Extremal points without compactness in $L^1(\mu)$

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Abstract. We investigate the existence of extremal points and the Krein-Milman representation $A = \overline{\text{co}} \text{Ext} A$ of bounded convex subsets of $L^1(\mu)$ which are only closed with respect to the topology of μ -a.e. convergence.

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1. Introduction

As shown in [2] convex bounded subsets of $L^1(\mu)$ that are closed with respect to the topology of μ -a.e. convergence enjoy several properties that are usually a consequence of the compactness in other vector topologies: Optimization without compactness, separation, Mazur's convergence.

In this paper we show that these properties also include the existence of extremal points (**Theorem 5** below), provided the set is bounded from below. We also investigate bounded, closed, convex subsets $A \subset L^1(\mu)$ that fulfill the Krein-Milman representation $A = \overline{\text{co}} \text{Ext} A$, and compare the result versus weak compactness of A ; more precisely we provide an example of a μ -closed, convex set bounded from below that is not weakly compact.

We also discuss the stability of μ -closedness with respect to the Kuratowski sum of sets, and derive some sharper properties concerning the extremal structure of sets in $L^1(\mu)$ having μ -closed extremal faces.

2. Preliminaries

Let (Ω, Σ, μ) be a σ -finite positive measure space.

In our notation, $\|\cdot\|_1$ will denote the usual norm on $L^1(\mu)$, B_1 will denote the usual unit ball.

We shall be concerned with the topology of convergence in measure in $L^1(\mu)$. We remind that this topology is induced by the metric

$$\rho(x, y) = \int_{\Omega} \frac{|x - y|}{1 + |x - y|} d\mu.$$

and it induces a vector topology on $L^1(\mu)$.

The convergence in this topology is equivalent to μ -a.e convergence.

Clearly $\rho(x, 0) \leq \|x\|_1$.

A set $A \subset L^1(\mu)$ is μ -closed if for every sequence (x_n) from A that μ -converges to some x in $L^1(\mu)$, $x \in A$.

It is then clear that if A is μ -closed then it is also $\|\cdot\|_1$ -closed, (and therefore complete).

In [2] several results substituting the compactness in the ρ -topology of $L^1(\mu)$ have been stated; we bound ourselves to those that we shall need in this paper.

Theorem 1. *Let A and B be two non-empty, μ -closed, convex sets in $L^1(\mu)$, such that $A \cap B = \emptyset$, and assume that one of them is norm-bounded. Then there exists a continuous linear functional that strictly separates A and B .*

Theorem 2. *If a collection of norm bounded, non empty convex sets in $L^1(\mu)$ which are μ -closed has the finite intersection property, then it has a non -empty intersection.*

Theorem 3. *Any convex functional, which is lower semicontinuous with respect to μ -convergence, attains its minimum on any convex, norm-bounded subset A of $L^1(\mu)$ which is closed in measure.*

Theorem 4. *If (x_n) is a norm-bounded sequence in $L^1(\mu)$, then there exist a function $x \in L^1(\mu)$ and a subsequence (x_{n_k}) such that any of its*

subsequences $(y_m)_m$ has the strong law of large numbers

$$\lim_{p \rightarrow \infty} \frac{1}{p} \sum_{i=1}^p y_i = x \text{ } \mu\text{-a.e.}$$

3. Extremal structure for sets closed in measure

Our next result extends the classical Krein-Milman Theorem in $L^1(\mu)$.

From now on by $[L^1(\mu)]^+$ we shall mean the usual cone of μ -almost everywhere non-negative functions

Theorem 5. *Let A be a non empty convex subset of $L^1(\mu)$ which is norm-bounded, μ -closed and order bounded from below (namely, there exists some $h \in L^1(\mu)$ such that $A \subset -|h| + [L^1(\mu)]^+$). Then $\text{Ext } A \neq \emptyset$.*

Proof. The proof goes along the same lines of the classical proof of Krein Milman Theorem, with slight adaptements to the present case; therefore we shall just briefly sketch it.

First define the class

$$\mathcal{P} = \{P \subseteq A, \text{ with } P \neq \emptyset, \mu\text{-closed, convex and extremal for } A\},$$

and note that it is non empty, partially ordered by inclusion, and according to **Theorem 2** contains the intersection of every totally ordered subfamily.

