A description of the Stone space of Banach lattice $C(K, E)$

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Abstract. We give a topological description of the Stone space of $C(K, E)$, Banach lattices of continuous functions from a compact Hausdorff space K into a Banach lattice E .

A compact Hausdorff space S is called *Stonean* if the closure of each open subset of K is also open. If S is a Stonean space then $C^\infty(S)$, the space of all extended real valued continuous functions f on S with $f^{-1}(\mathbb{R})$ dense in S , is a universal complete Riesz space. It is well known that for each Archimedean Riesz space E there exists unique (up to homeomorphism) Stonean space S_E such that E is order dense Riesz subspace of $C^\infty(S_E)$, this is known as Maeda-Ogasawara representation, and S_E is called *the Stone space* of E and $C^\infty(S)$ is called *the universal completion* of E . Let E be an Archimedean Riesz space. The set of all (linear) operators $T : E \rightarrow E$ such that $|T(x)| \leq \lambda|x|$ for each $x \in E$ and for some $\lambda \in \mathbb{R}$ is called the *center* of E and is denoted by $Z(E)$. $Z(E)$ is an AM-space with order unit I (the identity operator on E) which is due to Wickstead [3] and so from the Kakutani representation theorem $Z(E)$ is isometrically Riesz isomorphic to

$C(K)$ for some compact Hausdorff space K . Let E be a Banach lattice. Then we write

$$H_I(E) = \{T|T : Z(E) \rightarrow \mathbb{R} \text{ is lattice homomorphism, } T(I) = 1\}.$$

The Stone-Chech compactification of a completely regular Hausdorff space M is denoted by $\beta(M)$.

For the terminology for this paper we refer to [1]. Now we are ready to give a description the Stone space of Dedekind complete Riesz space $C(K, E)$ as follows.

Theorem 1. *Let K be a Stonean space and E be a Banach lattice with compact order intervals. Then the Stone space of $C(K, E)$ is $\beta(K \odot H_I(E))$, where $K \odot H_I(E)$ is a topological space with respect to some topology on $K \times H_I(E)$. That is, the universal completion of $C(K, E)$ is $C^\infty(\beta(K \odot H_I(E)))$.*

Proof. $C(K, E)$ is Dedekind complete if and only if K is Stonean and E has compact order intervals [2], so $C(K, E)$ is a Dedekind complete Banach lattice. Let $C_b(K, Z(E)_s)$ be the set of all functions f from K into $Z(E)$ with $\sup_{k \in K} \|f(k)\| < \infty$ and $f(k_\alpha)(e) \rightarrow f(k)(e)$ for each $e \in E$ whenever $k_\alpha \rightarrow k$ in K . Then $C_b(K, Z(E)_s)$ is a Dedekind complete Banach lattice and isometric Riesz isomorphic to $Z(C(K, E))$ under the operator

$$R : Z(C(K, E)) \rightarrow C_b(K, Z(E)_s), \quad R(\pi(k)(e)) = \pi(\mathbf{1} \otimes e)$$

where $\mathbf{1} \otimes e(k) = e$ for each $k \in K$. Let $K \times H_I(E)$ be topologized by

$$(k_\alpha, T_\alpha) \rightarrow (k, T) \iff T_\alpha(f(k_\alpha)) \rightarrow T(f(k)) \text{ for each } f \in C_b(K, Z(E)_s).$$

We denote this topological space by $K \odot H_I(E)$. That is, the smallest topology on $K \times H_I(E)$ such that

$$\{\pi_f : f \in C_b(K, Z(E)_s)\} \subset C(K \odot H_I(E))$$

where $\pi_f(k, T) = T(f(k))$. $K \odot H_I(E)$ is a completely regular Hausdorff space. Define

$$\pi : C_b(K, Z(E)_s) \rightarrow C_b(K \odot H_I(E))$$

by $\pi(f)(k, T) = \pi_f(k)(T)$, where $C_b(K \odot H_I(E))$ is the set of all real valued bounded continuous functions on $K \odot H_I(E)$. It is clear that π is a Riesz homomorphism and $\pi(i) = \mathbf{1}$ where $i \in C_b(K, Z(E)_s)$, $i(k) = I$ for each $k \in K$. For each $f \in C_b(K, E)$ it is clear that $\|\pi(f)\| \leq \|f\|$. Suppose that $\|\pi(f)\| < \|f\|$ for some $f \in C_b(K, Z(E)_s)$. Then there exists $k_0 \in K$ with $|T(f(k))| < \|f(k_0)\|$ for all $k \in K$ and for all $T \in H_I(E)$. We can

