The boundedness of multilinear commutators on locally compact Vilenkin groups

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Abstract. Let G be a locally compact Vilenkin group. In this paper, the authors investigate the boundedness of multilinear commutators of fractional integral operator on Lebesgue spaces on G . Furthermore, the boundedness on Hardy spaces are also obtained in this paper.

1. Introduction

The commutators have been studied by many authors for a long time. A well known result which is discovered by Coifman, Rocherg and Weiss ([3], [6], [9]) is that the commutators $[b, T]$ of singular integral operators are bounded on some $L^p(\mathbb{R}^n)$ ($1 < p < \infty$) if and only if $b \in BMO$, where $[b, T]$ is defined by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x).$$

A natural generalization of the commutator T_b^m is given by

$$(1.1) \quad T_b^m(f)(x) = \int_{\mathbb{R}^n} [b(x) - b(y)]^m K(x, y) f(y) dy,$$

where $m \in \mathbb{N}$. It was shown in [1] that it is bounded on $L^p(\omega)$ ($1 < p < \infty$) when $\omega \in A_p$. And in [2], the authors prove the $L^p(\omega)$ ($1 < p < \infty$)-boundedness for multilinear commutators. And similar results can be found in [10]. This is the motivation of considering the boundedness for multilinear commutators of fractional integral operator on locally compact Vilenkin group G .

In order to state our results more precisely we first introduce some notations and definitions.

Throughout this paper, G will denote a bounded locally compact Vilenkin group, that is, G is a locally compact Abelian group containing a strictly decreasing sequence of compact open subgroups $\{G_n\}_{n=-\infty}^{\infty}$ such that

- (a) $\cup_{n=-\infty}^{\infty} G_n = G$ and $\cap_{n=-\infty}^{\infty} G_n = \{0\}$;
- (b) $\sup\{\text{order}(G_n/G_{n+1}) : n \in \mathbb{Z}\} = B < \infty$.

Choose Haar measure dx on G so that $|G_0| = 1$, where $|A|$ denotes the measure of a measurable subset A of G . Let $|G_n| = (m_n)^{-1}$ for each $n \in \mathbb{Z}$. Since $2m_n \leq m_{n+1} \leq Bm_n$, it follows that

$$\sum_{n=k}^{\infty} (m_n)^{-\alpha} \leq c(m_k)^{-\alpha}$$

and

$$\sum_{n=-\infty}^k (m_n)^{\alpha} \leq c(m_k)^{\alpha}$$

for any $\alpha > 0, k \in \mathbb{Z}$, where c is a constant independent of k . For each $n \in \mathbb{Z}$ we choose elements $z_{l,n} \in G$ ($l \in \mathbb{Z}_+$) so that the subsets $G_{l,n} := z_{l,n} + G_n$ of G satisfy $G_{k,n} \cap G_{l,n} = \emptyset$ if $k \neq l$ and $\cup_{l=0}^{\infty} G_{l,n} = G$; moreover, we choose $z_{0,n}$ such that $G_{0,n} = G_n$. We now define the function $d : G \times G \rightarrow \mathbb{R}$ by $d(x, y) = 0$ if $x - y = 0$ and $d(x, y) = (m_n)^{-1}$ if $x - y \in G_n \setminus G_{n+1}$. Then d is a metric on G and the topology on G generated by this metric is the same as the original topology on G . For $x \in G$, set $|x| = d(x, 0)$. Then $|x| = (m_n)^{-1}$ if and only if $x \in G_n \setminus G_{n+1}$. Let $S(G)$ be the space of test functions and $S'(G)$ be the distribution space on G . And χ_{G_n} is the characteristic function of G_n . C can be denote various constants.

We also recall the definition of space of bounded mean oscillation. For $x_0 \in G$, set $I_j = x_0 + G_j$, we say a locally integrable function b has bounded

mean oscillation, $b \in BMO(G)$, if

$$\sup_{I_j \ni x} \inf_{c \in \mathbb{R}} \left(\frac{1}{|I_j|} \int_{I_j} |f(y) - c| dy \right) \leq \infty,$$

where the supremum is taken over all cosets $I_j \subset G$ (see[4]). Since the topological nature of G at any $x \in G$ is the same as it is at 0, we choose $x_0 = 0$ in our article.

