Q_K type spaces of analytic functions

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Abstract. For a nondecreasing function $K : [0, \infty) \rightarrow [0, \infty)$ and $0 < p < \infty$, $-2 < q < \infty$, we introduce $Q_K(p, q)$, a Q_K type space of functions analytic in the unit disk and study the characterizations of $Q_K(p, q)$. Necessary and sufficient conditions on K such that $Q_K(p, q)$ become some known spaces are given.

1. Introduction

The Green function in the unit disk Δ with singularity at $a \in \Delta$ is given by $g(z, a) = \log \frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = (a - z)/(1 - \bar{a}z)$ is a Möbius transformation of Δ . For $0 < r < 1$, let $\Delta(a, r) = \{z \in \Delta : |\varphi_a(z)| < r\}$ be the pseudo-hyperbolic disk with center $a \in \Delta$ and radius r .

Through this paper, we assume that $K : [0, \infty) \rightarrow [0, \infty)$ is a right-continuous and nondecreasing function. For $0 < p < \infty$, $-2 < q < \infty$, we

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say that a function f analytic in Δ belongs to the space $Q_K(p, q)$ if

$$(1.1) \quad \|f\|_{K,p,q}^p = \sup_{a \in \Delta} \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \infty,$$

where $dA(z)$ is the Euclidean area element on Δ .

It is clear that $Q_K(p, q)$ is a Banach space with the norm $\|f\| = |f(0)| + \|f\|_{K,p,q}$ when $p \geq 1$. If $q + 2 = p$, $Q_K(p, q)$ is Möbius-invariant, i.e., $\|f \circ \varphi_a\|_{K,p,q} = \|f\|_{K,p,q}$ for all $a \in \Delta$. Now we consider some special cases. If $p = 2, q = 0$, we have that $Q_K(p, q) = Q_K$ (cf. [2] and [4]). If $K(t) = t^s$, then $Q_K(p, q) = F(p, q, s)$ (cf. [7]). We know from [7] that $F(p, q, s)$ is contained in $\frac{q+2}{p}$ -Bloch space. Recall that the α -Bloch space $\mathcal{B}^\alpha$ ($0 < \alpha < \infty$) consists of analytic functions f on Δ such that

$$(1.2) \quad \|f\|_{\mathcal{B}^\alpha} = \sup\{(1 - |z|^2)^\alpha |f'(z)| : z \in \Delta\} < \infty.$$

In this paper we show some relations between $Q_K(p, q)$ and $\mathcal{B}^\alpha$ for a nondecreasing K , and give a general way to construct different spaces $Q_{K_1}(p, q)$ and $Q_{K_2}(p, q)$ by using some functions K_1 and K_2 . We assume throughout the paper that

$$(1.3) \quad \int_0^1 (1 - r^2)^q K\left(\log \frac{1}{r}\right) r dr < \infty.$$

Otherwise, $Q_K(p, q)$ is trivial, that is, $Q_K(p, q)$ contains constant functions only. In fact, if (1.3) does not hold and assume that f is a non-constant function. We show that f can not belong to $Q_K(p, q)$. By assumption there exists at least one point $a \in \Delta$ such that $f'(a) \neq 0$. By

$$(1 - |a|^2)^p |f'(a)|^p \leq \frac{1}{2\pi} \int_0^{2\pi} |(f \circ \varphi_a)'(re^{i\theta})|^p d\theta, \quad 0 < r < 1,$$

we conclude that

$$\begin{aligned} & \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q K(g(a, z)) dA(z) \\ &= \iint_{\Delta} |(f \circ \varphi_a)'(z)|^p |\varphi_a'(z)|^{-p} (1 - |\varphi_a(z)|^2)^{q+2} \\ & \quad \times (1 - |z|^2)^{-2} K\left(\log \frac{1}{|z|}\right) dA(z) \end{aligned}$$

$$\begin{aligned}
&\geq C(1 - |a|^2)^{|q+2-p|} \int_0^1 \left(\int_0^{2\pi} |(f \circ \varphi_a)'(re^{i\theta})|^p d\theta \right) \\
&\quad \times (1 - r^2)^q K \left(\log \frac{1}{r} \right) r dr \\
&\geq C(1 - |a|^2)^{|q+2-p|+p} |f'(a)|^p \int_0^1 (1 - r^2)^q K \left(\log \frac{1}{r} \right) r dr \\
&= \infty,
\end{aligned}$$

which means that $f \notin Q_K(p, q)$.

