Littlewood-Paley characterization for Campanato spaces

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Abstract. The Littlewood-Paley characterization for the local approximation Campanato spaces L_p^α is well known in the cases $\alpha \geq 0$ and $\alpha = -\frac{n}{p}$. We give in this paper a characterization of such a type for L_2^α spaces (and for Morrey-Campanato spaces $\mathcal{L}^{2,\lambda}$) for any $\alpha \geq -\frac{n}{2}$. These spaces contain as spacial cases the well known spaces BMO of John and Nirenberg and its local version bmo .

1. Introduction and results

In this paper, we make use of a partition of Littlewood-Paley type to get a dyadic characterization for the spaces BMO , bmo and more generally for Campanato spaces $\mathcal{L}^{2,\lambda}$ modulo polynomials and their local versions L_2^α . In this direction we mention the following classical characterization of BMO and bmo with the aid of Triebel-Lizorkin spaces $F_{p,q}^s(\mathbf{R}^n)$: $BMO = \dot{F}_{\infty,2}^0$, $bmo = F_{\infty,2}^0$ and $I^s(BMO) = \dot{F}_{\infty,2}^s$, where $I^s = \mathcal{F}^{-1}(|\cdot|^{-s}\mathcal{F})$ is the Riesz

potential operator, cf. [15, chapters 2 and 5]. The spaces $I^s(BMO)$ were studied by Strichartz [14].

If we denote L_p^α the local approximation Campanato spaces defined for instance in the book [16, Definition 1.7.2 (5)] for $\alpha \geq -\frac{n}{p}$ and $1 \leq p < +\infty$, then we recall that $L_p^\alpha = \mathcal{C}^\alpha$ for any $\alpha > 0$, $L_p^{-n/p} = L^p$ and $L_p^0 = bmo$, cf. [4], [12], [16] and [17] for the proof and more references. In the case $-\frac{n}{p} < \alpha < 0$, the spaces L_p^α coincide with Morrey spaces (introduced to complement the scale of L^p spaces) which are themselves equal to Campanato spaces $\mathcal{L}^{p,\lambda}$, $\lambda = n + \alpha p$ considered in this work, cf. Definition 5. The main result of this paper is the dyadic characterization for L_2^α spaces and $I^s(L_2^\alpha)$ which is of interest essentially in the case $-\frac{n}{2} < \alpha < 0$. By the way, we mention that this characterization is not true for the general spaces L_p^α , $1 \leq p < +\infty$, $p \neq 2$, cf. Remark 11.

We start by recalling the definition of the spaces $\mathcal{L}^{p,\lambda,s}(\mathbf{R}^n)$ and their homogeneous counterparts $\dot{\mathcal{L}}^{p,\lambda,s}(\mathbf{R}^n)$ which involve expressions with balls (or cubes) of $\mathbf{R}^n$ and a resolution of Littlewood-Paley. These spaces were introduced in the previous papers [9] and [5] in order to solve a conjecture of Hans Triebel regarding an isomorphism theorem for elliptic operators in BMO and bmo spaces, cf. [15, section 4.3.4].

Next, we show that for any $s \in \mathbf{R}$ and $0 \leq \lambda < n + 2$, the space $\dot{\mathcal{L}}^{2,\lambda,s}$ coincides with the Riesz-potential space $I^s(\mathcal{L}^{2,\lambda})$ built on Campanato space $\mathcal{L}^{2,\lambda}$ modulo polynomials (Theorem 10). In particular, for $s = 0$ we get the dyadic characterization for Campanato spaces $\mathcal{L}^{2,\lambda}$ modulo polynomials and their local versions L_2^α , cf. Proposition 8.

To prove these results we follow up the characterization of BMO given in [1].

To make this paper self contained we recall the definition of the spaces introduced in previous papers. For this we need a Littlewood-Paley partition of unity. We denote $x \in \mathbf{R}^n$ and ξ its dual variable. Let $\varphi \in C_0^\infty(\mathbf{R}^n)$, $\varphi \geq 0$ and φ equal to 1 for $|\xi| \leq 1$, 0 for $|\xi| \geq 2$; and set $\theta(\xi) = \varphi(\xi) - \varphi(2\xi)$, ($\text{supp } \theta \subset \{\frac{1}{2} \leq |\xi| \leq 2\}$). For $j \in \mathbf{Z}$ we set

$$\dot{\Delta}_j u = \theta(2^{-j} D_x) u$$

and denoting $\psi_j(\xi) = \sum_{k \leq j} \theta(2^{-k} \xi)$ ($= \varphi(2^{-j} \xi)$ for $\xi \neq 0$) we set

$$\dot{S}_j u = \psi_j(D_x) u$$

If $j \geq 1$ we set also

$$\Delta_j u = \dot{\Delta}_j u, \quad S_j u = \varphi(2^{-j} D_x) u$$

and

$$S_0 u = \Delta_0 u = \varphi(D_x) u$$

Remark 1. We have $\varphi(\xi) + \sum_{k \geq 1} \theta(2^{-k} \xi) = 1$ for any $\xi \in \mathbf{R}^n$, thus the nonhomogeneous partition of $u \in \mathcal{S}'(\mathbf{R}^n)$ is given by the formula

$$u = \sum_{k \geq 0} \Delta_k u = \Delta_0 u + \sum_{k \geq 1} \Delta_k u = S_j u + \sum_{k \geq j+1} \Delta_k u \text{ for } j \in \mathbf{N}.$$

Set $f(\xi) = \sum_{k \in \mathbf{Z}} \theta(2^{-k} \xi)$. For each fixed $\xi \neq 0$, $f(\xi)$ contains at most two non-vanishing terms, we deduce $f(\xi) = 1$ for any $|\xi| \geq 2$. We note that $f(2^{-j} \xi) = f(\xi)$ for any $j \in \mathbf{Z}$, and choosing j larger and larger, we obtain $f(\xi) = 1$ for any $\xi \neq 0$. Then, for $u \in \mathcal{S}'(\mathbf{R}^n)$, with $0 \notin \text{supp } \mathcal{F}u$, the homogeneous partition of u is given by the formula

$$u = \sum_{k \in \mathbf{Z}} \dot{\Delta}_k u = \dot{S}_j u + \sum_{k \geq j+1} \dot{\Delta}_k u \text{ for } j \in \mathbf{Z}.$$

Now we recall the definition of the nonhomogeneous spaces introduced in [5] and [6].

Definition 2. Let $s \in \mathbf{R}$, $\lambda \geq 0$ and $1 \leq p < +\infty$. The space $\mathcal{L}^{p,\lambda,s}(\mathbf{R}^n)$ denotes the set of all tempered distributions $u \in \mathcal{S}'(\mathbf{R}^n)$ such that

$$(1) \quad \|u\|_{\mathcal{L}^{p,\lambda,s}(\mathbf{R}^n)} = \left\{ \sup_B \frac{1}{|B|^{\frac{\lambda}{n}}} \sum_{j \geq J^+} 2^{jps} \|\Delta_j u\|_{L^p(B)}^p \right\}^{\frac{1}{p}} < +\infty$$

where $J^+ = \max(J, 0)$, $|B|$ is the measure of B and the supremum is taken over all $J \in \mathbf{Z}$ and all balls B of $\mathbf{R}^n$ of radius 2^{-J} .

The space $\mathcal{L}^{p,\lambda,s}(\mathbf{R}^n)$ equipped with the norm (1) is a Banach space.

To give the homogeneous counterpart of the spaces $\mathcal{L}^{p,\lambda,s}(\mathbf{R}^n)$, we recall the notation of [15, chapter 5]. Let

$$Z(\mathbf{R}^n) = \{\varphi \in \mathcal{S}(\mathbf{R}^n); (D^\alpha \mathcal{F}\varphi)(0) = 0 \text{ for every multi-index } \alpha\}$$

$Z(\mathbf{R}^n)$ is considered as a subspace of $\mathcal{S}(\mathbf{R}^n)$ with the same topology, and $Z'(\mathbf{R}^n)$ is the topological dual of $Z(\mathbf{R}^n)$. We may identify $Z'(\mathbf{R}^n)$ and

$\mathcal{S}'(\mathbf{R}^n)/\mathcal{P}$, where $\mathcal{P}$ is the set of all polynomials of $\mathbf{R}^n$ with complex coefficients, ie. $Z'(\mathbf{R}^n)$ is interpreted as $\mathcal{S}'(\mathbf{R}^n)$ modulo polynomials.

Definition 3. Let $s \in \mathbf{R}$, $\lambda \geq 0$ and $1 \leq p < +\infty$. The dotted space $\dot{\mathcal{L}}^{p,\lambda,s}(\mathbf{R}^n)$ denotes the set of all $u \in Z'(\mathbf{R}^n)$ such that

$$(2) \quad \|u\|_{\dot{\mathcal{L}}^{p,\lambda,s}(\mathbf{R}^n)} = \left\{ \sup_B \frac{1}{|B|^{\frac{\lambda}{n}}} \sum_{j \geq J} 2^{jps} \left\| \dot{\Delta}_j u \right\|_{L^p(B)}^p \right\}^{\frac{1}{p}} < +\infty$$

where the supremum is taken over all $J \in \mathbf{Z}$ and all balls B of $\mathbf{R}^n$ of radius 2^{-J} .

If P is a polynomial of $\mathcal{P}$ and $u \in \mathcal{S}'(\mathbf{R}^n)$, it follows immediatly that

$$\|u + P\|_{\dot{\mathcal{L}}^{p,\lambda,s}(\mathbf{R}^n)} = \|u\|_{\dot{\mathcal{L}}^{p,\lambda,s}(\mathbf{R}^n)}$$

This shows that the norm (2) is well defined. Further, the space $\dot{\mathcal{L}}^{p,\lambda,s}(\mathbf{R}^n)$ equipped with this norm is a Banach space.

Remark 4. The supremum in expressions (1) and (2) can be taken over all $J \in \mathbf{Z}$ and all cubes B of $\mathbf{R}^n$ of length side 2^{-J} .

We give in [5], [6] and [7] some properties of these spaces.

In order to compare $\dot{\mathcal{L}}^{p,\lambda,0}(\mathbf{R}^n)$ with Campanato space $\mathcal{L}^{p,\lambda}(\mathbf{R}^n)$ we must interpret the latter as a subspace of $Z'(\mathbf{R}^n)$.

Definition 5 (Campanato spaces). Let $\lambda \geq 0$ and $1 \leq p < +\infty$.

- (i) The space $\mathcal{L}^{p,\lambda}(\mathbf{R}^n)$ denotes the set of all functions $u \in L_{loc}^p(\mathbf{R}^n)$ such that

$$\|u\|_{\mathcal{L}^{p,\lambda}(\mathbf{R}^n)} = \left\{ \sup_B \frac{1}{|B|^{\frac{\lambda}{n}}} \int_B |u - m_B u|^p dx \right\}^{\frac{1}{p}} < +\infty$$

where $m_B u = \frac{1}{|B|} \int_B u(y) dy$ is the mean value of u and the supremum is taken over all balls B of $\mathbf{R}^n$.