2. Main results and proofs

Define multilinear commutator of fractional integral operator I_α as

$$[b, I_\alpha]f(x) = \int_G \prod_{j=1}^m \left[b_j(x) - b_j(y) \right] \frac{f(y)}{|x-y|^{1-\alpha}} dy$$

and maximal function

$$f_{\alpha,r}^*(x) = \sup_{x \in G_k} |G_k|^\alpha \left(\frac{1}{|G_k|} \int_{G_k} |f(y)|^r dy \right)^{1/r}.$$

Given any positive integer m , for all $1 \leq j \leq m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \dots, \sigma(j)\}$ of $\{1, \dots, m\}$ of j different elements. For any $\sigma \in C_j^m$, we associate the complementary sequence σ' given by $\sigma' = \{1, 2, \dots, m\} \setminus \sigma$, and

$$[b_\sigma, I_\alpha]f(x) = \int_G \left[b_{\sigma(1)}(x) - b_{\sigma(1)}(y) \right] \cdots \left[b_{\sigma(j)}(x) - b_{\sigma(j)}(y) \right] \frac{f(y)}{|x-y|^{1-\alpha}} dy.$$

suppose $\|b_i\|_{BMO} = 1$ for $i = 1, 2, \dots, m$. Let p' is adjoint index of p . We have the following theorem.

Theorem 2.1. *Let $b = (b_1, \dots, b_m)$, $b_i \in BMO$, $1/q = 1/p - \alpha$, $1 < p < 1/\alpha$, $0 < \alpha < 1$, then $[b, I_\alpha]$ maps $L^p(G)$ into $L^q(G)$.*

To prove this theorem, we need the following lemmas.

Lemma 2.1 (see [7]). *Let $p > 1$, $1/q = 1/p - \alpha$, $0 < \alpha < 1$, then*

$$\|I_\alpha f\|_q \leq C \|f\|_p.$$

Lemma 2.2 (see [5] and [8]). *Let $r < p < 1/\alpha$, $1/q = 1/p - \alpha$, $0 < \alpha < 1$, then*

$$\|f_{\alpha,r}^*\|_q \leq C \|f\|_p.$$

Lemma 2.3. *Let $b = (b_1, \dots, b_m)$, $b_i \in BMO$, $0 < \delta < \epsilon < 1$, then there exist a constant C depends on δ, ϵ , and choose p_1 that $1 < p_1 < p < 1/\alpha$ such that*

$$\begin{aligned} M_\delta^\sharp([b, I_\alpha]f)(x) \\ \leq C \left\{ M_\epsilon(I_\alpha f)(x) + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} M_\epsilon([b_{\sigma'}, I_\alpha]f)(x) + f_{\alpha, p_1}^*(x) \right\}. \end{aligned}$$

Proof. We firstly consider the case of $m = 1$, that is,

$$\begin{aligned} [b, I_\alpha]f(x) &= \int_G [b(x) - b(y)] \frac{f(y)}{|x - y|^{1-\alpha}} dy \\ &= [b(x) - \lambda] I_\alpha f(x) - I_\alpha [(b - \lambda)f](x). \end{aligned}$$

Fix x , let $x \in G_k$, since

$$||\alpha|^\delta - |\beta|^\delta| \leq |\alpha - \beta|^\delta$$

for any $\alpha, \beta \in R$, $0 < \delta < 1$, then we have

$$\begin{aligned} &\left(\frac{1}{|G_k|} \int_{G_k} ||[b, I_\alpha]f(y)|^\delta - |c|^\delta| dy \right)^{1/\delta} \\ &\leq \left(\frac{1}{|G_k|} \int_{G_k} |[b, I_\alpha]f(y) - c|^\delta dy \right)^{1/\delta} \\ &\leq C \left(\frac{1}{|G_k|} \int_{G_k} |[b(y) - \lambda] I_\alpha f(y)|^\delta dy \right)^{1/\delta} \\ &\quad + C \left(\frac{1}{|G_k|} \int_{G_k} |I_\alpha [(b - \lambda)f](y) - c|^\delta dy \right)^{1/\delta} \\ &= \text{I} + \text{II}. \end{aligned}$$