Hence, by Zorn's Lemma, there exists a minimal element P_0 of $\mathcal{P}$.

The rest of the proof consists in showing that P_0 is a singleton.

As usual, one assumes this to be false, and therefore suppose that P_0 contains two points $x \neq y$. Then, there would exist an element $x^* \in L^\infty(\mu)$ (the dual of our space) such that

$$\int_{\Omega} x^* x d\mu \neq \int_{\Omega} x^* y d\mu.$$

Furthermore, we can always choose $x^* \in [L^\infty(\mu)]^+$.

Let

$$m := \inf_{x \in P_0} \int_{\Omega} x^* x d\mu, \quad P_1 = \left\{ x \in A : \int_{\Omega} x^* x d\mu = m \right\}.$$

The map $\varphi : L^1(\mu) \rightarrow \mathbb{R}$ defined as $\varphi(x) = \int_{\Omega} x^* x d\mu$ is linear and therefore convex; moreover, by Fatou's Lemma it is lower semicontinuous with respect to the convergence in measure.

Then from **Theorem 3** the set P_1 is non empty.

Moreover, since either $x \notin P_1$ or $y \notin P_1$ it is clear that the inclusion $P_1 \subset P_0$ is a strict one. Since $P_1 = A \cap (x^*)^{-1}(\{m\})$ and x^* is linear, P_1 is immediately convex. Again an application of Fatou's Lemma proves that P_1 is closed in measure.

Finally, the standard “transitivity” argument:

$[P_1 \text{ is extremal for } P_0] + [P_0 \text{ is extremal for } A] \implies [P_1 \text{ is extremal for } A]$ leads to the conclusion that $P_1 \in \mathcal{P}$, and it is properly included in the minimal element P_0 ; a contradiction. $\square$

A natural objection would be: “*Is the class of subsets of $L^1(\mu)$ satisfying the assumptions of **Theorem 5** really larger than that of weakly compact sets?*”

The answer to this question is *Yes* as the following example shows:

Example 1. Let A be the intersection of B_1 with the positive cone of $L^1(\mu)$; then A is convex, and, from Fatou's Lemma, it is also μ -closed. However A is not weakly compact, otherwise the whole unit ball B_1 should be.

Krein-Milman Theorem is usually applied in the form of its Corollary, stating that weakly-compact convex sets coincide with the strong closure of the convex hull of their extremal points.

The above example shows that the same Corollary does not hold in general for sets that are μ -closed. It is therefore interesting to investigate which further property should A enjoy to obtain the same characterization. The remaining part of this section will be devoted to this question.

Definition 1. Let A be a subset of a normed space X ; we shall denote by $D_A \subset X^*$ the set $D_A = \{x^* \in X^* \setminus \{0\} \mid x^* \text{ attains its minimum on } A\}$. Let A be such that $D_A \neq \emptyset$; then, for $x^* \in D_A$ we shall call the set

$$F_A(x^*) = \left\{ x \in A \mid x^*(x) = \min_{y \in A} x^*(y) \right\}$$

a *face* of A .

We also recall the following result

Theorem 6. ([1]) *For a closed convex subset A of a Banach space X , let $x^* \in X^* \setminus \{0\}$, be bounded from above on A and let $\varepsilon > 0$ and $x_1 \in A$ be such that*

$$\sup\{x^*(x), x \in A\} < x^*(x_1) + \varepsilon.$$

Then for every $\lambda > 0$ there exists $x_o \in A$ and $x_o^* \in X^*$ such that

$$\begin{aligned} x_o^*(x_o) &= \sup\{x_o^*(x), x \in A\}; \\ \|x_1 - x_o\| &\leq \lambda; \end{aligned}$$

$$\|x^* - x_o^*\| \leq \frac{\varepsilon}{\lambda}.$$

Proposition 1. *Let E be a Banach space, and let A be a non empty, norm-bounded, closed and convex subset of E such that $D_A \neq \emptyset$. If for each $x^* \in D_A$ the face $F_A(x^*)$ has extreme points, then*

$$A = \overline{\text{co}} \text{Ext} A.$$

Proof. We first observe that, as a consequence of **Theorem 6** D_A is dense in $F(A) = \{x^* \in E^* \mid x^* \text{ is bounded from below in } A\}$.