2. $Q_K(p, q)$ -spaces

Theorem 2.1. *Let $0 < p < \infty$, $-2 < q < \infty$. Then*

- (i) $Q_K(p, q) \subset \mathcal{B}^{\frac{q+2}{p}}$.
 - (ii) $Q_K(p, q) = \mathcal{B}^{\frac{q+2}{p}}$ if and only if
- $$(2.1) \quad \int_0^1 (1 - r^2)^{-2} K \left(\log \frac{1}{r} \right) r dr < \infty.$$

Proof. (i) For a fixed $r \in (0, 1)$, let

$$E(a, r) = \{z \in \Delta, |z - a| < r(1 - |a|)\}.$$

We know that $E(a, r) \subset \Delta(a, r)$ and for any $z \in E(a, r)$, we have

$$(1 - r)(1 - |a|) \leq 1 - |z| \leq (1 + r)(1 - |a|),$$

which means that $1 - |z|^2 \approx 1 - |a|^2$ for any $z \in E(a, r)$. Since $|f'(z)|^p$ is subharmonic, for $a \in \Delta$,

$$\begin{aligned}
&\iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\
&\geq K \left(\log \frac{1}{r} \right) \iint_{\Delta(a, r)} |f'(z)|^p (1 - |z|^2)^q dA(z) \\
&\geq K \left(\log \frac{1}{r} \right) \iint_{E(a, r)} |f'(z)|^p (1 - |z|^2)^q dA(z) \\
&\geq \pi r^2 K \left(\log \frac{1}{r} \right) (1 - |a|^2)^{q+2} |f'(a)|^p.
\end{aligned}$$

If $f \in Q_K(p, q)$, then by estimate above we have

$$\sup_{a \in \Delta} (1 - |a|^2)^{q+2} |f'(a)|^p < \infty.$$

The proof of (i) is complete.

(ii) Let (2.1) hold. To prove $Q_K(p, q) = \mathcal{B}^{\frac{q+2}{p}}$, we only need to show that $\mathcal{B}^{\frac{q+2}{p}} \subset Q_K(p, q)$. For $f \in \mathcal{B}^{\frac{q+2}{p}}$ we notice that

$$\begin{aligned} & \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ & \leq \|f\|_{\mathcal{B}^{\frac{q+2}{p}}}^p \iint_{\Delta} (1 - |z|^2)^{-2} K(g(z, a)) dA(z) \\ & = \|f\|_{\mathcal{B}^{\frac{q+2}{p}}}^p \iint_{\Delta} (1 - |z|^2)^{-2} K(g(z, 0)) dA(z) \\ & = 2\pi \|f\|_{\mathcal{B}^{\frac{q+2}{p}}}^p \int_0^1 (1 - r^2)^{-2} K\left(\log \frac{1}{r}\right) r dr \\ & < \infty, \end{aligned}$$

which shows that $\mathcal{B}^{\frac{q+2}{p}} \subset Q_K(p, q)$.

Now we assume $\mathcal{B}^{\frac{q+2}{p}} = Q_K(p, q)$ and we show that (2.1) holds. In fact, by [5] there exist two functions f_1 and f_2 in $\mathcal{B}^{\frac{q+2}{p}}$ such that

$$|f_1'(z)| + |f_2'(z)| \approx (1 - |z|^2)^{-\frac{q+2}{p}}.$$

Then $f_1, f_2 \in Q_K(p, q)$ and

$$\begin{aligned} & \infty > 2^p \sup_{a \in \Delta} \iint_{\Delta} (|f_1'(z)|^p + |f_2'(z)|^p) (1 - |z|^2)^q K(g(z, a)) dA(z) \\ & \geq \iint_{\Delta} (|f_1'(z)| + |f_2'(z)|)^p (1 - |z|^2)^q K(g(z, 0)) dA(z) \\ & \approx \iint_{\Delta} (1 - |z|^2)^{-2} K\left(\log \frac{1}{|z|}\right) dA(z) \\ & = 2\pi \int_0^1 (1 - r^2)^{-2} K\left(\log \frac{1}{r}\right) r dr. \end{aligned}$$

Thus (2.1) holds. We complete the proof. $\square$

Moreover, we say that $f \in Q_K(p, q)$ belongs to the space $Q_{K,0}(p, q)$ if

$$(2.2) \quad \lim_{|a| \rightarrow 1} \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) = 0.$$