choose $\lambda = b_{G_k}$ and $1 < q < \epsilon/\delta$, using Hölder inequality and Jenson inequality, we can deduce that

$$\begin{aligned} \text{I} &\leq C \left(\frac{1}{|G_k|} \int_{G_k} |b(y) - \lambda|^{\delta q'} dy \right)^{1/\delta q'} \left(\frac{1}{|G_k|} \int_{G_k} |I_\alpha f(y)|^{\delta q} dy \right)^{1/\delta q} \\ &\leq C \|b\|_{BMO} M_\epsilon(I_\alpha f)(x). \end{aligned}$$

To estimate II, let $f = f_1 + f_2$, $f_1 = f \chi_{G_k}$, then

$$\begin{aligned} \text{II} &\leq C \left(\frac{1}{|G_k|} \int_{G_k} |I_\alpha [(b - \lambda)f_1](y)|^\delta dy \right)^{1/\delta} \\ &\quad + \left(\frac{1}{|G_k|} \int_{G_k} |I_\alpha [(b - \lambda)f_2](y) - c|^\delta dy \right)^{1/\delta} = \text{III} + \text{IV}. \end{aligned}$$

For III, we have

$$\begin{aligned}
 \text{III} &\leq C \frac{1}{|G_k|} \int_{G_k} |I_\alpha[(b-\lambda)f_1](y)| dy \\
 &\leq C \frac{1}{|G_k|} \int_{G_k} \int_{G_k} |b(y) - \lambda| \frac{|f(y)|}{|x-y|^{1-\alpha}} dy dx \\
 &\leq C \frac{1}{|G_k|^{1-\alpha}} \int_{G_k} |b(y) - \lambda| |f(y)| dy,
 \end{aligned}$$

choose p_1 such that $1 < p_1 < p < 1/\alpha$, then

$$\begin{aligned}
 \text{III} &\leq C \left(\frac{1}{|G_k|} \int_{G_k} |b(y) - \lambda|^{p'_1} dy \right)^{1/p'_1} |G_k|^\alpha \left(\frac{1}{|G_k|} \int_{G_k} |f(y)|^{p_1} dy \right)^{1/p_1} \\
 &\leq C \|b\|_{BMO f_{\alpha, p_1}^*}(x).
 \end{aligned}$$

Next, we turn to estimate IV. Let

$$c = (I_\alpha[(b-\lambda)f_2])_{G_k},$$

then

$$\begin{aligned}
 \text{IV} &\leq C \frac{1}{|G_k|} \int_{G_k} |I_\alpha[(b-\lambda)f_2](y) - (I_\alpha[(b-\lambda)f_2])_{G_k}| dy \\
 &= C \frac{1}{|G_k|} \int_{G_k} \left| \int_{G \setminus G_k} \frac{[b(\omega) - \lambda]f(\omega)}{|y-\omega|^{1-\alpha}} d\omega \right. \\
 &\quad \left. - \frac{1}{|G_k|} \int_{G_k} \left\{ \int_{G \setminus G_k} \frac{[b(\omega) - \lambda]f(\omega)}{|z-\omega|^{1-\alpha}} d\omega \right\} dz \right| dy \\
 &= C \frac{1}{|G_k|} \int_{G_k} \left| \frac{1}{|G_k|} \int_{G_k} \left\{ \int_{G \setminus G_k} [b(\omega) - \lambda]f(\omega) \right. \right. \\
 &\quad \left. \left. \times \left[\frac{1}{|y-\omega|^{1-\alpha}} - \frac{1}{|z-\omega|^{1-\alpha}} \right] d\omega \right\} dz \right| dy \\
 &\leq C \frac{1}{|G_k|} \int_{G_k} \frac{1}{|G_k|} \int_{G_k} \sum_{i=-\infty}^{k-1} \int_{G_i \setminus G_{i+1}} |b(\omega) - \lambda| \\
 &\quad \times |f(\omega)| \frac{|y-z|}{|z-\omega|^{2-\alpha}} d\omega dz dy \\
 &\leq C \sum_{i=-\infty}^{k-1} \frac{|G_k|}{|G_i|^{2-\alpha}} \int_{G_i} |b(\omega) - \lambda| |f(\omega)| d\omega \\
 &\leq C \sum_{i=-\infty}^{k-1} \frac{|G_k|}{|G_i|^{1-\alpha}} \left(\frac{1}{|G_i|} \int_{G_i} |b(\omega) - \lambda|^{p'_1} d\omega \right)^{1/p'_1} \\
 &\quad \times \left(\frac{1}{|G_i|} \int_{G_i} |f(\omega)|^{p_1} d\omega \right)^{1/p_1}
 \end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{i=-\infty}^{k-1} \frac{|G_k|}{|G_i|} f_{\alpha, p_1}^*(x) \left\{ \left(\frac{1}{|G_i|} \int_{G_i} |b(\omega) - b_{G_i}|^{p'_1} d\omega \right)^{1/p'_1} + |b_{G_i} - b_{G_k}| \right\} \\
&\leq C \sum_{i=-\infty}^{k-1} \frac{|G_k|}{|G_i|} \left[1 + \frac{1}{2}(k-i) \right] f_{\alpha, p_1}^*(x) \|b\|_{BMO} \\
&\leq C \sum_{i=-\infty}^{k-1} \frac{m_i}{m_k} (k-i) f_{\alpha, p_1}^*(x) \|b\|_{BMO} \\
&\leq C \|b\|_{BMO} f_{\alpha, p_1}^*(x),
\end{aligned}$$