We note that $\mathcal{L}^{p,\lambda}(\mathbf{R}^n)$ is a Banach space modulo constants equal to $\{0\}$ if $\lambda > n + p$.

- (ii) For $0 \leq \lambda < n + p$ we define the space $\dot{\mathcal{L}}^{p,\lambda}(\mathbf{R}^n)$ as the set of all equivalence classes U modulo $\mathcal{P}$ of elements of $\mathcal{L}^{p,\lambda}(\mathbf{R}^n)$, equipped with the norm $\|U\|_{\dot{\mathcal{L}}^{p,\lambda}(\mathbf{R}^n)} = \|u\|_{\mathcal{L}^{p,\lambda}(\mathbf{R}^n)}$ where u is the unique (modulo constants) element of $\mathcal{L}^{p,\lambda}(\mathbf{R}^n)$ belonging to the class U . Throughtout this paper we identify U with u .

We denote $BMO = \dot{\mathcal{L}}^{1,n}(\mathbf{R}^n)$.

Now we define the local approximation Campanato spaces $\mathcal{L}_p^\alpha(\mathbf{R}^n)$ which are connected to $\mathcal{L}^{p,\lambda}(\mathbf{R}^n)$. We follow Wallin [17] and Triebel [16, Definition 1.7.2 (5)].

Definition 6 (Local approximation spaces). Let $1 \leq p < +\infty$, $\alpha \geq -\frac{n}{p}$ and $N = \max(-1, [\alpha])$, $[\alpha]$ is the integer part of α . We say $u \in \mathcal{L}_p^\alpha(\mathbf{R}^n)$ if and only if $u \in L_{loc}^p(\mathbf{R}^n)$ and for some constant $M = M(u)$, for every cube B of length side δ , there exists a polynomial P_B of degree $\leq N$ ($P_B = 0$ if $N = -1$) such that

$$(3) \quad \left\{ \frac{1}{|B|^{1+\alpha p/n}} \int_B |u(x) - P_B|^p dx \right\}^{\frac{1}{p}} \leq M \quad \text{if } 0 < \delta \leq 1$$

and

$$(4) \quad \left\{ \frac{1}{|B|} \int_B |u(x)|^p dx \right\}^{\frac{1}{p}} \leq M \quad \text{if } \delta = 1$$

We denote by $\|u\|_{\mathcal{L}_p^\alpha}$ the infimum of the constants M as above.

We observe that (4) implies (4) for $\delta > 1$ which implies (3) for $\delta > 1$ with $P_B = 0$.

We denote $bmo = \mathcal{L}_1^0(\mathbf{R}^n)$ and $l^{p,\lambda}(\mathbf{R}^n) = \mathcal{L}_p^{\frac{\lambda-n}{p}}(\mathbf{R}^n)$.

Remark 7. (i) The space $\mathcal{L}^{p,\lambda}(\mathbf{R}^n)$, $\lambda = n + \alpha p$, coincides with the space defined by requiring (3) for $\delta > 0$ and omitting (4), cf. Lemma 17.

(ii) In the case $0 \leq \lambda < n$ the space $l^{p,\lambda}(\mathbf{R}^n) (= \mathcal{L}_p^{\frac{\lambda-n}{p}}(\mathbf{R}^n))$ coincides with Morrey space $L^{p,\lambda}(\mathbf{R}^n)$ (itself equal to $\mathcal{L}^{p,\lambda}(\mathbf{R}^n)$) defined as the set of all $u \in L_{loc}^p(\mathbf{R}^n)$ such that

$$\|u\|_{L^{p,\lambda}(\mathbf{R}^n)} = \left\{ \sup_B \frac{1}{|B|^{\frac{\lambda}{n}}} \int_B |u(x)|^p dx \right\}^{\frac{1}{p}} < +\infty$$

the sup being over all balls of $\mathbf{R}^n$.

The following proposition gives the dyadic characterization of $\dot{\mathcal{L}}^{2,\lambda}(\mathbf{R}^n)$ and $l^{2,\lambda}(\mathbf{R}^n)$.

Proposition 8. Let $0 \leq \lambda < n + 2$.

(i) The space $\dot{\mathcal{L}}^{2,\lambda,0}(\mathbf{R}^n)$ coincides algebraically and topologically with Campanato space $\dot{\mathcal{L}}^{2,\lambda}(\mathbf{R}^n)$. In particular, $\dot{\mathcal{L}}^{2,n,0}(\mathbf{R}^n) \equiv BMO$.

- (ii) The space $\mathcal{L}^{2,\lambda,0}(\mathbf{R}^n)$ coincides algebraically and topologically with the space $l^{2,\lambda}(\mathbf{R}^n)$. In particular, $\mathcal{L}^{2,n,0}(\mathbf{R}^n) \equiv bmo$.

This proposition allow us to deduce the link between the discrete scale built on the space $\dot{\mathcal{L}}^{2,\lambda}(\mathbf{R}^n)$ and the continuous one:

Corollary 9. Let $0 \leq \lambda < n + 2$ and $m \in \mathbf{N}$.

- (i) $\dot{\mathcal{L}}^{2,\lambda,m}(\mathbf{R}^n) \hookrightarrow \dot{\mathcal{H}}^{2,\lambda,m} := \{u \in \dot{\mathcal{L}}^{2,\lambda}(\mathbf{R}^n); D^\alpha u \in \dot{\mathcal{L}}^{2,\lambda}(\mathbf{R}^n), |\alpha| \leq m\}$.
(ii) $\dot{\mathcal{H}}^{2,\lambda,m} \cap \dot{H}^{m+\lambda/2}(\mathbf{R}^n) \hookrightarrow \dot{\mathcal{L}}^{2,\lambda,m}(\mathbf{R}^n)$, here $\dot{H}^{m+\lambda/2}(\mathbf{R}^n)$ is the classical homogeneous Sobolev space. Therefore,

$$\dot{\mathcal{H}}^{2,\lambda,m} \cap \dot{H}^{m+\lambda/2}(\mathbf{R}^n) \equiv \dot{\mathcal{L}}^{2,\lambda,m}(\mathbf{R}^n) \cap \dot{H}^{m+\lambda/2}(\mathbf{R}^n)$$

To state a more general result, we introduce the Riesz potential operator

$$I^s f = \mathcal{F}^{-1}\{|\cdot|^{-s} \mathcal{F}f\}, \quad f \in Z'(\mathbf{R}^n) \text{ and } s \in \mathbf{R}$$

Theorem 10. Let $s \in \mathbf{R}$ and $0 \leq \lambda < n + 2$.

- (i) The space $\dot{\mathcal{L}}^{2,\lambda,s}(\mathbf{R}^n)$ coincides algebraically and topologically with the space $I^s(\dot{\mathcal{L}}^{2,\lambda}(\mathbf{R}^n))$ image of $\dot{\mathcal{L}}^{2,\lambda}(\mathbf{R}^n)$ under I^s .
(ii) The space $\mathcal{L}^{2,\lambda,s}(\mathbf{R}^n)$ coincides algebraically and topologically with the space $I^s(l^{2,\lambda}(\mathbf{R}^n))$.

Remark 11. We mention that these results are not true in general for $p \neq 2$. G. Bourdaud has noted that $\dot{\mathcal{L}}^{p,0,0} = \dot{B}_{p,p}^0$ for any $1 \leq p < +\infty$, and it is classical that $\dot{\mathcal{L}}^{p,0} = L^p$, therefore the space $\dot{\mathcal{L}}^{p,\lambda,0}$ does not coincide in general with Campanato space $\dot{\mathcal{L}}^{p,\lambda}$. Nevertheless, we give some injections between $\dot{\mathcal{L}}^{p,\lambda}$ and $\dot{\mathcal{L}}^{p,\lambda,0}$ in Propositions 21, 22 and 26.

We start with some helpful lemmas needed in the further considerations.

2. Some preliminary lemmas

Throughtout this paper, $C, C_M \dots$ denote positive constants whose values may change from line to line. First of all, we recall the following lemmas proved in [6].

Lemma 12. Let $s \in \mathbf{R}$, $\lambda \geq 0$ and $1 \leq q \leq p < +\infty$. We have the following continuous embedding

$$F_{p,q}^{s+\frac{n}{p}}(\mathbf{R}^n) \hookrightarrow \mathcal{L}^{p,\lambda,s-\frac{\lambda-n}{p}}(\mathbf{R}^n)$$

This embedding remains true for the dotted spaces $\dot{F}$ and $\dot{\mathcal{L}}$.

Lemma 13. For any $u \in \mathcal{S}'(\mathbf{R}^n)$ we have the decomposition

$$u = \sum_{j \in \mathbf{Z}} \dot{\Delta}_j u \quad \text{in } Z'(\mathbf{R}^n).$$

Lemma 14. Let $1 \leq p < +\infty$, and A a real < 0 . If $(a_{j\nu})_{j,\nu}$ is a sequence of positive real numbers satisfying $(a_{j\nu})_j \in l^p$ for any $\nu \geq 1$, then there exists a constant $C > 0$ such that

$$\sum_{j \geq 1} \left(\sum_{\nu \geq 1} 2^{\nu A} a_{j\nu} \right)^p \leq C \sup_{\nu \geq 1} \sum_{j \geq 1} a_{j\nu}^p$$

Lemma 15. Let $1 \leq p < +\infty$, and $s \in \mathbf{R}$. Further, let λ and $\alpha \geq 0$. Then we have the following continuous embedding

$$\mathcal{L}^{p,\lambda,s+\frac{\alpha}{p}}(\mathbf{R}^n) \hookrightarrow \mathcal{L}^{p,\lambda+\alpha,s}(\mathbf{R}^n)$$

We have the same result for the dotted spaces $\dot{\mathcal{L}}$.

The following lemma is proved in [9] and [8].

Lemma 16. For any integer $M > 0$, there exists a constant $C_M > 0$ such that for any $J \in \mathbf{Z}, x_0 \in \mathbf{R}^n$, for any ball B centered at x_0 with radius 2^{-J} , for any $l \in \mathbf{Z}, k \in \mathbf{N}$ and $u \in W_{loc}^{k,p}(\mathbf{R}^n) \cap L^p(\mathbf{R}^n)$, $1 \leq p \leq +\infty$

$$(5) \quad \left\| \dot{\Delta}_l u \right\|_{L^p(B)} \leq C_k 2^{kJ} 2^{-kl} \sup_{|\alpha| \leq k} \|D_x^\alpha u\|_{L^p(3B)} \\ + C_M \sum_{\nu \geq -J+1} 2^{-(\nu+l)M} \|u\|_{L^p(F_\nu)}$$

and

$$(6) \quad \left\| \dot{\Delta}_l u \right\|_{L^p(B)} \leq C_k 2^{-kl} \sup_{|\alpha| \leq k} \|D_x^\alpha u\|_{L^p(3B)} + C_M 2^{-(l-J)M} \|u\|_{L^p(\mathbf{R}^n)}$$

hold with $F_\nu = \{x \in \mathbf{R}^n : 2^\nu \leq |x - x_0| \leq 2^{\nu+1}\}$.

