r}. \]

Then $\Phi$ define a linear and continuous mapping from $L^q(\mathbb{R}^*_+, L^p(X), \frac{dr}{r})$ to $\Lambda_{p,q}^\gamma(X)$.

Proof. Let us first prove that, for $g \in L^q(\mathbb{R}^*_+, L^p(X), \frac{dr}{r})$, $\Phi(g)$ define an element of $S^\gamma_*(X)$, that is for all $h \in S_*(X)$,

$$
\int_0^\infty \left| w(r)(g(r) \ast \phi_r, h) \right| \frac{dr}{r} < \infty.
$$

Take $\psi \in S_*(X)$ such that $F(\psi) = 1$ on $\text{Supp}F\phi$. Then, using Hölder’s and Young’s inequalities, we obtain

\[
\begin{align*}
\left| (g(r) \ast \phi_r, h) \right| & = \left| (g(r) \ast \phi_r, h \ast \psi_r) \right| \\
& \leq \|g(r) \ast \phi_r\|_\infty \|h \ast \psi_r\|_1 \\
& \leq Cr^{\frac{2\alpha+2}{p}} \|g(r)\|_p \|h \ast \psi_r\|_1.
\end{align*}
\]
On the other hand, using Lemma 3.2, we get
\[
\int_0^\infty \left| w(r) < g(r) \ast \phi_r, h > \right| \frac{d\tau}{r} \leq C \left\{ \int_0^1 w(r) r^{2k - \frac{2\alpha + 2}{p}} \| g(r) \|_p \frac{d\tau}{r} \right. \\
+ \int_1^\infty w(r) r^{-\frac{2\alpha + 2}{p}} \| g(r) \|_p \frac{d\tau}{r} \right\}^{1/q} \\
\leq C \left( \int_0^1 \| g(r) \|_p^q \frac{d\tau}{r} \right)^{1/q} \\
\times \left( \int_0^1 \left( w(r) r^{2k - \frac{2\alpha + 2}{p}} \right)^{\frac{q}{q'}} \frac{d\tau}{r} \right)^{1/q'} \\
+ C \left( \int_1^\infty \| g(r) \|_p^q \frac{d\tau}{r} \right)^{1/q} \\
\times \left( \int_1^\infty \left( w(r) r^{-\frac{2\alpha + 2}{p}} \right)^{\frac{q}{q'}} \frac{d\tau}{r} \right)^{1/q'},
\]
where $\frac{1}{q'}$ is the conjugate exponent of $q$. Using the fact that $w$ belongs to $W_{\frac{2\alpha + 2}{p}}$, we obtain for $k$ sufficiently large
\[
\int_0^\infty \left| w(r) < g(r) \ast \phi_r, h > \right| \frac{d\tau}{r} \leq C \| g \|_{L^q (\mathbb{R}^*, L^p (\mathcal{X}, \mathcal{H}))} < \infty.
\]
Now, let $\psi$ in $S_{*,0}^1$. From Minkowski’s inequality and Young’s inequality we have
\[
\frac{\| \Phi(g) \ast \psi_p \|_p}{w(\rho)} \leq C \int_0^\beta w(\rho) \left\| g(r) \|_p \right\|_p \frac{d\tau}{w(\rho)} \\
= C \int_0^\infty \mathbb{P}_{[\alpha, \beta]}(r) \left\| g(r) \|_p \right\|_p \frac{d\tau}{r} \\
= C (H \ast G)(\rho)
\]
with $H(s) = \mathbb{P}_{[\alpha, \beta]}(s), G(s) = \| g(s) \|_p$ and $H \ast G$ is the convolution of $H$ and $G$ on the group $(\mathbb{R}^*, \frac{d\tau}{r})$. Now, by Young’s inequality, it holds
\[
\| \Phi(g) \|_{\Lambda_{p,q}^\alpha (\mathcal{X})} \leq C \left\| H \ast G \right\|_{L^q (\mathbb{R}^*, \frac{d\tau}{r})} \\
\leq C \| H \|_{L^1 (\mathbb{R}^*, \frac{d\tau}{r})} \| G \|_{L^q (\mathbb{R}^*, \frac{d\tau}{r})} \\
= C \| g \|_{L^q (\mathbb{R}^*, L^p (\mathcal{X}, \mathcal{H}))}.
\]
The lemma is proved. \qed
**Proof of Proposition 3.3.** Let $\psi$ in $S^1_{*,0}$ and take $\phi = \psi$ in Lemma 3.3. Then $\Phi$ defined by (3.5) is a continuous linear mapping from $L^q(\mathbb{R}^*_+, L^p(\mathcal{X}), \frac{dr}{r})$ to $\hat{\Lambda}^\gamma_{p,q}(\mathcal{X})$. On the other hand the operator $\Psi$ associating to $f$ in $\hat{\Lambda}^\gamma_{p,q}(\mathcal{X})$ the function $\Psi(f)$ defined on $\mathbb{R}^*_+$ by:

$$\Psi(f)(r) = \frac{f \ast \psi_r}{w(r)}$$

is obviously a linear isometry from $\hat{\Lambda}^\gamma_{p,q}(\mathcal{X})$ to $L^q(\mathbb{R}^*_+, L^p(\mathcal{X}), \frac{dr}{r})$ and using the decomposition (3.1), we obtain $\Phi \circ \Psi = Id_{\hat{\Lambda}^\gamma_{p,q}(\mathcal{X})}$. This implies

$$(\Psi \circ \Phi) \circ \Psi = \Psi \quad \text{on} \quad \hat{\Lambda}^\gamma_{p,q}(\mathcal{X}).$$

So $\Psi\left(\hat{\Lambda}^\gamma_{p,q}(\mathcal{X})\right) = Ker\left(\Psi \circ \Phi - Id_{L^q(\mathbb{R}^*_+, L^p(\mathcal{X}), \frac{dr}{r})}\right)$ is a closed subspace of $L^q(\mathbb{R}^*_+, L^p(\mathcal{X}), \frac{dr}{r})$. Since $\Psi$ is an isometry, then $\hat{\Lambda}^\gamma_{p,q}(\mathcal{X})$ can be identified with a closed subspace of $L^q(\mathbb{R}^*_+, L^p(\mathcal{X}), \frac{dr}{r})$. The completeness of $\hat{\Lambda}^\gamma_{p,q}(\mathcal{X})$ follows.

**Proposition 3.4.** Let $1 \leq p \leq \infty$, $1 \leq q < \infty$ and $\omega$ a weight. Then the subspace $\hat{\Lambda}^\gamma_{p,q}(\mathcal{X}) \cap C^\infty(\mathcal{X})$ is dense in $\Lambda^\gamma_{p,q}(\mathcal{X})$.

**Proof.** Let $\phi \in S^1_{*,0}(\mathcal{X})$ and $f \in \hat{\Lambda}^\gamma_{p,q}(\mathcal{X})$. Then for $\varepsilon > 1$, the function

$$f_\varepsilon = \int_{1/\varepsilon}^{\varepsilon} f \ast \phi_r \ast \phi_r \frac{dr}{r}$$

is obviously $C^\infty$ and belongs to $\hat{\Lambda}^\gamma_{p,q}(\mathcal{X})$. Moreover the same reasoning given in Proposition 3.1 leads to

$$\|f_\varepsilon - f\|_{\hat{\Lambda}^\gamma_{p,q}(\mathcal{X})} \leq C \left\| \int \frac{f \ast \phi_r}{r^{\gamma}} \mathbb{1}_\varepsilon(r) \right\|_{L^q(\mathbb{R}^*_+, \frac{dr}{r})},$$

where $\mathbb{1}_\varepsilon$ is the characteristic function of the set $\mathbb{R} \setminus [1/\varepsilon, \varepsilon]$. And the right hand side of the above inequality tends to zero as $\varepsilon$ tends to $\infty$. □

**Proposition 3.5.** Let $1 \leq q \leq \infty$. Then, for $1 \leq p_1 \leq p_2 \leq \infty$ and $\omega_1, \omega_2$ two weights such that $r^{-\frac{2q+2}{p_1}}(\omega_1(r))^{-1} = r^{-\frac{2q+2}{p_2}}(\omega_2(r))^{-1}$, we have

$$\hat{\Lambda}^w_{p_1,q}(\mathcal{X}) \subseteq \hat{\Lambda}^w_{p_2,q}(\mathcal{X}) \quad \text{(with continuous embedding)}.$$

**Proof.** Let $\psi$ in $S^1_{*,0}$ and let $1 \leq p_3 \leq \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{p_3}$. We consider $\phi \in S_*(\mathcal{X})$ satisfying $\mathcal{F}\phi = 1$ on $supp(\mathcal{F}\psi)$. Then we obtain
for all \( r > 0 \) and \( f \in \dot{A}_{p_1,q}^{\omega_1}(\mathbb{X}) \)
\[
\| f \ast \psi_r \|_{p_2} = \| f \ast \psi_{r*} \|_{p_2} \\
\leq \| f \ast \psi_{r*} \|_{p_1} \| \phi_r \|_{p_3} \\
= C\| f \ast \psi_r \|_{p_1} r^{(2\alpha + 2)(\frac{1}{p_3} - 1)}.
\]
So, it holds
\[
\| f \ast \psi_r \|_{p_2} \leq C\| f \ast \psi_r \|_{p_1}.
\]
Which leads to \( A_{p_2,q}^{\omega_2}(f) \leq CA_{p_1,q}^{\omega_1}(f) \).