where

$$\begin{aligned}
|b_{G_n} - b_{G_{n+1}}| &\leq \frac{1}{|G_n|} \int_{G_n} |b - b_{G_{n+1}}| dx \\
&\leq \frac{|G_{n+1}|}{|G_n|} \frac{1}{|G_{n+1}|} \int_{G_{n+1}} |b(x) - b_{G_{n+1}}| dx \\
&\leq \frac{|G_{n+1}|}{|G_n|} \|b\|_{BMO} \\
&\leq \frac{1}{2} \|b\|_{BMO},
\end{aligned}$$

for any $n \in \mathbb{Z}$, and

$$\begin{aligned}
|b_{G_i} - b_{G_k}| &= |b_{G_i} - b_{G_{i+1}}| + |b_{G_{i+1}} - b_{G_{i+2}}| + \cdots + |b_{G_{k-1}} - b_{G_k}| \\
&= \frac{1}{2} (k-i) \|b\|_{BMO}.
\end{aligned}$$

Therefore, we have proved that

$$M_{\delta}^{\sharp}([b, I_{\alpha}]f)(x) \leq C \|b\|_{BMO} [M_{\epsilon}(I_{\alpha}f)(x) + f_{\alpha, p_1}^*(x)]$$

for $m = 1$.

If $m \geq 2$, let $\lambda = (\lambda_1, \dots, \lambda_m)$, we have

$$\begin{aligned}
[b, I_{\alpha}]f(x) &= \int_G \prod_{j=1}^m [(b_j(x) - b_j(y))] \frac{f(y)}{|x-y|^{1-\alpha}} dy \\
&= \int_G \prod_{j=1}^m [(b_j(x) - \lambda_j + \lambda_j - b_j(y))] \frac{f(y)}{|x-y|^{1-\alpha}} dy \\
&= \prod_{j=1}^m [(b_j(x) - \lambda_j) I_{\alpha}f(x) + (-1)^m I_{\alpha} \left[\prod_{j=1}^m (b_j(x) - \lambda_j) f \right](x) \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} C_{m,j} [b(x) - \lambda]_{\sigma} [b_{\sigma'}, I_{\alpha}](f)(x),
\end{aligned}$$

where $C_{m,j}$ are constants depending on m and j .