Lemma 17. Let $\lambda \geq 0$ and $1 \leq p < +\infty$. Let $u \in L_{loc}^p(\mathbf{R}^n)$. If for each ball B of $\mathbf{R}^n$, there exists a real c_B such that

$$\sup_B \frac{1}{|B|^{\frac{\lambda}{n}}} \int_B |u - c_B|^p dx = C < +\infty$$

then $u \in \mathcal{L}^{p,\lambda}(\mathbf{R}^n)$ and $\|u\| \leq 2C^{\frac{1}{p}}$.

Proof. We write

$$\left(\int_B |u - m_B u|^p dx \right)^{\frac{1}{p}} \leq \left(\int_B |u - c_B|^p dx \right)^{\frac{1}{p}} + \left(\int_B |c_B - m_B u|^p dx \right)^{\frac{1}{p}}$$

Now

$$\begin{aligned} |c_B - m_B u| &\leq \frac{1}{|B|} \int_B |u - c_B| dx \\ &\leq \left(\frac{1}{|B|} \int_B |u - c_B|^p dx \right)^{\frac{1}{p}} \\ &\leq \left(C |B|^{\frac{\lambda}{n}-1} \right)^{\frac{1}{p}}. \end{aligned} \quad \square$$

Lemma 18. Let $1 \leq p < +\infty$, $0 \leq \lambda < n+p$, and $\alpha = \begin{cases} 0 & \text{if } 0 \leq \lambda \leq n \\ 1 & \text{if } \lambda > n \end{cases}$.

There exists $C_n > 0$ such that

$$m_{2^j B} |u| \leq |m_B u| + C_n (j+1) |2^{\alpha j} B|^{\frac{1}{p}(\frac{\lambda}{n}-1)} \|u\|_{\dot{\mathcal{L}}^{p,\lambda}(\mathbf{R}^n)}$$

for all balls B of $\mathbf{R}^n$, j integer ≥ 1 and $u \in \dot{\mathcal{L}}^{p,\lambda}(\mathbf{R}^n)$.

Remark 19. In the case $\lambda > n$, we may remove the term $j+1$ from the right hand side of the last inequality as we will see in the proof.

Proof.

$$\begin{aligned} m_B |u| &= \frac{1}{|B|} \int_B |u(y)| dy \\ &\leq \frac{1}{|B|} \int_B |u(y) - m_B u| dy + |m_B u| \\ &\leq |B|^{\frac{1}{p}(\frac{\lambda}{n}-1)} \left\{ \frac{1}{|B|^{\frac{\lambda}{n}}} \int_B |u(y) - m_B u|^p dy \right\}^{\frac{1}{p}} + |m_B u| \\ &\leq |B|^{\frac{1}{p}(\frac{\lambda}{n}-1)} \|u\|_{\mathcal{L}^{p,\lambda}(\mathbf{R}^n)} + |m_B u| \end{aligned}$$

Replacing B by $2^j B$ yields

$$(7) \quad m_{2^j B} |u| - |2^j B|^{\frac{1}{p}(\frac{\lambda}{n}-1)} \|u\|_{\mathcal{L}^{p,\lambda}(\mathbf{R}^n)} \leq |m_{2^j B} u|$$

On the other hand

$$(8) \quad |m_{2^j B} u| \leq \sum_{k=1}^j |m_{2^k B} u - m_{2^{k-1} B} u| + |m_B u|$$

and

$$\begin{aligned}
 (9) \quad |m_{2^k B} u - m_{2^{k-1} B} u| &\leq 2^n \frac{1}{|2^k B|} \int_{2^k B} |u(y) - m_{2^k B} u| dy \\
 &\leq 2^n \left\{ \frac{1}{|2^k B|} \int_{2^k B} |u(y) - m_{2^k B} u|^p dy \right\}^{\frac{1}{p}} \\
 &\leq 2^n |2^k B|^{\frac{1}{p}(\frac{\lambda}{n}-1)} \|u\|_{\mathcal{L}^{p,\lambda}(\mathbf{R}^n)}
 \end{aligned}$$

In the case $\lambda > n$, inequalities (7), (8) and (9) give

$$\begin{aligned}
 m_{2^j B} |u| - |2^j B|^{\frac{1}{p}(\frac{\lambda}{n}-1)} \|u\|_{\mathcal{L}^{p,\lambda}(\mathbf{R}^n)} \\
 \leq 2^n |2^j B|^{\frac{1}{p}(\frac{\lambda}{n}-1)} \left(\sum_{k=1}^j 2^{\frac{1}{p}(k-j)(\lambda-n)} \right) \|u\|_{\mathcal{L}^{p,\lambda}(\mathbf{R}^n)} + |m_B u| \\
 \leq C 2^n |2^j B|^{\frac{1}{p}(\frac{\lambda}{n}-1)} \|u\|_{\mathcal{L}^{p,\lambda}(\mathbf{R}^n)} + |m_B u|
 \end{aligned}$$

In the case $0 \leq \lambda \leq n$, we use (9) to get

$$|m_{2^k B} u - m_{2^{k-1} B} u| \leq 2^n |B|^{\frac{1}{p}(\frac{\lambda}{n}-1)} \|u\|_{\mathcal{L}^{p,\lambda}(\mathbf{R}^n)}$$

and by (7) and (8)

$$m_{2^j B} |u| - |2^j B|^{\frac{1}{p}(\frac{\lambda}{n}-1)} \|u\|_{\mathcal{L}^{p,\lambda}(\mathbf{R}^n)} \leq j 2^n |B|^{\frac{1}{p}(\frac{\lambda}{n}-1)} \|u\|_{\mathcal{L}^{p,\lambda}(\mathbf{R}^n)} + |m_B u|$$

The proof of Lemma 18 is complete. $\square$

Lemma 20. *Let $s \in \mathbf{R}$ and $1 \leq p \leq +\infty$. Let $I^s = \mathcal{F}^{-1}\{|\cdot|^{-s}\mathcal{F}\}$ be the Riesz potential operator and $u \in \mathcal{Z}'(\mathbf{R}^n)$ such that $\dot{\Delta}_k u \in L^p(\mathbf{R}^n)$ for any $k \in \mathbf{Z}$. Then*

$$\left\| I^s \dot{\Delta}_k u \right\|_{L^p(\mathbf{R}^n)} \leq C 2^{-ks} \|u\|_{L^p(\mathbf{R}^n)}$$

holds and the constant C is independent of k and u .

Proof. Note that

$$\dot{\Delta}_k u = \sum_{l=-1}^1 \dot{\Delta}_{k+l} \dot{\Delta}_k u$$

holds for any k . Thus, Young's inequality gives

$$\left\| I^s \dot{\Delta}_k u \right\|_{L^p(\mathbf{R}^n)} \leq \sum_{l=-1}^1 \left\| I^s \dot{\Delta}_{k+l} \dot{\Delta}_k u \right\|_{L^p(\mathbf{R}^n)}$$

$$\begin{aligned}
&\leq \sum_{l=-1}^1 \left\| \mathcal{F}^{-1}(|\cdot|^{-s} \theta(2^{-k-l} \cdot)) \right\|_{L^1(\mathbf{R}^n)} \left\| \dot{\Delta}_k u \right\|_{L^p(\mathbf{R}^n)} \\
&\leq C \|u\|_{L^p(\mathbf{R}^n)} \sum_{l=-1}^1 \left\| \mathcal{F}^{-1}(|\cdot|^{-s} \theta(2^{-k-l} \cdot)) \right\|_{L^1(\mathbf{R}^n)}
\end{aligned}$$

Now

$$\mathcal{F}^{-1}(|\cdot|^{-s} \theta(2^{-k-l} \cdot))(x) = 2^{-(k+l)s} 2^{(k+l)n} \mathcal{F}^{-1}(|\cdot|^{-s} \theta)(2^{k+l} x)$$

yields

$$\left\| \mathcal{F}^{-1}(|\cdot|^{-s} \theta(2^{-k-l} \cdot)) \right\|_{L^1(\mathbf{R}^n)} = 2^{-(k+l)s} \left\| \mathcal{F}^{-1}(|\cdot|^{-s} \theta) \right\|_{L^1(\mathbf{R}^n)}$$

and the proof is complete. $\square$

3. Proof of Proposition 8

(i) Let $f \in \dot{\mathcal{L}}^{2,\lambda,0}(\mathbf{R}^n)$. Using Lemma 17 it suffices to show that for any ball B centered at $x_0 \in \mathbf{R}^n$ and with radius 2^{-J} , $J \in \mathbf{Z}$, there exists a constant $C_B \in \mathbf{R}$ such that

$$\sup_B \frac{1}{|B|^{\frac{\lambda}{n}}} \int_B |f - C_B|^2 dx \leq C < +\infty.$$

Let us take $C_B = \dot{S}_J f(x_0)$.

$$\begin{aligned}
(10) \quad &\left(\frac{1}{|B|^{\frac{\lambda}{n}}} \int_B |f - \dot{S}_J f(x_0)|^2 dx \right)^{\frac{1}{2}} \\
&\leq \left(\frac{1}{|B|^{\frac{\lambda}{n}}} \int_B |\dot{S}_J f(x) - \dot{S}_J f(x_0)|^2 dx \right)^{\frac{1}{2}} \\
&\quad + \left(\frac{1}{|B|^{\frac{\lambda}{n}}} \int_B \left| \sum_{j>J} \dot{\Delta}_j f(x) \right|^2 dx \right)^{\frac{1}{2}}
\end{aligned}$$

For any $x \in B$ we have

$$|\dot{S}_J f(x) - \dot{S}_J f(x_0)| \leq 2^{-J} \sup_{x \in B} |\nabla \dot{S}_J f(x)|$$

Lemma 13 yields in $Z'(\mathbf{R}^n)$

$$f = \sum_{j \in \mathbf{Z}} \dot{\Delta}_j f = \dot{S}_J f + \sum_{j > J} \dot{\Delta}_j f$$

so $\dot{S}_J f = \sum_{j \leq J} \dot{\Delta}_j f$ in $Z'(\mathbf{R}^n)$ and thus

$$\nabla \dot{S}_J f(x) = \sum_{j \leq J} \nabla \left(\dot{\Delta}_j f \right)(x), \quad x \in \mathbf{R}^n.$$

Let $\phi \in \mathcal{S}(\mathbf{R}^n)$ with $\mathcal{F}\phi = 1$ on $\frac{1}{2} \leq |\xi| \leq 2$.