To obtain more general properties we introduce an equivalent discrete norm on \( \dot{A}_{p,q}^{\omega_0}(\mathbb{X}) \) replacing the group \( \mathbb{R}_+^* \) by the 2-powers group \( \mathbb{D}_2 = \{2^j : j \in \mathbb{Z}\} \) equipped with the invariant counting measure.

**Theorem 3.1.** Let \( 1 \leq p, q \leq \infty \), \( \omega \) a weight and \( \theta \in S_{*,0}(\mathbb{X}) \) such that \( \mathcal{F} \theta \in D_*(0, \infty) \) and, for fixed \( \lambda_2 > 2\lambda_1 > 0 \), \( \mathcal{F} \theta(\lambda) \neq 0 \) on \( [\lambda_1, \lambda_2] \). For \( f \) in \( \dot{A}_{p,q}^{\omega_0}(\mathbb{X}) \) put
\[
D_{p,q}^{\omega,\theta}(f) = \begin{cases} \left( \sum_{j \in \mathbb{Z}} \left( \frac{\| f \ast \theta_{2^j} \|_{p}}{\omega(2^j)} \right)^q \right)^{\frac{1}{q}}, & \text{if } 1 \leq q < \infty, \\
\sup_{j \in \mathbb{Z}} \left( \frac{\| f \ast \theta_{2^j} \|_{p}}{\omega(2^j)} \right), & \text{if } q = \infty.
\end{cases}
\]
Then \( D_{p,q}^{\omega,\theta} \) is a norm on \( \dot{A}_{p,q}^{\omega}(\mathbb{X}) \) equivalent to the norm \( A_{p,q}^{\omega}(\mathbb{X}) \).

**Remark 3.3.** For \( \omega \) a weight, two different functions \( \theta_1 \) and \( \theta_2 \) satisfying the hypothesis of the above theorem give two equivalent norms.

**Proof of the Theorem 3.1.** To obtain the desired result we will compare the norms \( A_{p,q}^{\omega} \) and \( D_{p,q}^{\omega,\theta} \). Using the fact that \( \mathcal{F} \theta \neq 0 \) on \( [\lambda_1, \lambda_2] \) for \( \lambda_2 > 2\lambda_1 > 0 \), there exists \( \mathcal{F} \sigma \) belonging to \( D_*(0, \infty) \) such that \( (\mathcal{F} \theta)(\mathcal{F} \sigma) = 1 \) on \( [\lambda_1, \lambda_2] \). Let \( \psi \in S_{1,0} \) satisfying \( \text{supp}(\mathcal{F} \psi) \subset [2\lambda_1, 2\lambda_2] \). This gives, for all \( 1 \leq r \leq 2 \) and \( \lambda \in [0, \infty] \)
\[
(3.6) \quad \mathcal{F} \psi(2^j r \lambda) = \mathcal{F} \psi(2^j r \lambda) \mathcal{F} \theta(2^j \lambda) \mathcal{F} \sigma(2^j \lambda).
\]
Since \( \omega \) satisfies (2.4), it holds for \( 1 \leq q < \infty \) that
\[
A_{p,q}^{\omega}(f) = \left( \sum_{j \in \mathbb{Z}} \int_1^2 \left( \frac{\| f \ast \psi_{2^j} \|_{p}}{\omega(2^j)} \right)^q \frac{dr}{r} \right)^{\frac{1}{q}} \\
\leq C \left( \sum_{j \in \mathbb{Z}} \int_1^2 \left( \frac{\| f \ast \theta_{2^j} \|_{p}}{\omega(2^j)} \right)^q \frac{dr}{r} \right)^{\frac{1}{q}} = C \log(2) D_{p,q}^{\omega}(f).
\]
Conversely, let $\mathcal{F}\theta$ supported by $[\lambda_1, \lambda_2]; \ 0 < \lambda_1 < \lambda_2$ and let $\psi \in S_{1,0}(\mathbb{X})$ satisfying $\mathcal{F}\psi = 1$ on $[\lambda_1/2, \lambda_2]$. Then it holds, for $1/2 \leq r \leq 1$

\[ (3.7) \quad \mathcal{F}\theta(2^j \lambda) = \mathcal{F}\psi(2^j r \lambda) \mathcal{F}\theta(2^j \lambda) \quad \text{for all } \lambda \in [0, \infty[. \]