Now, for fixed $x \in G$, for any number c and $G_k \ni x$, since $0 < \delta < 1$, we have

$$\begin{aligned}
& \left(\frac{1}{|G_k|} \int_{G_k} ||[b, I_\alpha]f(y)|^\delta - |c|^\delta| dy \right)^{1/\delta} \\
& \leq \left(\frac{1}{|G_k|} \int_{G_k} |[b, I_\alpha]f(y) - c|^\delta dy \right)^{1/\delta} \\
& \leq C \left(\frac{1}{|G_k|} \int_{G_k} \left| \prod_{j=1}^m [b_j(y) - \lambda_j] I_\alpha(f)(y) \right|^\delta dy \right)^{1/\delta} \\
& \quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \left(\frac{1}{|G_k|} \int_{G_k} |[b(y) - \lambda]_\sigma [b_{\sigma'}, I_\alpha](f)(y)|^\delta dy \right)^{1/\delta} \\
& \quad + C \left(\frac{1}{|G_k|} \int_{G_k} \left| I_\alpha \left[\prod_{j=1}^m (b_j - \lambda_j) f \right] (y) - c \right|^\delta dy \right)^{1/\delta} \\
& = \text{I} + \text{II} + \text{III}.
\end{aligned}$$

For the first two parts I and II, choose $\lambda_j = (b_j)_{G_k}, j = 1, 2, \dots, m$, and $1 < q < \delta/\epsilon$, similar to the case of $m = 1$, we can deduce that

$$\text{I} \leq CM_\epsilon(I_\alpha f)(x),$$

and

$$\text{II} \leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} M_\epsilon([b_{\sigma'}, I_\alpha]f)(x).$$

For III, let $f = f_1 + f_2$, $f_1 = f\chi_{G_k}$, then

$$\begin{aligned}
\text{III} & \leq C \left\{ \frac{1}{|G_k|} \int_{G_k} \left| I_\alpha \left[\prod_{j=1}^m (b_j - \lambda_j) f_1 \right] (y) \right|^\delta dy \right\}^{1/\delta} \\
& \quad + C \left\{ \frac{1}{|G_k|} \int_{G_k} \left| I_\alpha \left[\prod_{j=1}^m (b_j - \lambda_j) f_2 \right] (y) - c \right|^\delta dy \right\}^{1/\delta} \\
& = \text{IV} + \text{V}.
\end{aligned}$$

By Jenson inequality, we have

$$\begin{aligned}
\text{III} & \leq C \frac{1}{|G_k|} \int_{G_k} \left| I_\alpha \left[\prod_{j=1}^m (b_j - \lambda_j) f_1 \right] (y) \right| dy \\
& \leq C \frac{1}{|G_k|} \int_{G_k} \int_{G_k} \prod_{j=1}^m \left| b_j(z) - \lambda_j \right| \frac{|f(z)|}{|y - z|^{1-\alpha}} dz dy
\end{aligned}$$

$$\begin{aligned}
&\leq C \frac{1}{|G_k|} \int_{G_k} \prod_{j=1}^m |b_j(z) - \lambda_j| |f(z)| \left(\int_{G_k} \frac{1}{|y-z|^{1-\alpha}} dy \right) dz \\
&\leq C \left(\frac{1}{|G_k|} \int_{G_k} \left| \prod_{j=1}^m (b_j(z) - \lambda_j) \right|^{p'_1} dz \right)^{1/p'_1} |G_k|^\alpha \left(\frac{1}{|G_k|} \int_{G_k} |f(z)|^{p_1} dz \right)^{1/p_1} \\
&\leq C \prod_{j=1}^m \left(\frac{1}{|G_k|} \int_{G_k} |b_j(z) - \lambda_j|^{mp'_1} dz \right)^{1/mp'_1} f_{\alpha, p_1}^*(x) \\
&\leq C \|b\|_{BMO} f_{\alpha, p_1}^*(x).
\end{aligned}$$