From $\mathcal{F}\dot{\Delta}_j u(\xi) = \mathcal{F}(\dot{\Delta}_j u)(\xi)\mathcal{F}\phi(2^{-j}\xi)$ we deduce

$$(11) \quad \dot{\Delta}_j u(x) = 2^{nj} \int_{\mathbf{R}^n} \dot{\Delta}_j u(y) \phi(2^j(x-y)) dy$$

Thus for j integer $\leq J$

$$\nabla \left(\dot{\Delta}_j f \right)(x) = 2^{(n+1)j} \int_{\mathbf{R}^n} \left(\dot{\Delta}_j f \right)(y) (\nabla \phi) (2^j(x-y)) dy$$

We divide the integration domain $\mathbf{R}^n = E_j \cup \bigcup_{\nu=-j+1}^{\infty} F_\nu$ with

$$E_j = \{y; |y - x_0| \leq 2^{-j+1}\} \text{ and } F_\nu = \{y; 2^\nu \leq |y - x_0| \leq 2^{\nu+1}\}, \nu \geq -j+1.$$

On E_j we write

$$|(\nabla \phi) (2^j(x-y))| \leq \|\nabla \phi\|_\infty.$$

Since $x \in B = B(x_0, 2^{-J})$ and $j \leq J$, if $y \in F_\nu, \nu \geq -j+1 \geq -J+1$ then $|y - x_0| \sim |y - x|$ and $|2^j(x-y)|$ is large. Hence for each $N \in \mathbf{N}$, there exists $C_N > 0$ such that

$$|(\nabla \phi) (2^j(x-y))| \leq C_N 2^{-(j+\nu)N}$$

Therefore

$$\begin{aligned} \sum_{j \leq J} \left| \nabla \left(\dot{\Delta}_j f \right)(x) \right| &\leq C \sum_{j \leq J} 2^{j(n+1)} \left\{ \int_{E_j} \left| \dot{\Delta}_j f(y) \right| dy + \right. \\ &\quad \left. \sum_{\nu=-j+1}^{\infty} 2^{-(j+\nu)N} \int_{F_\nu} \left| \dot{\Delta}_j f(y) \right| dy \right\} \end{aligned}$$

Now hypothesis $f \in \dot{\mathcal{L}}^{2,\lambda,0}(\mathbf{R}^n)$ yields in particular a constant $C > 0$ such that for any ball $\tilde{B}$ of $\mathbf{R}^n$ of radius $2^{-\tilde{J}}$, $\tilde{J} \in \mathbf{Z}$, we get for any $k \geq \tilde{J}$

$$\int_{\tilde{B}} \left| \dot{\Delta}_k f(y) \right| dy \leq C \left| \tilde{B} \right|^{\frac{1}{2} + \frac{\lambda}{2n}} \|f\|_{\dot{\mathcal{L}}^{2,\lambda,0}(\mathbf{R}^n)}$$

For $\int_{E_j} \left| \dot{\Delta}_j f(y) \right| dy$ the condition $k \geq \tilde{J}$ becomes $j \geq j-1$; and for $\int_{F_\nu} \left| \dot{\Delta}_j f(y) \right| dy$ the condition $k \geq \tilde{J}$ becomes $j \geq -\nu-1$. So we can apply the above inequality to these two terms, and we deduce for N large and $x \in B$ (since $0 \leq \lambda < n+2$)

$$\begin{aligned} \sum_{j \leq J} \left| \nabla \left(\dot{\Delta}_j f \right) (x) \right| &\leq C \|f\|_{\dot{\mathcal{L}}^{2,\lambda,0}(\mathbf{R}^n)} \left\{ \sum_{j \leq J} 2^{j \frac{n+2-\lambda}{2}} \right. \\ &\quad \left. + \sum_{j \leq J} 2^{j(n+1)} \sum_{\nu=-j+1}^{\infty} 2^{-(j+\nu)N} 2^{\nu n \left(\frac{1}{2} + \frac{\lambda}{2n} \right)} \right\} \\ &\leq C \|f\|_{\dot{\mathcal{L}}^{2,\lambda,0}(\mathbf{R}^n)} \sum_{j \leq J} 2^{j \frac{n+2-\lambda}{2}} \\ &\leq C \|f\|_{\dot{\mathcal{L}}^{2,\lambda,0}(\mathbf{R}^n)} 2^{\frac{n+2-\lambda}{2} J} \end{aligned}$$

Therefore

$$(12) \quad \left(\frac{1}{|B|^{\frac{\lambda}{n}}} \int_B \left| \dot{S}_J f(x) - \dot{S}_J f(x_0) \right|^2 dx \right)^{\frac{1}{2}} \leq C \|f\|_{\dot{\mathcal{L}}^{2,\lambda,0}(\mathbf{R}^n)}$$

Now we show the following inequality

$$(13) \quad \frac{1}{|B|^{\frac{\lambda}{n}}} \int_B \left| \sum_{j > J} \dot{\Delta}_j f(x) \right|^2 dx \leq C \|f\|_{\dot{\mathcal{L}}^{2,\lambda,0}(\mathbf{R}^n)}^2$$

For this, let $\theta \in C_0^\infty(\mathbf{R}^n)$ be a positive function equal 0 on $|\xi| \geq \frac{1}{2}$ such that $\theta(-\xi) = \theta(\xi)$ and $\int \theta(\xi) d\xi = 2(2\pi)^n$. Then $\mathcal{F}^{-1}\theta(x) \geq 1$ for $|x| \leq 1$. [For this let us take $n=1$: $\mathcal{F}^{-1}\theta(x) = \frac{1}{2\pi} \int e^{ix\xi} \theta(\xi) d\xi = \frac{1}{\pi} \int_0^{1/2} e^{ix\xi} \theta(\xi) d\xi$, so $\mathcal{F}^{-1}\theta(0) = \frac{1}{\pi} \int_0^{1/2} \theta(\xi) d\xi = 2$ and $\overline{\mathcal{F}^{-1}\theta(x)} = \mathcal{F}^{-1}\theta(x)$. We deduce for $|x| \leq 1$,

$$|\mathcal{F}^{-1}\theta(x) - \mathcal{F}^{-1}\theta(0)| = \frac{1}{\pi} \left| \int_0^{1/2} (e^{ix\xi} - 1) \theta(\xi) d\xi \right|$$

$$\begin{aligned}
&= \frac{|x|}{\pi} \left| \int_0^{1/2} \xi e^{ic\xi} \theta(\xi) d\xi \right| \\
&\leq \frac{1}{2\pi} \int_0^{1/2} |\theta(\xi)| d\xi = 1.
\end{aligned}$$

Thus $|\mathcal{F}^{-1}\theta(x) - 2| \leq 1$ for all $|x| \leq 1$.

We set now

$$\psi(x) = 2^{J\frac{\lambda}{2}} \mathcal{F}^{-1}\theta(2^J(x - x_0)).$$

We have $\text{supp } \mathcal{F}\psi \subset \{|\xi| \leq \frac{1}{2}2^J\}$ and $\psi(x)^2 \geq \frac{C}{|B|^{\frac{\lambda}{n}}}$ for $x \in B$. Thus

$$\begin{aligned}
(14) \quad \frac{1}{|B|^{\frac{\lambda}{n}}} \int_B \left| \sum_{j>J} \dot{\Delta}_j f(x) \right|^2 dx &\leq C \int_B \left| \sum_{j>J} \psi(x) \dot{\Delta}_j f(x) \right|^2 dx \\
&\leq C \int_{\mathbf{R}^n} \left| \sum_{j>J} \mathcal{F}(\psi \dot{\Delta}_j f)(\xi) \right|^2 d\xi
\end{aligned}$$

For $j > J$

$$\text{supp } \mathcal{F}(\psi \dot{\Delta}_j f) \subset \left\{ \frac{1}{2}(2^j - 2^J) \leq |\xi| \leq \frac{5}{2}2^j \right\}$$

implies for each fixed $\xi \in \mathbf{R}^n$ satisfying $|\xi| < \frac{1}{2}(2^j - 2^J)$, $\mathcal{F}(\psi \dot{\Delta}_j f)(\xi) = 0$.

Then, $\sum_{j>J} \mathcal{F}(\psi \dot{\Delta}_j f)(\xi) = \sum_{j=J+1}^N \mathcal{F}(\psi \dot{\Delta}_j f)(\xi)$ where $N = N(\xi, J)$ is any integer $> \log_2(2|\xi| + 2^J)$. Consequently,

$$\begin{aligned}
(15) \quad \left| \sum_{j>J} \mathcal{F}(\psi \dot{\Delta}_j f)(\xi) \right|^2 &\leq \left\{ \sum_{j=J+1}^N \left| \mathcal{F}(\psi \dot{\Delta}_j f)(\xi) \right| \right\}^2 \\
&= \sum_{j=J+1}^N \left| \mathcal{F}(\psi \dot{\Delta}_j f)(\xi) \right|^2 \\
&\quad + 2 \sum_{\substack{j,k=J+1 \\ j < k}}^N \left| \mathcal{F}(\psi \dot{\Delta}_j f)(\xi) \right| \left| \mathcal{F}(\psi \dot{\Delta}_k f)(\xi) \right| \\
&= \sum_{j=J+1}^N \left| \mathcal{F}(\psi \dot{\Delta}_j f)(\xi) \right|^2
\end{aligned}$$

$$\begin{aligned}
& +2 \sum_{j=J+1}^N \left| \mathcal{F}(\psi \dot{\Delta}_j f)(\xi) \right| \left| \mathcal{F}(\psi \dot{\Delta}_{j+1} f)(\xi) \right| \\
& +2 \sum_{j=J+1}^N \left| \mathcal{F}(\psi \dot{\Delta}_j f)(\xi) \right| \left| \mathcal{F}(\psi \dot{\Delta}_{j+2} f)(\xi) \right| \\
& +2 \sum_{j=J+1}^N \left| \mathcal{F}(\psi \dot{\Delta}_j f)(\xi) \right| \left| \mathcal{F}(\psi \dot{\Delta}_{j+3} f)(\xi) \right| \\
& \leq 7 \sum_{j=J+1}^N \left| \mathcal{F}(\psi \dot{\Delta}_j f)(\xi) \right|^2 \\
& = 7 \sum_{j>J} \left| \mathcal{F}(\psi \dot{\Delta}_j f)(\xi) \right|^2
\end{aligned}$$

With (14), (15) and Plancherel theorem there exists a constant C independent of J, B and ξ such that

$$\begin{aligned}
\frac{1}{|B|^{\frac{\lambda}{n}}} \int_B \left| \sum_{j>J} \dot{\Delta}_j f(x) \right|^2 dx & \leq C \int_{\mathbf{R}^n} \sum_{j>J} \left| \mathcal{F}(\psi \dot{\Delta}_j f)(\xi) \right|^2 d\xi \\
& \leq C \sum_{j>J} \int_{\mathbf{R}^n} \left| \psi(x) \dot{\Delta}_j f(x) \right|^2 dx
\end{aligned}$$

We decompose $\mathbf{R}^n$ into disjoint cubes $Q_\nu = \{x \in \mathbf{R}^n / \|x - x_0 - 2^{-J}\nu\|_\infty \leq 2^{-J-1}\}, \nu \in \mathbf{Z}^n$, and we set $a_\nu = \sup_{\|x-\nu\|_\infty \leq \frac{1}{2}} |\mathcal{F}^{-1}\theta(x)|^2$ which is a rapidly decreasing sequence.