As a subspace of $\mathcal{B}^\alpha$, the little α -Bloch space $\mathcal{B}_0^\alpha$ consists of analytic functions f on Δ such that

$$(2.3) \quad \lim_{|z| \rightarrow 1} (1 - |z|^2)^\alpha |f'(z)| = 0.$$

Thus we have

Theorem 2.2. *Let $0 < p < \infty$, $-2 < q < \infty$. Then*

- (i) $Q_{K,0}(p, q) \subset \mathcal{B}_0^{\frac{q+2}{p}}$.
- (ii) $Q_{K,0}(p, q) = \mathcal{B}_0^{\frac{q+2}{p}}$ if and only if (2.1) holds.

Proof. (i) Without loss of generality, we assume that $K(1) > 0$, from the proof of Theorem 2.1 we have

$$\begin{aligned} \pi(1/e)^2 K(1)(1 - |a|)^{q+2} |f'(a)|^p &\leq K(1) \iint_{E(a)} |f'(z)|^p (1 - |z|^2)^q dA(z) \\ &\leq K(1) \iint_{\Delta(a, 1/e)} |f'(z)|^p (1 - |z|^2)^q dA(z) \\ &\leq \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z), \end{aligned}$$

where $E(a) = \{z \in \Delta, |z - a| < \frac{1}{e}(1 - |a|)\}$. If $f \in Q_{K,0}(p, q)$, we have

$$\lim_{|a| \rightarrow 1} (1 - |a|^2)^{\frac{q+2}{p}} |f'(a)| = 0.$$

- (ii) We only need to prove that $\mathcal{B}_0^{\frac{q+2}{p}} \subset Q_{K,0}(p, q)$. Assume that

$$A = \int_0^1 (1 - r^2)^{-2} K\left(\log \frac{1}{r}\right) r dr < \infty.$$

For given $\epsilon > 0$ there exists an r_1 , $0 < r_1 < 1$, such that

$$(2.4) \quad \int_{r_1}^1 (1 - r^2)^{-2} K\left(\log \frac{1}{r}\right) r dr < \epsilon,$$

Then we have

$$\begin{aligned} &\iint_{\Delta \setminus \Delta(a, r_1)} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ &\leq \|f\|_{\mathcal{B}^{\frac{q+2}{p}}}^p \iint_{\Delta \setminus \Delta(a, r_1)} (1 - |z|^2)^{-2} K(g(z, a)) dA(z) \\ (2.5) \quad &= \|f\|_{\mathcal{B}^{\frac{q+2}{p}}}^p \iint_{r_1 < |w| < 1} (1 - |w|^2)^{-2} K\left(\log \frac{1}{|w|}\right) dA(w) \\ &= 2\pi \|f\|_{\mathcal{B}^{\frac{q+2}{p}}}^p \int_{r_1}^1 (1 - r^2)^{-2} K\left(\log \frac{1}{r}\right) r dr \\ &\leq 2\pi \epsilon \|f\|_{\mathcal{B}^{\frac{q+2}{p}}}^p. \end{aligned}$$

Similarly, if $f \in \mathcal{B}_0^{\frac{q+2}{p}}$, we have that $|f'(\varphi_a(w))|(1 - |\varphi_a(w)|^2)^{\frac{q+2}{p}} \rightarrow 0$ uniformly for $|w| \leq r$ if $|a| \rightarrow 1$, where r is fixed and $0 < r < 1$. Then

$$\begin{aligned}
 (2.6) \quad & \lim_{|a| \rightarrow 1} \iint_{\Delta(a, r_1)} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\
 &= \lim_{|a| \rightarrow 1} \iint_{|w| \leq r_1} |f'(\varphi_a(w))|^p (1 - |\varphi_a(w)|^2)^{q+2} (1 - |w|^2)^{-2} \\
 &\quad \times K\left(\log \frac{1}{|w|}\right) dA(w) \\
 &\leq A \lim_{|a| \rightarrow 1} \sup_{|w| \leq r_1} |f'(\varphi_a(w))|^p (1 - |\varphi_a(w)|^2)^{q+2} = 0.
 \end{aligned}$$

By (2.5) and (2.6) it is easy to see that

$$(2.7) \quad \lim_{|a| \rightarrow 1} \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) = 0.$$