To estimate V , let

$$c = \left(I_\alpha \left[\prod_{j=1}^m (b_j - \lambda_j) f_2 \right] \right)_{G_k},$$

similar to the case in $m = 1$, it can follow that

$$\begin{aligned}
\text{IV} &\leq C \frac{1}{|G_k|} \int_{G_k} \left| I_\alpha \left[\prod_{j=1}^m (b_j - \lambda_j) f_2 \right] (y) - \left[I_\alpha \left(\prod_{j=1}^m (b_j - \lambda_j) f_2 \right) \right]_{G_k} \right| dy \\
&\leq C \sum_{i=-\infty}^{k-1} \frac{|G_k|}{|G_i|^{2-\alpha}} \int_{G_i} \prod_{j=1}^m |b_j(\omega) - \lambda_j| |f(\omega)| d\omega \\
&\leq C \sum_{i=-\infty}^{k-1} \frac{|G_k|}{|G_i|^{1-\alpha}} \left(\frac{1}{|G_i|} \int_{G_i} \left| \prod_{j=1}^m (b_j(\omega) - \lambda_j) \right|^{p'_1} d\omega \right)^{1/p'_1} \\
&\quad \times \left(\frac{1}{|G_i|} \int_{G_i} |f(\omega)|^{p_1} d\omega \right)^{1/p_1} \\
&\leq C \sum_{i=-\infty}^{k-1} \frac{|G_k|}{|G_i|} f_{\alpha, p_1}^*(x) \times \prod_{j=1}^m \left[\left(\frac{1}{|G_i|} \int_{G_i} |b_j(\omega) - (b_j)_{G_i}|^{mp'_1} d\omega \right)^{1/mp'_1} \right. \\
&\quad \left. + |(b_j)_{G_i} - (b_j)_{G_k}| \right] \\
&\leq C \sum_{i=-\infty}^{k-1} \frac{m_i}{m_k} (k-i)^m f_{\alpha, p_1}^*(x) \|b\|_{BMO} \\
&\leq C \|b\|_{BMO} f_{\alpha, p_1}^*(x).
\end{aligned}$$

Combine with I, II, IV, V, we finish the proof of Lemma 2.3. $\square$

Proof of Theorem 2.1. We first take it for granted that $M([b, I_\alpha]f) \in L^q(G)$ and we'll check this to the end of the proof.

We proceed by induction on m . For $m = 1$, by lemma 2.1, 2.2 and 2.3, we have

$$\begin{aligned} \|[b, I_\alpha]f\|_q &\leq C\|M_\delta^\#([b, I_\alpha]f)\|_q \\ &\leq C\|M_\epsilon(I_\alpha f)\|_q + C\|f_{\alpha, p_1}^*\|_q \\ &\leq C\|I_\alpha f\|_q + C\|f\|_p \\ &\leq C\|f\|_p. \end{aligned}$$

Suppose now that for $m - 1$ the theorem is true, and let us prove it for m . The same argument as used above and the induction hypothesis give

$$\begin{aligned} \|[b, I_\alpha]f\|_q &\leq C\|(M_\epsilon(I_\alpha f))\|_q + C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \|M_\epsilon([b_{\sigma'}, I_\alpha]f)\|_q + C\|f\|_p \\ &\leq C\|f\|_p. \end{aligned}$$

Now let us check that $M([b, I_\alpha]f) \in L^q(G)$. By the boundedness of maximal operator, we only need to prove $[b, I_\alpha]f \in L^q(G)$. Suppose for any $j(j = 1, 2, \dots, m)$, b_j is bounded, since $C_c^\infty(G)$ is dense in $L^p(G)(p > 1)$, we only need to consider the function f with compact support. Suppose $\text{supp } f \subset G_k$, we have

$$\begin{aligned} \|[b, I_\alpha]f\|_q &= \left(\int_{G_k} |[b, I_\alpha](f)(x)|^q dx \right)^{1/q} + \left(\int_{G \setminus G_k} |[b, I_\alpha](f)(x)|^q dx \right)^{1/q} \\ &= A_1 + A_2. \end{aligned}$$

Using the boundedness of b_i , we get

$$\begin{aligned} A_1 &\leq C \left(\int_{G_k} \left| \int_G [b(x) - b(y)] \frac{f(y)}{|x - y|^{1-\alpha}} dy \right|^q dx \right)^{1/q} \\ &\leq C \left(\int_{G_k} |I_\alpha(f)(x)|^q dx \right)^{1/q} \\ &\leq C\|f\|_p. \end{aligned}$$