Thus,

$$\begin{aligned}
\frac{1}{|B|^{\frac{\lambda}{n}}} \int_B \left| \sum_{j>J} \dot{\Delta}_j f(x) \right|^2 dx & \leq C \sum_{\nu} \sum_{j \geq J} \int_{Q_\nu} \left| \psi(x) \dot{\Delta}_j f(x) \right|^2 dx \\
& \leq C \sum_{\nu} a_\nu \frac{1}{|Q_\nu|^{\frac{\lambda}{n}}} \int_{Q_\nu} \sum_{j \geq J} \left| \dot{\Delta}_j f(x) \right|^2 dx \\
& \leq C \|f\|_{\dot{\mathcal{L}}^{2,\lambda,0}(\mathbf{R}^n)}^2 \sum_{\nu} a_\nu
\end{aligned}$$

Inequality (13) is proved. Finally combining (10), (12) and (13) we obtain

$$\left(\frac{1}{|B|^{\frac{\lambda}{n}}} \int_B |f - \dot{S}_J f(x_0)|^2 dx \right)^{\frac{1}{2}} \leq C \|f\|_{\dot{\mathcal{L}}^{2,\lambda,0}(\mathbf{R}^n)}$$

The injection $\dot{\mathcal{L}}^{2,\lambda,0}(\mathbf{R}^n) \hookrightarrow \dot{\mathcal{L}}^{2,\lambda}(\mathbf{R}^n)$ is proved for $0 \leq \lambda < n+2$.

Conversely let $f \in \dot{\mathcal{L}}^{2,\lambda}(\mathbf{R}^n)$.

We decompose $f = f_1 + f_2 + f_3$ with $f_1 = m_{2B}f$, $f_2 = (f - m_{2B}f)\chi_{2B}$ and $f_3 = (f - m_{2B}f)\chi_{\mathbf{R}^n \setminus 2B}$ where B is a ball of $\mathbf{R}^n$ of radius 2^{-J} , $J \in \mathbf{Z}$, and χ_Ω is the characteristic function of the set Ω .

The function f_1 is constant, $\dot{\Delta}_k f_1 = 0$ for any k .

For f_2 we have

$$\begin{aligned} \|f_2\|_{L^2(\mathbf{R}^n)} &= \left(\int_{2B} |f - m_{2B}f|^2 dx \right)^{1/2} \\ &\leq C |B|^{\frac{\lambda}{2n}} \|f\|_{\dot{\mathcal{L}}^{2,\lambda}(\mathbf{R}^n)} \end{aligned}$$

Now $\dot{F}_{2,2}^0(\mathbf{R}^n) \equiv L^2(\mathbf{R}^n)$ yields

$$\begin{aligned} (16) \quad \left(\sum_{j \geq J} \left\| \dot{\Delta}_j f_2 \right\|_{L^2(B)}^2 \right)^{\frac{1}{2}} &\leq \left(\sum_{j \in \mathbf{Z}} \left\| \dot{\Delta}_j f_2 \right\|_{L^2(\mathbf{R}^n)}^2 \right)^{\frac{1}{2}} \\ &\leq C \|f_2\|_{L^2(\mathbf{R}^n)} \\ &\leq C |B|^{\frac{\lambda}{2n}} \|f\|_{\dot{\mathcal{L}}^{2,\lambda}(\mathbf{R}^n)} \end{aligned}$$

For f_3 we will show that for any $x \in B$, $j \geq J$ and $0 \leq \lambda < n+2$

$$(17) \quad \left| \dot{\Delta}_j f_3(x) \right| \leq C 2^{J-j} |B|^{\frac{1}{2}(\frac{\lambda}{n}-1)} \|f\|_{\dot{\mathcal{L}}^{2,\lambda}(\mathbf{R}^n)}$$

holds.

From inequality (17) we deduce

$$(18) \quad \sum_{j \geq J} \left\| \dot{\Delta}_j f_3 \right\|_{L^2(B)}^2 \leq C |B|^{\frac{\lambda}{n}} \|f\|_{\dot{\mathcal{L}}^{2,\lambda}(\mathbf{R}^n)}^2$$

and the proof of the injection $\dot{\mathcal{L}}^{2,\lambda}(\mathbf{R}^n) \hookrightarrow \dot{\mathcal{L}}^{2,\lambda,0}(\mathbf{R}^n)$ for $0 \leq \lambda < n+2$ will be complete.

Proof of (17). For $x \in B$

$$\dot{\Delta}_j f_3(x) = 2^{jn} \int_{\mathbf{R}^n \setminus 2B} f_3(y) \mathcal{F}^{-1} \theta(2^j(x-y)) dy$$

Now $\mathcal{F}^{-1}\theta \in \mathcal{S}(\mathbf{R}^n)$ yields a constant $C > 0$ such that

$$|\mathcal{F}^{-1}\theta(x)| \leq C|x|^{-n-1} \text{ for } |x| \geq 1.$$

If $x \in B, y \in \mathbf{R}^n \setminus 2B$ and $j \geq J$, then $2^j|x-y| \geq 1$ and $|x_0 - y| \leq 2|x-y|$. So

$$\begin{aligned} \left| \dot{\Delta}_j f_3(x) \right| &\leq C2^{-j} \int_{\mathbf{R}^n \setminus 2B} |f - m_{2B}f|(y) |x_0 - y|^{-n-1} dy \\ &\leq C2^{-j} \sum_{l=1}^{+\infty} \int_{2^{l+1}B \setminus 2^lB} |f - m_{2B}f|(y) |x_0 - y|^{-n-1} dy \\ &\leq C2^{J-j} \sum_{l=1}^{+\infty} 2^{-l} m_{2^{l+1}B}(|f - m_{2B}f|) \end{aligned}$$

Using Lemma 18 we obtain

$$\begin{aligned} \left| \dot{\Delta}_j f_3(x) \right| &\leq C2^{J-j} \sum_{l=1}^{+\infty} 2^{-l} \left\{ |m_{2B}(f - m_{2B}f)| \right. \\ &\quad \left. + C(l+1) |2^{\alpha l} B|^{\frac{1}{2}(\frac{\lambda}{n}-1)} \|f - m_{2B}f\|_{\dot{L}^{2,\lambda}(\mathbf{R}^n)} \right\} \\ &\leq C2^{J-j} \sum_{l=1}^{+\infty} 2^{-l(1-\frac{\alpha}{2}n(\frac{\lambda}{n}-1))} (l+1) \|f\|_{\dot{L}^{2,\lambda}(\mathbf{R}^n)} \end{aligned}$$

where $\alpha = \begin{cases} 0 & \text{if } 0 \leq \lambda \leq n \\ 1 & \text{if } \lambda > n \end{cases}$. The series converges when $\lambda < n+2$.

Inequality (17) and then the first part of Proposition 8 are proved.

(ii) Let $f \in \mathcal{L}^{2,\lambda,0}(\mathbf{R}^n)$. We show that for any ball B centered at $x_0 \in \mathbf{R}^n$ and with radius $2^{-J}, J \in \mathbf{Z}$, there exists a constant $C_B \in \mathbf{R}$ such that

$$\sup_{B, J \geq 0} \frac{1}{|B|^{\frac{\lambda}{n}}} \int_B |f - C_B|^2 dx + \sup_{B, J=0} \frac{1}{|B|} \int_B |f|^2 dx = C < +\infty.$$

To estimate $\sup_{B, J \geq 0} \frac{1}{|B|^{\frac{\lambda}{n}}} \int_B |f - C_B|^2 dx$, we write

$$f = \sum_{j \geq 0} \Delta_j f = \Delta_0 f + \tilde{S}_{J^+} f + \sum_{j > J^+} \Delta_j f, \text{ here } J^+ = J \geq 0.$$

where $\tilde{S}_{J^+}f = \sum_{j=1}^{J^+} \Delta_j f$. Set $C_B = \tilde{S}_{J^+}f(x_0)$. Thus

$$\begin{aligned} & \left(\frac{1}{|B|^{\frac{\lambda}{n}}} \int_B |f - \tilde{S}_{J^+}f(x_0)|^2 dx \right)^{\frac{1}{2}} \\ & \leq \left(\frac{1}{|B|^{\frac{\lambda}{n}}} \int_B |\Delta_0 f|^2 dx \right)^{\frac{1}{2}} + \left(\frac{1}{|B|^{\frac{\lambda}{n}}} \int_B |\tilde{S}_{J^+}f(x) - \tilde{S}_{J^+}f(x_0)|^2 dx \right)^{\frac{1}{2}} \\ & \quad + \left(\frac{1}{|B|^{\frac{\lambda}{n}}} \int_B \left| \sum_{j>J^+} \Delta_j f(x) \right|^2 dx \right)^{\frac{1}{2}} \end{aligned}$$

The first term of the right hand side is bounded from above by $\|f\|_{\mathcal{L}^{2,\lambda,0}(\mathbf{R}^n)}$. For the two other terms we argue in the same way as for inequalities (12) and (13).

To estimate the remainder term $\sup_{B,J=0} \frac{1}{|B|} \int_B |f|^2 dx$ we write

$$\left(\int_B |f|^2 dx \right)^{1/2} \leq C \left\{ \|\Delta_0 f\|_{L^2(B)} + \left(\int_B \left| \sum_{j \geq 1} \Delta_j f(x) \right|^2 dx \right)^{1/2} \right\}$$

In the same way as for inequality (13) we deduce

$$\left(\int_B |f|^2 dx \right)^{1/2} \leq C \|f\|_{\mathcal{L}^{2,\lambda,0}(\mathbf{R}^n)}.$$

Conversely, let $f \in l^{2,\lambda}$. We use the same decomposition $f = f_1 + f_2 + f_3$ as in the first part.

For f_1 , if $J \geq 1$ then $\Delta_j f_1 = 0$ for $j \geq 1$, and $\left(\frac{1}{|B|^{\frac{\lambda}{n}}} \sum_{j \geq J^+} \|\Delta_j f_1\|_{L^2(B)}^2 \right)^{1/2} = 0 \leq \|f\|_{l^{2,\lambda}}$.

In the case $J \leq 0$, we remark that $\Delta_0 f_1 = f_1$ and $J^+ = 0$, so

$$\begin{aligned} \left(\frac{1}{|B|^{\frac{\lambda}{n}}} \sum_{j \geq 0} \|\Delta_j f_1\|_{L^2(B)}^2 \right)^{1/2} &= \left(\frac{1}{|B|^{\frac{\lambda}{n}}} \int_B |\Delta_0 f_1|^2 dx \right)^{1/2} \\ &= \frac{1}{|B|^{\frac{\lambda}{2n} - \frac{1}{2}}} |m_{2B} f| \end{aligned}$$

$$\leq C \left(\frac{1}{|2B|^{\frac{\lambda}{n}}} \int_{2B} |f|^2 dx \right)^{1/2}$$

this last term is bounded from above by $C \|f\|_{l^{2,\lambda}}$ since (3) is valid with $P_B = 0$ when $\delta = 2^{-J} \geq 1$.