Moreover, $\text{Ext} A \neq \emptyset$. In fact, since by assumption every D_A -face of A has extreme points, and again it is an extremal set for A , the usual transitivity argument shows that the extreme points of the faces of A are extreme points of A .

Let $K = \overline{\text{co}} \text{Ext} A$; clearly $K \subset A$. Assume by contradiction that there exists $x_o \in A \setminus K$.

From the Separation Theorem, and the density of D_A , there shall exist $x^* \in D_A$ and $\alpha \in \mathbb{R}$ such that

$$(1) \quad x^*(x_o) < \alpha < \inf_{x \in K} x^*(x).$$

From the assumptions the set $F_A(x^*)$ is non-empty and has extreme points.

Now $F_A(x^*) \cap \text{Ext} A = \emptyset$. In fact, assume by contradiction that $x_1 \in F_A(x^*) \cap \text{Ext} A$, exists. As $x_1 \in \text{Ext} A \subset K$ then $x^*(x_1) > \alpha$, while $x^*(x_o) < \alpha$ and so $x^*(x_1 - x_o) > 0$. Hence, writing $x_1 = x_o + (x_1 - x_o)$ we find $x^*(x_1) > x^*(x_o)$ thus contradicting the assumption that $x_1 \in F_A(x^*)$.

But $F_A(x^*)$ is extremal for A , and therefore, $\text{Ext} F_A(x^*) \subset \text{Ext} A$, a contradiction. $\square$

Let now consider again $X = L^1(\mu)$.

We can now give a partial answer to the original question, with the following

Corollary 1. *Let A be a non empty, norm-bounded, μ -closed, convex subset of $L^1(\mu)$ which is also order bounded from below. If the D_A -faces of A are μ -closed, then*

$$A = \overline{\text{co}} \text{Ext} A.$$

Proof. We first note that A is also closed. Moreover from **Theorem 3** and Fatou's Lemma, the dual cone $(X^*)^+ \subset D_A$ and thus $D_A \neq \emptyset$.

It is also clear that each D_A -face of A fulfills the assumptions of **Theorem 5** and therefore all the conditions of **Proposition 1** are satisfied. $\square$

It is interesting to note that in force of **Theorem 1** under the same assumptions of **Proposition 1** it could have also been proven along the same proof that

$$A = \text{cl}_\mu \text{co Ext } A.$$

Again it is natural to ask whether the conditions in **Corollary 1** do not in fact imply that A is weakly compact. The answer is also in this case in the negative as the following example shows.

Example 2. Let (Ω, Σ, μ) be the Lebesgue space $([0, 1], \mathcal{B}, \mu)$ with μ denoting the usual Lebesgue measure.

We shall apply a procedure similar to [4], (**Theorem 3.1.6**, p. 107) to equip $L^1(\mu)$ with an equivalent rotund norm $|\cdot|$.

Consider the countable family

$$\mathcal{F} = \{\mathbf{1}_{[a,b]}, a, b \in \mathbf{Q} \cap [0, 1], a \leq b\}$$

as a subset of $L^\infty(\mu)$.

It is then clear that $\mathcal{F}$ is total on $L^1(\mu)$, namely, if $x \in L^1(\mu)$ is such that $\int_0^1 f \cdot x d\mu = 0$ for every $f \in \mathcal{F}$, then $x = 0$: in fact, the family of intervals $\mathcal{I} = \{[a, b], a, b \in \mathbf{Q} \cap [0, 1], a \leq b\}$ is a semi-ring generating the whole Borel σ -algebra $\mathcal{B}$, and so we can apply, for instance, [3] (p. 25, Corollary).

Let $\mathcal{F} = \{x_1^*, x_2^*, \dots\}$ be any enumeration of the family $\mathcal{F}$.

Now, as in [4], consider the mapping $L : L^1(\mu) \rightarrow \ell^2$ defined by

$$L(x) = \left(\frac{1}{2} x_1^*(x), \frac{1}{2^2} x_2^*(x), \dots, \frac{1}{2^n} x_n^*(x), \dots \right).$$

Then L is linear and continuous, and since $\mathcal{F}$ is total, L is one-to-one.