Conversely, suppose that (2.1) does not hold; that is,

$$\int_0^1 (1 - r^2)^{-2} K\left(\log \frac{1}{r}\right) r dr = \infty.$$

Thus we find a continuous strictly decreasing function $h : [0, 1) \rightarrow [0, \infty)$ tending to zero at 1 such that

$$(2.8) \quad \int_0^1 h(r) (1 - r^2)^{-2} K\left(\log \frac{1}{r}\right) r dr = \infty.$$

It is easy to see that

$$(2.9) \quad r^{2^{k+1}-2} \geq \exp\{-2^{k+2}(1-r)\}, \quad r \in \left[\frac{1}{2}, 1\right).$$

We know that for $\alpha > 0$ $t^{2\alpha} \exp\{-4t\}$ is decreasing in $[\alpha/2, (\alpha+1)/2]$ and

$$\sup_{t>0} t^{2\alpha} \exp\{-4t\} = t^{2\alpha} \exp\{-4t\}|_{t=\frac{\alpha}{2}} = \left(\frac{\alpha}{2}\right)^{2\alpha} e^{-2\alpha}.$$

There exists an integer k for $3/4 \leq r < 1$ such that $\frac{\alpha}{2} \leq 2^k(1-r) < \frac{\alpha+1}{2}$ and

$$\begin{aligned}
 & 2^{2\alpha k} \exp\{-2^{k+2}(1-r)\} \\
 (2.10) \quad & = (1-r)^{-2\alpha} [2^k(1-r)]^{2\alpha} \exp\{-4 \cdot 2^k(1-r)\} \\
 & \geq \left(\frac{\alpha+1}{2}\right)^{2\alpha} e^{-2(\alpha+1)} (1-r)^{-2\alpha}.
 \end{aligned}$$

For $3/4 \leq r < 1$ and k above we define

$$f_0(z) = \sum_{j=0}^{\infty} a_j 2^{\frac{j(q+2-p)}{p}} z^{2^j},$$

where $a_j = (h(1 - \frac{p+q+2}{p} 2^{-j-1}))^{1/p}$, $j = 0, 1, 2, \dots$. By (2.9) and (2.10) we conclude that

$$\begin{aligned}
 (2.11) \quad M_2^2(r, f'_0) &= \int_0^{2\pi} |f'_0(re^{i\theta})|^2 d\theta = 2\pi \sum_{k=1}^{\infty} a_k^2 2^{\frac{2k(q+2)}{p}} r^{2^{k+1}-2} \\
 &\geq 2\pi h^{\frac{2}{p}}(r) 2^{\frac{2k(q+2)}{p}} \exp\{-2^{k+2}(1-r)\} \\
 &\geq C h^{\frac{2}{p}}(r) (1-r)^{\frac{-2(q+2)}{p}}.
 \end{aligned}$$

Since f_0 is defined by a gap series with Hadamard condition, we have

$$M_2(r, f'_0) \approx M_p(r, f'_0),$$

where

$$M_p(r, f'_0) = \left(\int_0^{2\pi} |f'_0(re^{i\theta})|^p d\theta \right)^{1/p}.$$

Therefore,

$$\begin{aligned}
 & \sup_{a \in \Delta} \iint_{\Delta} |f'_0(z)|^p (1-|z|^2)^q K(g(z, a)) dA(z) \\
 & \geq \int_0^1 M_p^p(r, f'_0) (1-r^2)^q K(\log 1/r) r dr \\
 & \approx \int_0^1 M_2^p(r, f'_0) (1-r^2)^q K(\log 1/r) r dr \\
 & \geq C \int_{\frac{3}{4}}^1 h(r) (1-r^2)^{-2} K\left(\log \frac{1}{r}\right) r dr \\
 & = \infty.
 \end{aligned}$$

It means that $f_0 \in \mathcal{B}_0^{\frac{q+2}{p}} \setminus Q_{K_0}(p, q)$, which is a contradiction. Hence (2.1) holds. $\square$

3. More results on $Q_K(p, q)$ -spaces

The following result means that the kernel functions K can be chosen as bounded.