For the second term, according to the boundedness of b_i and $x \notin G_k$, $y \in G_k$, $|x - y| \sim |x|$, we have

$$\begin{aligned} \left| [b, I_\alpha]f \right| &\leq C \int_{G_k} \prod_{j=1}^m |b_j(x) - b_j(y)| \frac{|f(y)|}{|x - y|^{1-\alpha}} dy \\ &\leq C \int_{G_k} \frac{|f(y)|}{|x - y|^{1-\alpha}} dy \end{aligned}$$

$$\begin{aligned}
&\leq C \frac{1}{|x|^{1-\alpha}} \int_{G_k} |f(y)| dy \\
&\leq C |G_k|^\alpha \left(\frac{1}{|G_k|} \int_{G_k} |f(y)|^{p_1} dy \right)^{1/p_1} \\
&\leq C f_{\alpha, p_1}^*(x),
\end{aligned}$$

Thus,

$$A_2 \leq C \left(\int_{G \setminus G_k} |f_{\alpha, p_1}^*(x)|^q dx \right)^{1/q} \leq C \|f_{\alpha, p_1}^*\|_q,$$

which is finite by lemma 2.2.

For the general case, we will truncate the symbols b_j as follows. Denote $b_j^N = \min\{b_j, N\}$, take into account the fact that f has compact support, we deduce that any product $b_{j_1}^N \cdots b_{j_i}^N f$ converges in any $L^q(G)$ ($q > 1$) to $b_{j_1} \cdots b_{j_i} f$ as $N \rightarrow \infty$. By the above discussion, we have

$$\|[b^N, I_\alpha](f)\|_q \leq C \|b^N\|_{BMO} \|f\|_p.$$

By Fatou's lemma, we conclude that the theorems holds for this general case. The theorem is proved. $\square$

Furthermore, we can discuss the boundedness on Hardy spaces.

Definition 2.1. Let $0 < p \leq 1 \leq q \leq \infty$, $p \neq q$, a nonnegative integer $s \geq [1/p - 1]$, $b = (b_1, \dots, b_m)$, $b_i \in BMO(G)$, $i = 1, 2, \dots, m$. A function $a(x) \in L_{loc}^q(G)$ is said to be a (p, q, b) -atom if it satisfies:

- (i) for some $k \in \mathbb{Z}$, $\text{supp } a \subset G_k$;
- (ii) $\|a\|_q \leq |G_k|^{1/q-1/p}$;
- (iii) $\int_{G_k} a(y) dy = \int_{G_k} a(y) \prod_{l \in \sigma} b_l(y) dy = 0$, for any $\sigma \in C_j^m$, $1 \leq j \leq m$.

Definition 2.2. The Hardy spaces $H_b^p(G)$ are defined by

$$\begin{aligned}
H_b^p(G) = & \left\{ f : f \in S'(G), f = \sum_{k=-\infty}^{\infty} \lambda_k a_k, a_k \text{ is } (p, q, b) \text{-atom}, \right. \\
& \left. \sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty \right\}.
\end{aligned}$$

and

$$\|f\|_{H_b^p(G)} \sim \inf \left\{ \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p} \right\},$$

where the infimum is taken over all the decomposition of f as above.

Theorem 2.2. Let b is the same as Theorem 2.1. Let $1/2 < p \leq 1$, $1/q = 1/p - \alpha$, $0 < \alpha < 1$, then $[b, I_\alpha]$ is a bounded operator from $H_b^p(G)$ into $L^q(G)$.

Proof. We only need to prove theorem for the (p, ∞, b) -atom. Suppose $\text{supp } a \subset G_k$, then

$$\|[b, I_\alpha]a\|_q \leq \left(\int_{G_k} |[b, I_\alpha]a(x)|^q dx \right)^{1/q} + \left(\int_{G \setminus G_k} |[b, I_\alpha]a(x)|^q dx \right)^{1/q} = \text{I} + \text{II}.$$