The same reasoning as above leads to

\[ \mathbf{D}_{p,q}^\omega(f) \leq C \mathbf{A}_{p,q}^\omega(f). \]

For the case $q = \infty$, assume $f \in \mathbf{\Lambda}_{p,\infty}^\omega(\mathbb{X})$ such that $\mathbf{D}_{p,\infty}^\omega(f) < \infty$ and let $r > 0$ and $j \in \mathbb{Z}$ such that $2^j \leq r < 2^{j+1}$, then, from (3.6), we get

\[ \| f \ast \psi_r \|_p \leq C \| f \ast \theta_{2^j} \|_p \leq C \mathbf{D}_{p,\infty}^\omega(f) \omega(2^j) \leq C \mathbf{D}_{p,\infty}^\omega(f) \omega(r) \]

which implies that $\mathbf{A}_{p,\infty}^\omega(f) \leq C \mathbf{D}_{p,\infty}^\omega(f)$.

Conversely let us take $f \in \mathbf{\Lambda}_{p,\infty}^\omega(\mathbb{X})$. Then, from (3.7) we obtain for $1/2 \leq r \leq 1$ the following estimation

\[ \| f \ast \theta_{2^j} \|_p \leq C \| f \ast \psi_{2^j} \|_p \leq C \mathbf{A}_{p,\infty}^\omega(f) \omega(2^j r) \leq C \mathbf{A}_{p,\infty}^\omega(f) \omega(2^j). \]

This completes the proof of the theorem. \hfill \Box

**Proposition 3.6.** Let $1 \leq p \leq \infty$ and $\omega$ a weight. Then, for $1 \leq q_1 \leq q_2 < \infty$ we have

\[ (3.8) \quad \mathbf{\Lambda}_{p,q_1}^\omega(\mathbb{X}) \subseteq \mathbf{\Lambda}_{p,q_2}^\omega(\mathbb{X}) \quad \text{(with continuous embedding)}. \]

**Proof.** The result holds using the discrete norm and the fact that $l^{q_1} \subset l^{q_2}$ for all $1 \leq q_1 \leq q_2 \leq \infty$ and we have

\[ \left( \sum |u_j|^{q_2} \right)^{1/q_2} \leq \left( \sum |u_j|^{q_1} \right)^{1/q_1} \]

for all $(u_j) \in l^{q_1}$. \hfill \Box

**4. Characterization of the Spaces $\mathbf{\Lambda}_{p,q}^\omega(\mathbb{X}) = \mathbf{\Lambda}_{p,q}^\omega(\mathbb{X}) \cap L^p$**

In this section we study the spaces $\mathbf{\Lambda}_{p,q}^\omega(\mathbb{X}) = \mathbf{\Lambda}_{p,q}^\omega(\mathbb{X}) \cap L^p(d\mu_\omega)$ endowed with the norm $\mathbf{A}_{p,q}^\omega$ and we give some characterizations using equivalent norms for some classes of weights $\omega$. Let us first start with the following definition.

**Definition 4.1.** Let $0 \leq \varepsilon, \delta < \infty$, $1 \leq q, q' \leq \infty$; $\frac{1}{q} + \frac{1}{q'} = 1$ and $\omega$ be a weight.
ω is said to be a $d_\varepsilon$-weight if there exists $C > 0$ such that
\[ \int_0^s r^\varepsilon \omega(r) \frac{dr}{r} \leq Cs^\varepsilon \omega(s), \quad \text{for all } s > 0. \]

(ii) $\omega$ is said to be a $b_\delta$-weight if there exists $C > 0$ such that
\[ \int_s^{\infty} \frac{\omega(r) dr}{r^{\delta}} \leq C s^{\delta} \omega(s), \quad \text{for all } s > 0. \]

If $(d_\varepsilon)$ (resp. $(b_\delta)$) denotes the class of $d_\varepsilon$-weight (resp. $b_\delta$-weight) we write
\[ W_{\varepsilon,\delta} = (d_\varepsilon) \cap (b_\delta) \]

and
\[ W^q_{\varepsilon,\delta} = \{ \omega : \text{weight } / \omega(r) = \chi^\frac{1}{2} (r) r^{-\frac{1}{2}} (r^{-1}); \chi, \nu \in W_{\varepsilon,\delta} \}. \]

The following properties hold

**P1.** If $\omega \in W^q_{\varepsilon,\delta}$ then $\omega \in W^q_{\varepsilon',\delta'}$, for any $\varepsilon' > \varepsilon$ and $\delta' > \delta$.