For f_2 and f_3 we argue in the same way as for (16) and (18) and we use the injection $l^{2,\lambda} \hookrightarrow \mathcal{L}^{2,\lambda}(\mathbf{R}^n)$. $\square$

In order to compare between the spaces $\dot{\mathcal{L}}^{p,\lambda,0}(\mathbf{R}^n)$ and $\dot{\mathcal{L}}^{p,\lambda}(\mathbf{R}^n)$ for $p \geq 1$, we consider two cases $p \geq 2$ and $1 \leq p \leq 2$:

Proposition 21. *Let $p \geq 2$, and $0 \leq \lambda < n + p$. The Campanato space $\dot{\mathcal{L}}^{p,\lambda}(\mathbf{R}^n)$ is continuously embedded in the space $\dot{\mathcal{L}}^{p,\lambda,0}(\mathbf{R}^n)$.*

Proof. The proof is similar to the one of Proposition 8. Let $f \in \dot{\mathcal{L}}^{p,\lambda}(\mathbf{R}^n)$. We decompose $f = f_1 + f_2 + f_3$ with $f_1 = m_{2B}f$, $f_2 = (f - m_{2B}f)\chi_{2B}$ and $f_3 = (f - m_{2B}f)\chi_{\mathbf{R}^n \setminus 2B}$ where B is a ball of $\mathbf{R}^n$ of radius 2^{-J} , $J \in \mathbf{Z}$.

The function f_1 is constant, $\dot{\Delta}_k f_1 = 0$ for any k . For the function f_2 we have $\|f_2\|_{L^p(\mathbf{R}^n)} \leq |2B|^{\frac{\lambda}{np}} \|f\|_{\dot{\mathcal{L}}^{p,\lambda}(\mathbf{R}^n)}$ then $f_2 \in L^p(\mathbf{R}^n)$ and since $p \geq 2$ we obtain

$$\begin{aligned} \left(\sum_{j \geq J} \left\| \dot{\Delta}_j f_2 \right\|_{L^p(B)}^p \right)^{\frac{1}{p}} &\leq \left(\sum_{j \in \mathbf{Z}} \left\| \dot{\Delta}_j f_2 \right\|_{L^p(\mathbf{R}^n)}^p \right)^{\frac{1}{p}} \\ &= \|f_2\|_{\dot{B}_{p,p}^0(\mathbf{R}^n)} \\ &\leq C \|f_2\|_{L^p(\mathbf{R}^n)} \\ &\leq C |B|^{\frac{\lambda}{np}} \|f\|_{\dot{\mathcal{L}}^{p,\lambda}(\mathbf{R}^n)} \end{aligned}$$

For f_3 we show in the same way as for inequality (17) that for any $x \in B$, $j \geq J$ and $\lambda < n + p$

$$(19) \quad \left| \dot{\Delta}_j f_3(x) \right| \leq C 2^{J-j} |B|^{\frac{1}{p}(\frac{\lambda}{n}-1)} \|f\|_{\dot{\mathcal{L}}^{p,\lambda}(\mathbf{R}^n)}$$

holds. So

$$\sum_{j \geq J} \left\| \dot{\Delta}_j f_3 \right\|_{L^p(B)}^p \leq C |B|^{\frac{\lambda}{n}} \|f\|_{\dot{\mathcal{L}}^{p,\lambda}(\mathbf{R}^n)}^p$$

which complete the proof. $\square$

Proposition 22. *Let $1 \leq p \leq 2$ and $n - \frac{n}{2}p \leq \lambda < n + p$. The space $\dot{\mathcal{L}}^{p,\lambda,0}(\mathbf{R}^n) \cap \dot{F}_{2,q}^{\frac{n}{2} + \frac{\lambda-n}{p}}(\mathbf{R}^n)$ is continuously embedded in Campanato space $\dot{\mathcal{L}}^{p,\lambda}(\mathbf{R}^n)$ for $1 \leq q \leq 2$.*

Remark 23. If $n \leq \lambda < n + p$ and $1 \leq p \leq 2$ then

$$\dot{\mathcal{L}}^{p,\lambda,0}(\mathbf{R}^n) \cap \dot{F}_{2,q}^{\frac{n}{2} + \frac{\lambda-n}{p}}(\mathbf{R}^n) \hookrightarrow \dot{\mathcal{L}}^{p,\lambda,0}(\mathbf{R}^n) \cap \dot{\mathcal{L}}^{2,\lambda}(\mathbf{R}^n),$$

for $1 \leq q \leq 2$.

For this, Lemma 12 yields $\dot{F}_{2,q}^{\frac{n}{2} + \frac{\lambda-n}{p}}(\mathbf{R}^n) \hookrightarrow \dot{\mathcal{L}}^{2,\lambda, \frac{\lambda-n}{p} - \frac{\lambda-n}{2}}(\mathbf{R}^n)$, and $\lambda \geq n$ implies that $\dot{\mathcal{L}}^{2,\lambda, \frac{\lambda-n}{p} - \frac{\lambda-n}{2}}(\mathbf{R}^n) \hookrightarrow \dot{\mathcal{L}}^{2,\lambda,0}(\mathbf{R}^n) = \dot{\mathcal{L}}^{2,\lambda}(\mathbf{R}^n)$.

Proof. It suffices to prove the proposition for $q = 2$.

Let $f \in \dot{\mathcal{L}}^{p,\lambda,0}(\mathbf{R}^n) \cap \dot{F}_{2,2}^{\frac{n}{2} + \frac{\lambda-n}{p}}(\mathbf{R}^n)$. Using Lemma 17 it suffices to show that for any ball B centered at $x_0 \in \mathbf{R}^n$ and with radius 2^{-J} , $J \in \mathbf{Z}$, there exists a constant $C_B \in \mathbf{R}$ such that

$$\sup_B \frac{1}{|B|^{\frac{\lambda}{n}}} \int_B |f - C_B|^p dx \leq C < +\infty.$$

Set $C_B = \dot{S}_J f(x_0)$. As in the proof of Proposition 8 we write

$$\begin{aligned} \left(\frac{1}{|B|^{\frac{\lambda}{n}}} \int_B |f - \dot{S}_J f(x_0)|^p dx \right)^{\frac{1}{p}} &\leq \left(\frac{1}{|B|^{\frac{\lambda}{n}}} \int_B |\dot{S}_J f(x) - \dot{S}_J f(x_0)|^p dx \right)^{\frac{1}{p}} \\ &\quad + \left(\frac{1}{|B|^{\frac{\lambda}{n}}} \int_B \left| \sum_{j>J} \dot{\Delta}_j f(x) \right|^p dx \right)^{\frac{1}{p}} \end{aligned}$$

In the same way as for inequality (12) we obtain

$$(20) \quad \left(\frac{1}{|B|^{\frac{\lambda}{n}}} \int_B |\dot{S}_J f(x) - \dot{S}_J f(x_0)|^p dx \right)^{\frac{1}{p}} \leq C \|f\|_{\dot{\mathcal{L}}^{p,\lambda,0}(\mathbf{R}^n)}$$

We will show

$$(21) \quad \left\{ \frac{1}{|B|^{\frac{\lambda}{n}}} \int_B \left| \sum_{j>J} \dot{\Delta}_j f(x) \right|^p dx \right\}^{1/p} \leq C \|f\|_{\dot{F}_{2,2}^{\frac{n}{2} + \frac{\lambda-n}{p}}(\mathbf{R}^n)}$$

For this, $p \leq 2$ and Plancherel theorem yield

$$\begin{aligned}
& \left\{ \frac{1}{|B|^{\frac{\lambda}{n}}} \int_B \left| \sum_{j>J} \dot{\Delta}_j f(x) \right|^p dx \right\}^{1/p} \\
& \leq \frac{1}{|B|^{\frac{\lambda}{np}}} \left\{ \int_B \left| \sum_{j \geq J} \dot{\Delta}_j f(x) \right|^2 dx \right\}^{1/2} |B|^{\frac{1}{p} - \frac{1}{2}} \\
& \leq C |B|^{-\frac{\lambda}{np} + \frac{1}{p} - \frac{1}{2}} \left\{ \int_{\mathbf{R}^n} \left| \sum_{j \geq J} \mathcal{F} \dot{\Delta}_j f(\xi) \right|^2 d\xi \right\}^{1/2}
\end{aligned}$$

We argue as for (15), and next we use the assumption $n - \frac{n}{2}p \leq \lambda$ to obtain

$$\begin{aligned}
& \left\{ \frac{1}{|B|^{\frac{\lambda}{n}}} \int_B \left| \sum_{j>J} \dot{\Delta}_j f(x) \right|^p dx \right\}^{1/p} \\
& \leq C |B|^{-\frac{\lambda}{np} + \frac{1}{p} - \frac{1}{2}} \left\{ \int_{\mathbf{R}^n} \sum_{j>J} \left| \dot{\Delta}_j f(x) \right|^2 dx \right\}^{1/2} \\
& \leq C \left\{ \int_{\mathbf{R}^n} \sum_{j>J} 2^{2J(\frac{n}{2} + \frac{\lambda-n}{p})} \left| \dot{\Delta}_j f(x) \right|^2 dx \right\}^{1/2} \\
& \leq C \left\{ \int_{\mathbf{R}^n} \sum_{j>J} 2^{2j(\frac{n}{2} + \frac{\lambda-n}{p})} \left| \dot{\Delta}_j f(x) \right|^2 dx \right\}^{1/2} \\
& \leq C \left\{ \int_{\mathbf{R}^n} \sum_{j \in \mathbf{Z}} 2^{2j(\frac{n}{2} + \frac{\lambda-n}{p})} \left| \dot{\Delta}_j f(x) \right|^2 dx \right\}^{1/2} \\
& = C \|f\|_{\dot{F}_{2,2}^{\frac{n}{2} + \frac{\lambda-n}{p}}(\mathbf{R}^n)}
\end{aligned}$$

Finally, with (20) and (21) we deduce

$$\left(\frac{1}{|B|^{\frac{\lambda}{n}}} \int_B |f - \dot{S}_J f(x_0)|^p dx \right)^{\frac{1}{p}} \leq C \left(\|f\|_{\dot{L}^{p,\lambda,0}(\mathbf{R}^n)} + \|f\|_{\dot{F}_{2,2}^{\frac{n}{2} + \frac{\lambda-n}{p}}(\mathbf{R}^n)} \right)$$

which complete the proof. $\square$

Proof of Corollary 9. We recall that for $m = 0$, $\dot{\mathcal{L}}^{2,\lambda,0}(\mathbf{R}^n) = \dot{\mathcal{L}}^{2,\lambda}(\mathbf{R}^n) = \dot{\mathcal{H}}^{2,\lambda,0}$.

(i) If $u \in \dot{\mathcal{L}}^{2,\lambda,m}(\mathbf{R}^n)$ then $D^\alpha u \in \dot{\mathcal{L}}^{2,\lambda,m-|\alpha|}(\mathbf{R}^n) \hookrightarrow \dot{\mathcal{L}}^{2,\lambda,0}(\mathbf{R}^n) = \dot{\mathcal{L}}^{2,\lambda}(\mathbf{R}^n)$, $|\alpha| \leq m$.