Choose p_1, q_1 such that $1 < p_1 < q_1 < \infty$ and $1/q = 1/p - \alpha$, then $q < q_1$. By Hölder inequality and Theorem 2.1, we have

$$\begin{aligned} \text{I} &\leq C \left(\int_{G_k} \left| [b, I_\alpha]a(x) \right|^{q_1} dx \right)^{1/q_1} |G_k|^{1/q - 1/q_1} \\ &\leq C \|a\|_{p_1} |G_k|^{1/q - 1/q_1} \\ &\leq C \|a\|_\infty |G_k|^{1/p_1 + 1/q - 1/q_1} \\ &\leq C |G_k|^{1/p_1 - 1/p + 1/q - 1/q_1} \\ &= C. \end{aligned}$$

Note that $x \in G \setminus G_k$, and $y \in G_k$, let $\lambda_j = (b_j)_{G_k}$ for any j , then

$$\begin{aligned} |[b, I_\alpha]a(x)| &= \left| \int_{G_k} \prod_{j=1}^m [b_j(x) - b_j(y)] \frac{a(y)}{|x - y|^{1-\alpha}} dy \right| \\ &= \left| \sum_{j=0}^m \sum_{\sigma \in C_j^m} C[b(x) - \lambda]_\sigma \int_{G_k} [b(x) - \lambda]_{\sigma'} \frac{a(y)}{|x - y|^{1-\alpha}} dy \right|. \end{aligned}$$

Using the vanishing moment condition (iii), we get

$$\begin{aligned} |[b, I_\alpha]a(x)| &= \left| \sum_{j=0}^m \sum_{\sigma \in C_j^m} C[b(x) - \lambda]_\sigma \int_{G_k} [b(x) - \lambda]_{\sigma'} a(y) \right. \\ &\quad \times \left. \left[\frac{1}{|x - y|^{1-\alpha}} - \frac{1}{|x|^{1-\alpha}} \right] dy \right| \\ &\leq C \sum_{j=0}^m \sum_{\sigma \in C_j^m} |[b(x) - \lambda]_\sigma| \left(\frac{|G_k|}{|x|^{2-\alpha}} \int_{G_k} |[b(x) - \lambda]_{\sigma'} a(y)| dy \right). \end{aligned}$$

Similar to the proof of Theorem 2.1, we deduce that

$$\begin{aligned}
\text{II} &\leq C \sum_{j=0}^m \sum_{\sigma \in C_j^m} \int_{G_k} |[b(x) - \lambda]_{\sigma'} a(y)| dy \\
&\quad \times \left[\sum_{i=-\infty}^{k-1} \int_{G_i \setminus G_{i+1}} |[b(x) - \lambda]_{\sigma}|^q \left\{ \frac{|G_k|}{|x|^{2-\alpha}} \right\}^q dx \right]^{1/q} \\
&\leq C \sum_{j=0}^m \sum_{\sigma \in C_j^m} \int_{G_k} |[b(x) - \lambda]_{\sigma'} a(y)| dy \\
&\quad \times \sum_{i=-\infty}^{k-1} \frac{|G_k|}{|G_i|^{2-\alpha}} \left(\int_{G_i \setminus G_{i+1}} |[b(x) - \lambda]_{\sigma}|^q dx \right)^{1/q} \\
&\leq C \|a\|_{\infty} \sum_{j=0}^m \sum_{\sigma \in C_j^m} |G_k| \left(\frac{1}{|G_k|} \int_{G_k} |[b(x) - \lambda]_{\sigma'}| dy \right) \\
&\quad \times \sum_{i=-\infty}^{k-1} \frac{|G_k|}{|G_i|^{2-\alpha}} |G_i|^{1/q} \left(\frac{1}{|G_i|} \int_{G_i} |[b(x) - \lambda]_{\sigma}|^q dx \right)^{1/q} \\
&\leq C \|a\|_{\infty} |G_k| \|b\|_{BMO} \sum_{i=-\infty}^{k-1} \frac{|G_k|}{|G_i|^{2-\alpha-1/q}} (k-i)^m \\
&\leq C \|b\|_{BMO} \sum_{i=-\infty}^{k-1} \frac{|G_k|^{2-1/p}}{|G_i|^{2-\alpha-1/q}} (k-i)^m \\
&\leq C \|b\|_{BMO},
\end{aligned}$$

Here $m \geq 1$, $1/2 < p \leq 1$ and $1/q = 1/p - \alpha$.

Therefore, Theorem 2.2 is proved. $\square$

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