**P2.** Let $\tilde{\omega}(r) = \omega(r^{-1})$. Then $\omega \in (b_\varepsilon)$ if and only if $\tilde{\omega} \in (d_\varepsilon)$.

Note that, $w(r) = r^\gamma$ belongs to $W^q_{0,2}$ for $0 < \gamma < 2$.

**Theorem 4.1** (First characterization theorem). Let $1 \leq p, q \leq \infty$ and $\omega$ a weight belonging to $W^q_{0,2}$. For $f$ in $W_{p,q}$ put
\[ B^\omega_{p,q}(f) = \begin{cases} \left( \int_X \left( \frac{\|\Delta_x f\|_p}{\omega(x)} \right)^q \frac{dx}{x} \right)^{\frac{1}{q}}, & \text{if } 1 \leq q < \infty, \\ \text{esssup}_{x \in X} \left( \frac{\|\Delta_x f\|_p}{\omega(x)} \right), & \text{if } q = \infty \end{cases} \]

where $\Delta_x f = T_x^{(a)} f - f$, for all $x \in \mathbb{X}$. Then $B^\omega_{p,q}$ is a norm on $W_{p,q}$ equivalent to $A^\omega_{p,q}$.

**Remark 4.1.** Note that the case $1 < q < \infty$ could have been shown by interpolation with the extreme cases ($q = 1$ and $q = \infty$), but a direct proof is presented in this paper.

Let us first start with some basic lemmas that will be useful for our purposes.

**Lemma A** (Schur lemma) (see [2]). Let $1 < q < \infty$ and $q'$ its conjugate exponent. Let $(\Omega_1, \mathcal{M}_1, \mu_1)$ and $(\Omega_2, \mathcal{M}_2, \mu_2)$ be a pair of $\sigma$-finite measure spaces and let $F : \Omega_1 \times \Omega_2 \to \mathbb{R}_+$ be a measurable function. Define $T_{F,f}$
for all measurable positive function $f$ on $\Omega_1$ by

$$T_f \omega_2 = \int_{\Omega_1} F(\omega_1, \omega_2) f(\omega_1) d\mu_1(\omega_1), \quad \text{for all } \omega_2 \in \Omega_2.$$  

If there exist $C > 0$ and measurable functions $h_i : \Omega_i \to [0, +\infty]$; $(i = 1, 2)$ such that

$$\int_{\Omega_1} F(\omega_1, \omega_2) h_1^q (\omega_1) d\mu_1(\omega_1) \leq Ch_2^q (\omega_2) \quad \mu_2 - \text{a.e.}$$

$$\int_{\Omega_2} F(\omega_1, \omega_2) h_2^q (\omega_2) d\mu_2(\omega_2) \leq Ch_1^q (\omega_1) \quad \mu_1 - \text{a.e.}$$

Then $T_f$ can be extended as a bounded operator from $L^q(\Omega_1, \mu_1)$ into $L^q(\Omega_2, \mu_2)$.

**Lemma B.** Let $1 < q, q' < \infty$ such that $\frac{1}{q} + \frac{1}{q'} = 1$ and $\omega \in W^q_{\varepsilon, \delta}$; $\omega(r) = \chi_{\frac{1}{q'}}(r) \nu^{-\frac{1}{q'}}(r^{-1})$. For $0 \leq \varepsilon < \delta < \infty$ and $a, b \in \mathbb{R}^*_+$ let us set

$$R^{\varepsilon, \delta}(a, b) = \frac{\omega(a)}{\omega(b)} \min \left( (\frac{a}{b})^{\varepsilon}, (\frac{b}{a})^{\delta} \right).$$

Then there exist $C > 0$ and a measurable function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\int_{0}^{\infty} R^{\varepsilon, \delta}(a, b) g^{q'}(a) \frac{da}{a} \leq C g^{q'}(b) \quad \text{a.e. } b > 0$$

and

$$\int_{0}^{\infty} R^{\varepsilon, \delta}(a, b) g^{q}(b) \frac{db}{b} \leq C g^{q}(a) \quad \text{a.e. } a > 0.$$  

Proof. It suffices to take $g(r) = \chi_{\frac{1}{q'}}(r) \nu^{-\frac{1}{q'}}(r^{-1})$. \hfill \Box

The main idea in the proof of the above Theorem is based on the following lemmas.