Theorem 3.1. *Assume that $K(1) > 0$. Let $K_1(r) = \inf(K(r), K(1))$, then $Q_K(p, q) = Q_{K_1}(p, q)$.*

Proof. Since $K_1 \leq K$ and K_1 is nondecreasing, it is clear that $Q_K(p, q) \subset Q_{K_1}(p, q)$. It remains to prove that $Q_{K_1}(p, q) \subset Q_K(p, q)$. We note that

$$\begin{aligned} g(z, a) &> 1, \quad z \in \Delta(a, 1/e), \\ g(z, a) &\leq 1, \quad z \in \Delta \setminus \Delta(a, 1/e). \end{aligned}$$

Thus $K(g(z, a)) = K_1(g(z, a))$ in $\Delta \setminus \Delta(a, 1/e)$. It suffices to deal with integrals over $\Delta(a, 1/e)$. If $f \in Q_{K_1}(p, q)$, f is a $\frac{q+2}{p}$ -Bloch function by Theorem 2.1 and it follows that

$$\begin{aligned} &\iint_{\Delta(a, 1/e)} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ &\leq \|f\|_{\mathcal{B}^{\frac{q+2}{p}}}^p \iint_{\Delta(a, 1/e)} (1 - |z|^2)^{-2} K(g(z, a)) dA(z) \\ &= \|f\|_{\mathcal{B}^{\frac{q+2}{p}}}^p \iint_{\Delta(0, 1/e)} (1 - |w|^2)^{-2} K(\log \frac{1}{|w|}) dA(w) \leq C \|f\|_{\mathcal{B}^{\frac{q+2}{p}}}^p. \end{aligned}$$

Thus, $f \in Q_K(p, q)$ and Theorem 3.1 is proved. $\square$

Corollary 3.2. *Let $0 < p < \infty$, $-2 < q < \infty$. Then $f \in Q_K(p, q)$ if and only if*

$$(3.1) \quad \sup_{a \in \Delta} \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q K(1 - |\varphi_a(z)|^2) dA(z) < \infty.$$

Theorem 3.3. *Let $0 < p < \infty$, $-2 < q < \infty$. Assume $K_1(r) \leq K_2(r)$ for $r \in (0, 1)$ and $\frac{K_1(r)}{K_2(r)} \rightarrow 0$ as $r \rightarrow 0$. If (2.1) is divergent for K_2 , then $Q_{K_2}(p, q) \subsetneq Q_{K_1}(p, q)$.*

To prove Theorem 3.3 we need the following lemma, which is also an characterization of α -Bloch function.

Lemma 3.4. *Let $0 < p < \infty$, $-2 < q < \infty$. Then*

(i) $f \in \mathcal{B}^{\frac{q+2}{p}}$ if and only if there exists $\rho \in (0, 1)$ such that

$$(3.2) \quad \sup_{a \in \Delta} \iint_{\Delta(a, \rho)} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \infty;$$

(ii) $f \in \mathcal{B}_0^{\frac{q+2}{p}}$ if and only if there exists $\rho \in (0, 1)$ such that

$$(3.3) \quad \lim_{|a| \rightarrow 1} \iint_{\Delta(a, \rho)} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) = 0.$$

Proof. (i) Assume $f \in \mathcal{B}^{\frac{q+2}{p}}$. For any $\rho \in (0, 1)$ and $a \in \Delta$, we have

$$\begin{aligned} & \iint_{\Delta(a, \rho)} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ &= \iint_{\Delta(0, \rho)} |f'(\varphi_a(z))|^p (1 - |\varphi_a(z)|^2)^{q+2} (1 - |z|^2)^{-2} \\ & \quad \times K\left(\log \frac{1}{|z|}\right) dA(z) \\ &\leq \|f\|_{\mathcal{B}^{\frac{q+2}{p}}}^p \iint_{\Delta(0, \rho)} (1 - |z|^2)^{-2} K\left(\log \frac{1}{|z|}\right) dA(z) \\ &\leq C \|f\|_{\mathcal{B}^{\frac{q+2}{p}}}^p. \end{aligned}$$

Conversely, suppose that (3.2) holds for some $\rho, 0 < \rho < 1$. By the proof of Theorem 2.1 (i) we have

$$\begin{aligned} & \iint_{\Delta(a, \rho)} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ &\geq K\left(\log \frac{1}{\rho}\right) \iint_{\Delta(a, \rho)} |f'(z)|^p (1 - |z|^2)^q dA(z) \\ &\geq K\left(\log \frac{1}{\rho}\right) \iint_{E(a, \rho)} |f'(z)|^p (1 - |z|^2)^q dA(z) \\ &\geq \rho^2 K\left(\log \frac{1}{\rho}\right) (1 - |a|^2)^{q+2} |f'(a)|^p, \end{aligned}$$

which shows that $f \in \mathcal{B}^{\frac{q+2}{p}}$.