(ii) Let $u \in \dot{\mathcal{H}}^{2,\lambda,m} \cap \dot{H}^{m+\lambda/2}(\mathbf{R}^n)$. Inequality (6) yields

$$\begin{aligned}
 \frac{1}{|B|^{\frac{\lambda}{n}}} \sum_{l \geq J} 2^{2lm} \left\| \dot{\Delta}_l u \right\|_{L^2(B)}^2 &\leq \frac{C}{|B|^{\frac{\lambda}{n}}} \sum_{l \geq J} 2^{2lm} \sum_{j \sim l} \left\| \dot{\Delta}_j \dot{\Delta}_l u \right\|_{L^2(B)}^2 \\
 &\leq \frac{C}{|B|^{\frac{\lambda}{n}}} \sum_{l \geq J} 2^{2lm} 2^{-2lm} \sup_{|\alpha| \leq m} \left\| D^\alpha \dot{\Delta}_l u \right\|_{L^2(3B)}^2 \\
 &\quad + \frac{C}{|B|^{\frac{\lambda}{n}}} \sum_{l \geq J} 2^{2lm} 2^{-2(l-J)M} \left\| \dot{\Delta}_l u \right\|_{L^2(\mathbf{R}^n)}^2 \\
 &\leq C \sup_{|\alpha| \leq m} \left\| D^\alpha u \right\|_{\dot{\mathcal{L}}^{2,\lambda,0}(\mathbf{R}^n)}^2 \\
 &\quad + C \sum_{l \geq J} 2^{2l(m+\frac{\lambda}{2})} \left\| \dot{\Delta}_l u \right\|_{L^2(\mathbf{R}^n)}^2 \\
 &\leq C \|u\|_{\dot{\mathcal{H}}^{2,\lambda,m}}^2 + C \|u\|_{\dot{H}^{m+\frac{\lambda}{2}}(\mathbf{R}^n)}^2. \quad \square
 \end{aligned}$$

4. Proof of Theorem 10

The first part of Theorem 10 follows from these following lemmas.

Lemma 24. *There exists a constant $C > 0$ such that for any $f \in I^s(\dot{\mathcal{L}}^{2,\lambda})$, there exists a sequence of functions $(f_j)_{j \in \mathbf{Z}}$ of $C^\infty(\mathbf{R}^n)$ satisfying*

$$\frac{1}{|B|^{\frac{\lambda}{n}}} \sum_{j \geq J} 2^{2js} \|f_j\|_{L^2(B)}^2 \leq C \|f\|_{I^s(\dot{\mathcal{L}}^{2,\lambda})}^2$$

for any ball B of $\mathbf{R}^n$ with radius 2^{-J} , $J \in \mathbf{Z}$.

Lemma 25. *There exists a constant $C > 0$ such that for any functions $(f_j)_{j \in \mathbf{Z}}$ with $\text{supp } \mathcal{F}f_j \subset \{\frac{1}{2}2^j \leq |\xi| \leq 2 \cdot 2^j\}$, we get*

$$\left\| \sum_j f_j \right\|_{I^s(\dot{\mathcal{L}}^{2,\lambda})}^2 \leq C \sup_B \left\{ \frac{1}{|B|^{\frac{\lambda}{n}}} \sum_{j \geq J} 2^{2js} \|f_j\|_{L^2(B)}^2 \right\}$$

where the supremum is taken over all balls B of $\mathbf{R}^n$ of radius 2^{-J} , $J \in \mathbf{Z}$.

Proof of Lemma 24. Let $f \in I^s(\mathcal{L}^{\dot{2},\lambda})$ and $g = I^{-s}f$. If we set $f_j = \dot{\Delta}_j f$ then $f_j = I^s \dot{\Delta}_j g$.

The proof of Lemma 24 is equivalent to show that

$$(22) \quad \frac{1}{|B|^{\frac{\lambda}{n}}} \sum_{j \geq J} 2^{2js} \left\| I^s \dot{\Delta}_j g \right\|_{L^2(B)}^2 \leq C \|g\|_{\mathcal{L}^{\dot{2},\lambda}(\mathbf{R}^n)}^2$$

We decompose $g = g_1 + g_2 + g_3$ with $g_1 = m_{2B}g$, $g_2 = (g - m_{2B}g)\chi_{2B}$ and $g_3 = (g - m_{2B}g)\chi_{\mathbf{R}^n \setminus 2B}$ where B is the ball of $\mathbf{R}^n$ centered at x_0 and of radius 2^{-J} , $J \in \mathbf{Z}$.

The function g_1 is constant, $\dot{\Delta}_j g_1 = 0$ for any j . Then inequality (22) is valid for g_1 .

On the other hand $g_2 \in L^2(\mathbf{R}^n)$ and $\|g_2\|_{L^2(\mathbf{R}^n)} \leq C|B|^{\frac{\lambda}{2n}} \|g\|_{\mathcal{L}^{\dot{2},\lambda}(\mathbf{R}^n)}$.

We deduce from Lemma 20

$$\begin{aligned} \sum_{j \geq J} 2^{2js} \left\| I^s \dot{\Delta}_j g_2 \right\|_{L^2(B)}^2 &\leq \sum_j 2^{2js} \left\| I^s \dot{\Delta}_j g_2 \right\|_{L^2(\mathbf{R}^n)}^2 \\ &\leq C \sum_j \left\| \dot{\Delta}_j g_2 \right\|_{L^2(\mathbf{R}^n)}^2 \\ &\leq C \|g_2\|_{L^2(\mathbf{R}^n)}^2 \\ &\leq C|B|^{\frac{\lambda}{n}} \|g\|_{\mathcal{L}^{\dot{2},\lambda}(\mathbf{R}^n)}^2 \end{aligned}$$

Inequality (22) holds for g_2 .

We set $h_j = I^s \dot{\Delta}_j g_3$, $j \in \mathbf{Z}$. We will show that

$$(23) \quad |h_j(x)| \leq C \|g\|_{\mathcal{L}^{\dot{2},\lambda}(\mathbf{R}^n)} 2^{J-j-j_s} |B|^{\frac{1}{2}(\frac{\lambda}{n}-1)}$$

holds for any $x \in B$ and $j \geq J$, and therefore inequality (22) is valid for g_3 .

To show (23) we write

$$h_j(x) = \int_{\mathbf{R}^n \setminus 2B} g_3(y) (\mathcal{F}^{-1} \psi_j^s)(x-y) dy$$

where $\psi_j^s(\xi) = |\xi|^{-s} \theta(2^{-j}\xi)$, $\theta(\xi) = \varphi(\xi) - \varphi(2\xi)$, φ is the function defining the partition of unity. We have

$$(24) \quad (\mathcal{F}^{-1} \psi_j^s)(x) = 2^{j(n-s)} (\mathcal{F}^{-1} \psi^s)(2^j x)$$

where $\psi^s(\xi) = |\xi|^{-s} \theta(\xi)$.

Since $\psi^s \in \mathcal{S}(\mathbf{R}^n)$, for each nonnegative integer N there exists $A_N > 0$ such that

$$(25) \quad |(\mathcal{F}^{-1}\psi^s)(x)| \leq A_N(1+|x|^2)^{-N}$$

Now if $x \in B$, $y \in \mathbf{R}^n \setminus 2B$ and $j \geq J$ then $|2^j(x-y)| \geq 1$ and $|x_0 - y| \leq 2|x-y|$. Thus

$$\begin{aligned} |h_j(x)| &\leq A_N 2^{j(n-s-2N)} \int_{\mathbf{R}^n \setminus 2B} |g_3(y)| |x-y|^{-2N} dy \\ &\leq A_N 2^{j(n-s-2N)} \sum_{l=1}^{+\infty} \int_{2^{l+1}B \setminus 2^l B} |g_3(y)| |x_0-y|^{-2N} dy \\ &\leq A_N 2^{j(n-s-2N)} \sum_{l=1}^{+\infty} 2^{-2N(l-J)} m_{2^{l+1}B}(|g_3|) |2^{l+1}B| \\ &\leq A_N 2^{j(n-s-2N)} \sum_{l=1}^{+\infty} 2^{-2N(l-J)} 2^{(l+1-J)n} m_{2^{l+1}B}(|g - m_{2B}g|) \end{aligned}$$

Set $\alpha = \begin{cases} 0 & \text{if } 0 \leq \lambda \leq n \\ 1 & \text{if } \lambda > n \end{cases}$. Lemma 18 yields

$$\begin{aligned} |h_j(x)| &\leq A_N 2^{j(n-s-2N)} \sum_{l=1}^{+\infty} 2^{(n-2N)(l-J)} \{ |m_{2B}(g - m_{2B}g)| \\ &\quad + C(l+1) |2^{\alpha l} B|^{\frac{1}{2}(\frac{\lambda}{n}-1)} \|g - m_{2B}g\|_{\dot{\mathcal{L}}^{2,\lambda}(\mathbf{R}^n)} \} \\ &\leq A_N 2^{J-j-j_s} 2^{(j-J)(n+1-2N)} |B|^{\frac{1}{2}(\frac{\lambda}{n}-1)} \|g\|_{\dot{\mathcal{L}}^{2,\lambda}(\mathbf{R}^n)} \\ &\quad \times \sum_{l=1}^{+\infty} 2^{l(n-2N+\frac{\alpha}{2}(\lambda-n))} (l+1) \end{aligned}$$

For $j \geq J$ and N large we obtain inequality (23). The proofs of Lemma 24 and the injection $I^s(\dot{\mathcal{L}}^{2,\lambda}) \hookrightarrow \dot{\mathcal{L}}^{2,\lambda,s}$ are complete. $\square$

Proof of Lemma 25. Proposition 8 yields

$$\left\| \sum_j f_j \right\|_{I^s(\dot{\mathcal{L}}^{2,\lambda})}^2 = \left\| I^{-s} \sum_j f_j \right\|_{\dot{\mathcal{L}}^{2,\lambda}(\mathbf{R}^n)}^2 \leq C \left\| \sum_j I^{-s} f_j \right\|_{\dot{\mathcal{L}}^{2,\lambda,0}(\mathbf{R}^n)}^2$$

$$\begin{aligned}
&\leq C \sup_B \frac{1}{|B|^{\frac{\lambda}{n}}} \sum_{k \geq J} \left\| \sum_j I^{-s} \dot{\Delta}_k f_j \right\|_{L^2(B)}^2 \\
&\leq C \sup_B \frac{1}{|B|^{\frac{\lambda}{n}}} \sum_{k \geq J} \sum_{j=k-1}^{k+1} \left\| I^{-s} \dot{\Delta}_k f_j \right\|_{L^2(B)}^2
\end{aligned}$$

The *supremum* is taken over all balls B of $\mathbf{R}^n$ of radius 2^{-J} , $J \in \mathbf{Z}$.

Let $\phi \in C_0^\infty(\mathbf{R}^n)$, $\phi = 1$ on $2B$ and $\text{supp} \phi \subset 3B$.