**Lemma 4.1.** Let $\psi, \phi \in S_{*, 0}(\mathbb{R})$. Then

$$\|\Delta_x \phi\|_1 \leq C \min \left( 1, \left( \frac{2}{\rho} \right)^2 \right).$$

Proof. Using the fact that $\|\Delta_x \phi\|_1 \leq C x^2$ (see [11]) and the contraction property of the generalized translation operators we obtain

$$\|\Delta_x \phi\|_1 \leq C \min(1, x^2).$$
On the other hand, from the expression of the kernel $W_\alpha$, one can see easily that
\[ \| \Delta_x \phi_r \|_1 = \| \Delta_{\frac{x}{r}} \phi \|_1. \]
So it holds
\[ (4.4) \quad \| \Delta_x \phi_r \|_1 \leq C \min \left( 1, \left( \frac{x}{r} \right)^2 \right). \]
The lemma is proved. \( \square \)

**Lemma 4.2.** Let $\psi, \phi \in S_{*,0}(X)$. Then
\[ (4.5) \quad \| \phi \# \psi_r \|_1 \leq C \min \left( \left( \frac{r}{\rho} \right)^2, \left( \frac{\rho}{r} \right)^2 \right), \quad \text{for all } r, \rho > 0. \]

**Proof.** Let us take $r \leq \rho$. Since $\int_X \phi_r(y) d\mu_\alpha(y) = 0$ we have
\[ \phi \# \psi_r(y) = \int_X \psi_r(x) \Delta_x \phi_r(y) d\mu_\alpha(x). \]
From Minkowski’s inequality one gets
\[ (4.6) \quad \| \psi_r \# \phi \|_1 \leq \int_X |\psi_r(x)| \| \Delta_x \phi_r \|_1 d\mu_\alpha(x) = C \int_X \left( \frac{x}{r} \right)^{2\alpha+2} \left| \psi \left( \frac{x}{r} \right) \right| \| \Delta_x \phi_r \|_1 \frac{dx}{x}. \]
Taking into account that $\psi$ belongs to $S_*(X)$, then there exists $C > 0$ such that
\[ \left( \frac{x}{r} \right)^{2\alpha+2} \left| \psi \left( \frac{x}{r} \right) \right| \leq C \left( \frac{x}{r} \right)^{-3} \quad \text{and} \quad \left| \psi \left( \frac{x}{r} \right) \right| \leq C. \]
So we get
\[ (4.7) \quad \left( \frac{x}{r} \right)^{2\alpha+2} \left| \psi \left( \frac{x}{r} \right) \right| \leq C \min \left( \left( \frac{x}{r} \right)^{2\alpha+2}, \left( \frac{\rho}{r} \right)^3 \right). \]
Hence from (4.6), (4.7) and (4.4) we obtain
\[ \| \psi_r \# \phi \|_1 \leq C \int_0^\infty \left( \frac{x}{\rho} \right)^2 \min \left( \left( \frac{x}{r} \right)^{2\alpha+2}, \left( \frac{\rho}{x} \right)^3 \right) \frac{dx}{x} = C \left( \frac{r}{\rho} \right)^2. \]
The lemma is proved. \( \square \)

One of the main tools used in this paper is the Calderón reproducing formula that we recall in the following lemma (see [17]).
Lemma 4.3. Let \( \psi \in S^1_{*,0} \), and \( 1 < \varepsilon < \infty \). For \( f \in L^p(d\mu_\alpha) \) \((1 \leq p < \infty)\) define
\[
 f_\varepsilon = \int_{1/\varepsilon}^\varepsilon f \circ \psi_r \circ \psi \frac{dr}{r}.
\]
Then \( f_\varepsilon \) converges to \( f \) in \( L^p(d\mu_\alpha) \) as \( \varepsilon \to \infty \).

Lemma 4.4. (i) Let \( 1 \leq p \leq \infty \) and \( \psi \in S_{*,0} \). Then we have for all \( \beta > 0 \) and \( f \) such that \( B_{\omega,p,q}(f) < \infty \)
\[
\| f \circ \psi_r \|_p \leq C \int_X \min \left( \left( \frac{x}{r} \right)^{2\alpha+2}, \left( \frac{r}{x} \right)^\beta \right) \| \Delta_x f \|_{p} \frac{dx}{x}, \text{ a.e. } r > 0.
\]
(ii) For all \( f \in \Lambda_{\omega,p,q}(X) \) it holds
\[
\| \Delta_x f \|_p \leq C \int_0^\infty \min \left( 1, \left( \frac{x}{r} \right)^2 \right) \| f \circ \psi_r \|_p \frac{dr}{r}, \text{ a.e. } x \in X.
\]

Proof. The result follows from Lemmas 4.1, 4.2 and 4.3.