(ii) We omit the proof since its proof is similar to that of (i) above. $\square$

Proof of Theorem 3.3. It is clear that $Q_{K_2}(p, q) \subset Q_{K_1}(p, q)$. Suppose that $Q_{K_2}(p, q) = Q_{K_1}(p, q)$. By the Open Mapping Theorem we know that the identity map from one of these spaces into the other one is continuous. Thus there exists a constant C such that $\|\cdot\|_{K_2, p, q} \leq C\|\cdot\|_{K_1, p, q}$. Since $\frac{K_1(r)}{K_2(r)} \rightarrow 0$ as $r \rightarrow 0$, then there exists $r_0 \in (0, 1)$ such that $K_1(r) \leq (2C)^{-1}K_2(r)$ for $0 < r \leq r_0$. Choose $t_0 = e^{-r_0}$ and deduce that if $f \in Q_{K_2}(p, q)$, then

$$\begin{aligned} & \sup_{a \in \Delta} \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q K_2(g(z, a)) dA(z) \\ & \leq C \sup_{a \in \Delta} \iint_{\Delta(a, t_0)} |f'(z)|^p (1 - |z|^2)^q K_1(g(z, a)) dA(z) \\ & \quad + \frac{1}{2} \sup_{a \in \Delta} \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q K_2(g(z, a)) dA(z). \end{aligned}$$

Therefore,

$$\begin{aligned} & \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q K_2(g(z, a)) dA(z) \\ & \leq 2C \sup_{a \in \Delta} \iint_{\Delta(a, t_0)} |f'(z)|^p (1 - |z|^2)^q K_1(g(z, a)) dA(z). \end{aligned}$$

By Lemma 3.4 there exists a constant C' such that for $f \in Q_{K_2}(p, q)$,

$$(3.4) \quad \sup_{a \in \Delta} \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q K_2(g(z, a)) dA(z) \leq C' \|f\|_{\mathcal{B}^{\frac{q+2}{p}}}^p.$$

If $h \in \mathcal{B}^{\frac{q+2}{p}}$ and $h_r(z) = h(rz)$, $0 < r < 1$, then $\|h_r\|_{\mathcal{B}^{\frac{q+2}{p}}} \leq \|h\|_{\mathcal{B}^{\frac{q+2}{p}}}$. Since $h_r \in Q_{K_2}(p, q)$, $0 < r < 1$, we can choose $f = h_r$ in the inequality (3.4). Using Fatou's lemma, we deduce that

$$\sup_{a \in \Delta} \iint_{\Delta} |h'(z)|^p (1 - |z|^2)^q K_2(g(z, a)) dA(z) \leq C' \|h\|_{\mathcal{B}^{\frac{q+2}{p}}}^p.$$

We have proved that $h \in Q_{K_2}(p, q)$. It means that $Q_{K_2}(p, q) = \mathcal{B}^{\frac{q+2}{p}}$. It follows from Theorem 2.1 that the integral (2.1) with $K = K_2$ must be convergent, a contradiction. We obtain that $Q_{K_2}(p, q) \subsetneq Q_{K_1}(p, q)$. We finish the proof of Theorem 3.3. $\square$

Corollary 3.5. *Let $0 < p < \infty$ and $-2 < q < \infty$. Then $F(p, q, 0) \subseteq Q_K(p, q)$ and the equality holds if and only if $K(0) > 0$.*

Proof. By Theorem 3.1 the kernel function K can be viewed as bounded, which means that $F(p, q, 0) \subseteq Q_K(p, q)$. If $K(0) > 0$, $F(p, q, 0) = Q_K(p, q)$ holds immediately by the following estimates.

$$\begin{aligned} K(0) \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q dA(z) \\ \leq \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ \leq K(\infty) \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q dA(z). \end{aligned}$$

Conversely, if $K(0) = 0$, we assume that $K(r) \leq 1$ without loss of generality. We choose $K_1(r) = K(r)$ and $K_2(r) = K(1)$ and apply Theorem 3.3 to conclude that $F(p, q, 0) \subsetneq Q_K(p, q)$. Thus the proof of Corollary 3.4 is complete. $\square$

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