Lemma 20 gives for any $j \in \mathbf{Z}$

$$\begin{aligned}
(26) \quad \left\| I^{-s} \dot{\Delta}_j f_j \right\|_{L^2(B)} &= \left\| \phi I^{-s} \dot{\Delta}_j f_j \right\|_{L^2(B)} \\
&\leq \left\| I^{-s} \dot{\Delta}_j \phi f_j \right\|_{L^2(B)} + \left\| [I^{-s} \dot{\Delta}_j, \phi] f_j \right\|_{L^2(B)} \\
&\leq C 2^{js} \left\| \phi f_j \right\|_{L^2(\mathbf{R}^n)} + \left\| [I^{-s} \dot{\Delta}_j, \phi] f_j \right\|_{L^2(B)} \\
&\leq C 2^{js} \left\| f_j \right\|_{L^2(3B)} + \left\| [I^{-s} \dot{\Delta}_j, \phi] f_j \right\|_{L^2(B)}
\end{aligned}$$

On the other hand

$$\begin{aligned}
I^{-s} \dot{\Delta}_j f_j(x) &= \int_{\mathbf{R}^n} f_j(y) (\mathcal{F}^{-1} \psi_j^{-s})(x-y) dy \\
&= 2^{j(n+s)} \int_{\mathbf{R}^n} f_j(y) (\mathcal{F}^{-1} \psi^{-s})(2^j(x-y)) dy
\end{aligned}$$

where $\psi_j^{-s}(\xi)$ and ψ^{-s} are defined in (24). Therefore, if $x \in B$

$$\begin{aligned}
[I^{-s} \dot{\Delta}_j, \phi] f_j(x) &= -2^{j(n+s)} \int_{\mathbf{R}^n \setminus 2B} f_j(y) (\mathcal{F}^{-1} \psi^{-s})(2^j(x-y)) dy \\
&\quad + 2^{j(n+s)} \int_{3B \setminus 2B} \phi(y) f_j(y) (\mathcal{F}^{-1} \psi^{-s})(2^j(x-y)) dy
\end{aligned}$$

We decompose $\mathbf{R}^n \setminus 2B = \bigcup_{\nu \geq -J+1} F_\nu$ with $F_\nu = \{y \in \mathbf{R}^n : 2^\nu \leq |x_0 - y| \leq 2^{\nu+1}\}$. For $\nu \geq -J+1$, if $x \in B$ and $y \in F_\nu$, then $|x-y| \sim |x_0-y| \sim 2^\nu$. With the aid of inequality (25) we obtain

$$\begin{aligned}
&\left| -2^{j(n+s)} \int_{\mathbf{R}^n \setminus 2B} f_j(y) (\mathcal{F}^{-1} \psi^{-s})(2^j(x-y)) dy \right| \\
&\leq A_N 2^{j(n+s)} \sum_{\nu \geq -J+1} \int_{F_\nu} 2^{-2jN} |f_j(y)| |x-y|^{-2N} dy
\end{aligned}$$

$$\begin{aligned}
&\leq A_N 2^{j(n+s-2N)} \sum_{\nu \geq -J+1} 2^{-2\nu N} \int_{F_\nu} |f_j(y)| dy \\
&\leq A_N 2^{j(n+s-2N)} \sum_{\nu \geq -J+1} 2^{-2\nu N} \|f_j\|_{L^2(F_\nu)} |F_\nu|^{\frac{1}{2}}
\end{aligned}$$

For $x \in B$ and $y \in 3B \setminus 2B$, $|x - y| \sim 2^{-J}$, so for any integer N , there exists $C_N > 0$ such that $|(\mathcal{F}^{-1}\psi^{-s})(2^j(x - y))| \leq C_N 2^{-(j-J)2N}$. Thus

$$\begin{aligned}
&\left| 2^{j(n+s)} \int_{3B \setminus 2B} \phi(y) f_j(y) (\mathcal{F}^{-1}\psi^{-s})(2^j(x - y)) dy \right| \\
&\leq C_N 2^{j(n+s)} 2^{-(j-J)2N} \int_{3B \setminus 2B} |f_j(y)| dy \\
&\leq C_N 2^{j(n+s-2N)} 2^{-2(-J+1)N} \|f_j\|_{L^2(F_{-J+1})} |F_{-J+1}|^{\frac{1}{2}}
\end{aligned}$$

Hence, for any integer N there exists $C_N > 0$ such that

$$\left| \left[I^{-s} \dot{\Delta}_j, \phi \right] f_j(x) \right| \leq C_N 2^{j(n+s-2N)} \sum_{\nu \geq -J+1} 2^{-2\nu N} \|f_j\|_{L^2(F_\nu)} |F_\nu|^{\frac{1}{2}}$$

Therefore

$$\begin{aligned}
(27) \quad \left\| \left[I^{-s} \dot{\Delta}_j, \phi \right] f_j \right\|_{L^2(B)} &\leq C_N 2^{j(n-2N)} 2^{js} |B|^{\frac{1}{2}} \\
&\quad \times \sum_{\nu \geq -J+1} 2^{\nu(-2N + \frac{\lambda+n}{2})} \frac{1}{|F_\nu|^{\frac{\lambda}{2n}}} \|f_j\|_{L^2(F_\nu)} \\
&\leq C_N 2^{(j-J)(n-2N)} 2^{js} 2^{-J\frac{\lambda}{2}} \sum_{\nu \geq 1} 2^{\nu(-2N + \frac{\lambda+n}{2})} \\
&\quad \times \frac{1}{|F_{\nu-J}|^{\frac{\lambda}{2n}}} \|f_j\|_{L^2(F_{\nu-J})}
\end{aligned}$$

We choose N large and we use inequalities (26) and (27) together with Lemma 14 to deduce

$$\begin{aligned}
&\frac{1}{|B|^{\frac{\lambda}{n}}} \sum_{j \geq J} \sum_{k=j-1}^{j+1} \left\| I^{-s} \dot{\Delta}_j f_k \right\|_{L^2(B)}^2 \\
&\leq C \frac{1}{|B|^{\frac{\lambda}{n}}} \sum_{j \geq J} 2^{2js} \|f_j\|_{L^2(3B)}^2 + \sum_{j \geq J} C_N 2^{2(j-J)(n-2N)} 2^{2js} \\
&\quad \times \left\{ \sum_{\nu \geq 1} 2^{\nu(-2N + \frac{\lambda+n}{2})} \frac{1}{|F_{\nu-J}|^{\frac{\lambda}{2n}}} \|f_j\|_{L^2(F_{\nu-J})} \right\}^2
\end{aligned}$$

$$\begin{aligned}
&\leq C \frac{1}{|B|^{\frac{\lambda}{n}}} \sum_{j \geq J} 2^{2js} \|f_j\|_{L^2(3B)}^2 \\
&\quad + C_N \sum_{j \geq J} \left\{ \sum_{\nu \geq 1} 2^{\nu(-2N + \frac{\lambda+n}{2})} \frac{2^{js}}{|F_{\nu-J}|^{\frac{\lambda}{2n}}} \|f_j\|_{L^2(F_{\nu-J})} \right\}^2 \\
&\leq C \frac{1}{|B|^{\frac{\lambda}{n}}} \sum_{j \geq J} 2^{2js} \|f_j\|_{L^2(3B)}^2 + C'_N \sup_{\nu \geq 1} \frac{1}{|F_{\nu-J}|^{\frac{\lambda}{n}}} \\
&\quad \times \sum_{j \geq J-\nu-1} 2^{2js} \|f_j\|_{L^2(F_{\nu-J})}^2
\end{aligned}$$

Finally

$$\frac{1}{|B|^{\frac{\lambda}{n}}} \sum_{j \geq J} \sum_{k=j-1}^{j+1} \left\| I^{-s} \dot{\Delta}_j f_k \right\|_{L^2(B)}^2 \leq C \sup_B \frac{1}{|B|^{\frac{\lambda}{n}}} \sum_{j \geq J} 2^{2js} \|f_j\|_{L^2(B)}^2$$

The proof of Lemma 25 is complete. $\square$

Proposition 26. *Let $s \in \mathbf{R}, 0 \leq \lambda < n+p$ and $2 \leq p < +\infty$. The space $I^s(\dot{\mathcal{L}}^{\cdot, \lambda})$ is continuously embedded in the space $\dot{\mathcal{L}}^{\cdot, \lambda, s}(\mathbf{R}^n)$.*

Proof. Let $f \in I^s(\dot{\mathcal{L}}^{\cdot, \lambda})$. If we set $g = I^{-s}f$ and $f_j = \dot{\Delta}_j f$, then $f_j = I^s \dot{\Delta}_j g$. We show that

$$(28) \quad \frac{1}{|B|^{\frac{\lambda}{n}}} \sum_{j \geq J} 2^{pjs} \left\| I^s \dot{\Delta}_j g \right\|_{L^p(B)}^p \leq C \|g\|_{\dot{\mathcal{L}}^{\cdot, \lambda}(\mathbf{R}^n)}^p$$

As for Lemma 24 we decompose $g = g_1 + g_2 + g_3$ with $g_1 = m_{2B}g$, $g_2 = (g - m_{2B}g)\chi_{2B}$ and $g_3 = (g - m_{2B}g)\chi_{\mathbf{R}^n \setminus 2B}$ where B is a ball of $\mathbf{R}^n$ of radius $2^{-J}, J \in \mathbf{Z}$.

The function g_1 is constant, $\dot{\Delta}_j g_1 = 0$ for any j . Then inequality (28) is valid for g_1 . On the other hand $g_2 \in L^p(\mathbf{R}^n)$ and $\|g_2\|_{L^p(\mathbf{R}^n)} \leq C|B|^{\frac{\lambda}{pn}} \|g\|_{\dot{\mathcal{L}}^{\cdot, \lambda}(\mathbf{R}^n)}$, we deduce from Lemma 20

$$\begin{aligned}
\sum_{j \geq J} 2^{pjs} \left\| I^s \dot{\Delta}_j g_2 \right\|_{L^p(B)}^p &\leq \sum_{j \in \mathbf{Z}} 2^{pjs} \left\| I^s \dot{\Delta}_j g_2 \right\|_{L^p(\mathbf{R}^n)}^p \\
&\leq C \sum_{j \in \mathbf{Z}} \left\| \dot{\Delta}_j g_2 \right\|_{L^p(\mathbf{R}^n)}^p \\
&\leq C \|g_2\|_{\dot{B}_{p,p}^0(\mathbf{R}^n)}^p \\
&\leq C \|g_2\|_{L^p(\mathbf{R}^n)}^p \quad (p \geq 2)
\end{aligned}$$

$$\leq C|B|^{\frac{\lambda}{n}} \|g\|_{\mathcal{L}^{p,\lambda}(\mathbf{R}^n)}^p$$

Inequality (28) holds for g_2 .

Setting $h_j = I^s \Delta_j g_3$, we show in the same way as for inequality (23) that

$$|h_j(x)| \leq C \|g\|_{\mathcal{L}^{p,\lambda}(\mathbf{R}^n)} 2^{J-j-j_s} |B|^{\frac{1}{p}(\frac{\lambda}{n}-1)}$$

holds for any $x \in B$ and $j \geq J$. Therefore inequality (28) holds for g_3 . $\square$

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