Proof of Theorem 4.1. (i) Let us start with the case \( q = \infty \) which follows immediately from (4.10) and (4.11). Assume \( f \) such that \( B_{\omega,p,\infty}(f) < \infty \).

Using (4.10) with \( \beta \geq 2 \), we get
\[
\| f \circ \psi_r \|_p \leq C \int_X \min \left( \left( \frac{x}{r} \right)^{2\alpha+2}, \left( \frac{r}{x} \right)^\beta \right) \| \Delta_x f \|_{p} \frac{dx}{x} \\
\leq C B_{\omega,p,\infty}(f) \int_X \omega(x) \min \left( \left( \frac{x}{r} \right)^{2\alpha+2}, \left( \frac{r}{x} \right)^\beta \right) \frac{dx}{x} \\
\leq C B_{\omega,p,\infty}(f) \omega(r).
\]

That is \( A_{\omega,\infty}(f) \leq C B_{\omega,p,\infty}(f) \).

Take now \( f \in \Lambda_{\omega,p,\infty}(X) \). Then, from (4.11) it holds
\[
\| \Delta_x f \|_p \leq C \int_0^\infty \min \left( 1, \left( \frac{x}{r} \right)^2 \right) \| f \circ \psi_r \|_p \frac{dr}{r} \\
\leq C A_{\omega,p,\infty}(f) \int_0^\infty \omega(r) \min \left( 1, \left( \frac{x}{r} \right)^2 \right) \frac{dr}{r} \\
\leq C A_{\omega,p,\infty}(f) \omega(x).
\]

(ii) Let us prove the case \( q = 1 \). Assume \( f \) such that \( B_{\omega,p,1}(f) < \infty \). We shall prove that \( A_{\omega,p,1}(f) \leq C B_{\omega,p,1}(f) \). From (4.10), with \( \beta \geq 2 \) we obtain
\[
A_{\omega,p,1}(f) = \int_0^\infty \frac{\| f \circ \psi_r \|_p}{\omega(r)} \frac{dr}{r}
\]
Conversely, let us take \( f \) in \( \Lambda^{\omega}_{p,1}(X) \). Then using (4.11), we get

\[
B^{\omega}_{p,1}(f) = \int_X \| \Delta_x f \|_p \frac{dx}{\omega(x)}.
\]

Using the inequality (4.10) we obtain, for \( \beta \geq 2 \)

\[
\frac{\| f \ast \psi_r \|_p}{\omega(r)} \leq C \int_X \frac{\omega(x)}{\omega(r)} \min \left( \left( \frac{x}{r} \right)^{2q+2}, \left( \frac{r}{x} \right)^{\beta} \right) \frac{\| \Delta_x f \|_p}{\omega(x)} \frac{dx}{x}.
\]

\[
= C \int_X \frac{\| \Delta_x f \|_p}{\omega(x)} \frac{dx}{x}.
\]

\[
= C \| f \ast \psi_r \|_p \frac{dr}{r}.
\]

(iii) Let us now prove the case \( 1 < q < \infty \). Using the inequality (4.10) we obtain, for \( \beta \geq 2 \)

\[
\frac{\| f \ast \psi_r \|_p}{\omega(r)} \leq C \int_X \frac{\omega(x)}{\omega(r)} \min \left( \left( \frac{x}{r} \right)^{2q+2}, \left( \frac{r}{x} \right)^{\beta} \right) \frac{\| \Delta_x f \|_p}{\omega(x)} \frac{dx}{x}.
\]

\[
= C \int_X R^{2q+2,\beta}(x, r) \frac{\| \Delta_x f \|_p}{\omega(x)} \frac{dx}{x}.
\]

\[
= C T_{R^{2q+2,\beta}} \left( \frac{\| \Delta_x f \|_p}{\omega(x)} \right)(r).
\]

where \( R^{2q+2,\beta}(x, r) = \frac{\omega(x)}{\omega(r)} \min(\left( \frac{x}{r} \right)^{2q+2}, \left( \frac{r}{x} \right)^{\beta}) \). By Lemma A and Lemma B, we obtain the boundedness of the operator \( T_{R^{2q+2,\beta}} \) from \( L^q(X, \frac{dx}{x}) \) into \( L^q(X, \frac{dx}{x}) \). Moreover the hypothesis \( B^{\omega}_{p,1}(f) < \infty \) means that the function \( x \mapsto \frac{\| \Delta_x f \|_p}{\omega(x)} \) belongs to \( L^q(X, \frac{dx}{x}) \). Hence