choose a lattice homomorphism $T_0 \in H_I(K)$ with $T_0(f(k_0)) = \|f(k_0)\|$. This shows that π is an isometry so $\pi(C_b(K, Z(E)_s))$ is a closed subspace of $C_b(K \odot H_I(E))$. Let $(k_1, T_1) \neq (k_2, T_2)$ in $K \odot H_I(E)$. If $k_1 \neq k_2$ choose $f \in C(K)$ with $f(k_1) \neq f(k_2)$. Let $g(k) = f(k)I$ for each $k \in K$. Then $g \in C_b(K, Z(E)_s)$ and $\pi(g)(k_1, T_1) \neq \pi(g)(k_2, T_2)$. If $T_1 \neq T_2$ choose $f \in Z(E)$ with $T_1(f) \neq T_2(f)$. Let $h(k) = f$ for each $k \in K$, so $f \in C_b(K, Z(E)_s)$ and $T_1(h)(k_1, T_1) \neq T_2(h)(k_1, T_1)$. This shows that $\pi(C_b(K, Z(E)_s))$ separates the points $K \odot H_I(E)$. By the Stone-Weierstrass theorem π is isometry Riesz isomorphism onto $C_b(K \odot H_I(E))$. Since $Z(C(K, E))$ is Dedekind complete, $\beta(K \odot H_I(E))$ is a Stonean space. It is well known that if F is Dedekind complete Banach lattice then $Z(F)$ is isometric Riesz isomorphism to $C(K)$, where K is the Stone space of F (see [1] for a proof), so $\beta(K \odot H_I(E))$ is the Stone space of $C(K, E)$. Hence, $C^\infty(\beta(K \odot H_I(E)))$ is the universal completion of $C(K, E)$. $\square$

Let E be a Banach lattice and consider the Hull-Kernel topology of $Z(E)$. That is a compact Hausdorff topological space on $H_I(E)$ defined by

$$T_\alpha \rightarrow T \iff T_\alpha(f) \rightarrow T(f) \quad \text{for each } f \in Z(E).$$

For a compact Hausdorff space K let us denote the product topology of K and the Hull-Kernel topology $H_I(E)$ by $K \times_p H_I(E)$. The relation of $K \times_p H_I(E)$ and $K \odot H_I(E)$ as follows.

Theorem 2. *Let K be an infinite Stonean space and E be a Banach lattice with compact order intervals. Then the topological spaces $K \times_p H_I(E)$ and $K \odot H_I(E)$ coincide if and only if E is finite dimensional.*

Proof. Let $(k_\alpha, T_\alpha) \rightarrow (k, T)$ in $K \odot H_I(E)$. Let $f \in C(K)$ and $z \in Z(E)$ be given. Define $g(k) = f(k)I$ and $h(k) = z$ in $C_b(K, Z(E)_s)$. Then $f(k_\alpha) = T_\alpha(g(k_\alpha)) \rightarrow T(g(k)) = f(k)$. Since f is arbitrary $k_\alpha \rightarrow k$ in K . Similarly, $T_\alpha(z) = T_\alpha(h(k_\alpha)) \rightarrow T(h(k)) = T(z)$. This shows that $K \times_p H_I(E) \subset K \odot H_I(E)$. Suppose that $K \times_p H_I(E) = K \odot H_I(E)$. Then $K \times_p H_I(E)$ is Stonean so $H_I(E)$ must be finite. Since E is order dense in $C^\infty(H_I(E)) = C(H_I(E))$, E is finite dimensional. The other direction of the proof is clear since in this case $K \times_p H_I(E)$ is the Stone space of $C(K, E)$. $\square$

Remark. Let $C_0(K, E)$ be the space of all continuous functions f from a locally compact Hausdorff space into a Banach lattice E which are vanishing at infinity. It is easy to check that all results in above are also true for Dedekind complete Banach lattice $C_0(K, E)$ (see [2] for conditions under which this occurs). For example the Stone space of $C_0(K, E)$ is $\beta(K \odot H_I(E))$.

References

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