\[
A^{\omega}_{p,q}(f) = \left\| \frac{\| f \ast \psi_r \|_p}{\omega(r)} \right\|_{L^q(X, \frac{dx}{x})} \leq C \left\| T_F \left( \frac{\| \Delta_x f \|_p}{\omega(x)} \right) \right\|_{L^q(X, \frac{dx}{x})},
\]

\[
\leq C \left\| \frac{\| \Delta_x f \|_p}{\omega(x)} \right\|_{L^q(X, \frac{dx}{x})} = C B^{\omega}_{p,1}(f).
\]
Conversely, let us take $f \in \Lambda_{p,q}^{\omega}(X)$. From (4.11) we get
\[
\frac{\|\Delta_x f\|_p}{\omega(x)} \leq C \int_0^{\infty} \frac{\omega(r)}{\omega(x)} \min \left(1, \left(\frac{x}{r}\right)^2\right) \frac{\|f \ast \psi_r\|_p}{\omega(r)} \frac{dr}{r}
\]
\[
= C \int_0^{\infty} R^{0.2}(r,x) \frac{\|f \ast \psi_r\|_p}{\omega(r)} \frac{dr}{r}
\]
\[
= C \int_0^{\infty} \left(\frac{\|f \ast \psi_r\|_p}{\omega(r)}\right) (x)
\]
where $R^{0.2}(r,x) = \frac{\omega(r)}{\omega(x)} \min(1, (\frac{x}{r})^2)$. We proceed as above with adequate changes to obtain
\[
B_{p,q}^{\omega}(f) \leq C \|T_{\omega}^{0.2} \left(\frac{\|f \ast \psi_r\|_p}{\omega(r)}\right)\|_{L^q(\mathbb{R})}
\]
\[
\leq C \left\|\frac{\|f \ast \psi_r\|_p}{\omega(r)}\right\|_{L^q(\mathbb{R})}
\]
\[
= C A_{p,q}^{\omega}(f)
\]
which gives the desired result.

**Theorem 4.2** (Second characterization theorem). Let $1 \leq p, q \leq \infty$ and $\omega \in \mathcal{W}_{0,q}^q$. For $f$ in $\Lambda_{p,q}^{\omega}(X)$ put
\[
C_{p,q}^{\omega}(f) = \begin{cases} 
\left(\int_X \left(\frac{m_p(f,x)}{\omega(x)}\right)^{q} \frac{dx}{x}\right)^{\frac{1}{q}}, & \text{if } 1 \leq q < \infty, \\
\text{esssup}_{x \in X} \left(\frac{m_p(f,x)}{\omega(x)}\right), & \text{if } q = \infty
\end{cases}
\]
where $m_p(f,x) = \sup_{0 \leq y \leq x} \|\Delta_y f\|_p$ is the generalized modulus of continuity associated with the generalized translation operators $T_x^{(\alpha)}$. Then $C_{p,q}^{\omega}$ is a norm on $\Lambda_{p,q}^{\omega}(X)$ equivalent to $A_{p,q}^{\omega}$.

**Proof.** To compare $A_{p,q}^{\omega}(f)$ and $C_{p,q}^{\omega}(f)$ we proceed as in the proof of the Theorem 4.1 using the following lemma instead of Lemma 4.4. □

**Lemma 4.5.** (i) Let $1 \leq p \leq \infty$ and $\psi \in S_{*,0}$. Then we have for all $\beta > 0$ and $f$ such that $B_{p,q}^{\omega}(f) < \infty$
\[
(4.10) \ \|f \ast \psi_r\|_p \leq C \int_X \min \left(\left(\frac{x}{r}\right)^{2\alpha+2}, \left(\frac{r}{x}\right)^{\beta}\right) m_p(f,x) \frac{dx}{x}, \ a.e. \ r > 0.
\]
(ii) For all \( f \in \Lambda_{p,q}^\omega(X) \) it holds

\[
(4.11) \quad m_p(f, x) \leq C \int_0^\infty \min\left(1, \left(\frac{x}{r}\right)^2\right) \|f \ast \psi_r\|_p \frac{dr}{r}, \quad \text{a.e. } x \in X.
\]

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Département de Mathématiques IPEIN  
Campus Universitaire Mrezka 8000  
Nabeul  
Tunisia  
(E-mail : Miloud.Assal@fst.rnu.tn)

Département de Mathématiques  
Faculté des Sciences de Bizerte  
Zarzouna 7021  
Bizerte  
Tunisia  
(E-mail : Hacen.Benabdallah@fst.rnu.tn)

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