Since $(\ell^2, \|\cdot\|_2)$ is rotund, the norm on $L^1(\mu)$ defined by

$$|x| = \|x\|_1 + \|L(x)\|_2$$

is an equivalent rotund norm for $L^1(\mu)$.

Let $B = \{x \in L^1(\mu) : |x| \leq 1\}$ and let A be the intersection of B with the positive cone of $L^1(\mu)$.

First note that A is μ -closed. Indeed, if $(x_k)_k \subset A$ μ -converges to $x_o \in L^1(\mu)$, then x_o is the limit μ -a.e. of functions in the cone, namely of μ -a.e. non-negative functions, and thus x_o is in the cone also.

Also, by the μ -a.e. convergence, the Fatou's Lemma yields that

$$(2) \quad \|x_o\|_1 = \int_0^1 x_o d\mu \leq \liminf_k \int_0^1 x_k d\mu = \liminf_k \|x_k\|_1$$

and, for each $n \in \mathbb{N}$ fixed, and $x_n^* \in \mathcal{F}$ also

$$0 \leq \int_0^1 x_n^* x_o d\mu \leq \liminf_k \int_0^1 x_n^* x_k d\mu,$$

or else

$$(3) \quad |x_n^*(x_o)| \leq \liminf_k |x_n^*(x_k)|.$$

Let $(x_{k_p})_p$ be a subsequence such that $\liminf_k \|L(x_k)\|_2^2 = \lim_p \|L(x_{k_p})\|_2^2$.

We shall show that

$$(4) \quad \|L(x_o)\|_2^2 \leq \lim_p \|L(x_{k_p})\|_2^2$$

which together with (2) will prove that $|x_o| \leq 1$.

Clearly $(x_n^*(x_{k_p}))_p$ converges for every $n \in \mathbb{N}$, and from (3) we have that

$$|x_n^*(x_o)| \leq \lim_p |x_n^*(x_{k_p})|$$

for every $n \in \mathbb{N}$. Thus, for every $r \in \mathbb{N}$

$$\lim_p \|L(x_{k_p})\|_2^2 = \lim_p \sum_{n=1}^{\infty} \frac{|x_n^*(x_{k_p})|^2}{2^{2n}} \geq \sum_{n=1}^r \frac{\lim_p |x_n^*(x_{k_p})|^2}{2^{2n}} \geq \sum_{n=1}^r \frac{|x_n^*(x_o)|^2}{2^{2n}}$$

whence (4) follows.

As $0 \in A$, for every $x^* \in D_A$ we have necessarily that $\min\{x^*(x), x \in A\} \leq 0$. If x^* is such that $\min\{x^*(x), x \in A\} = 0$ then x^* is a positive linear functional, and therefore the face $F_A(x^*)$ is μ -closed by means of Fatou's Lemma.

Let now $x^* \in D_A$ be such that $m(x^*) := \min\{x^*(x), x \in A\} < 0$; then the face $F_A(x^*)$ reduces to a singleton, by means of the rotundity of $|\cdot|$. In fact if $x \in F_A(x^*)$ then $x \neq 0$ and necessarily $|x| = 1$, otherwise considering $u_o = \frac{x}{|x|} \in A$ we would find

$$x^*(u_o) = \frac{1}{|x|} x^*(x) = \frac{1}{|x|} m(x^*) < m(x^*).$$

Therefore, if $x_1, x_2 \in F_A(x^*)$ then necessarily $\frac{x_1 + x_2}{2} \in F_A(x^*)$ which in turn implies $\left| \frac{x_1 + x_2}{2} \right| = 1$ thus contradicting the rotundity of $|\cdot|$.

Therefore for every functional in D_A the corresponding face is μ -closed. Thus all the assumptions of **Corollary 1** are satisfied, but, as in **Example 1**, A cannot be weakly compact.

4. Structure theorems for convex μ -closed sets

In this section we collect some properties of sets that are convex and μ -closed. It is well known that the sum of two compact sets is compact. We shall now examine whether μ -closedness is preserved by the sum.

Proposition 2. *Let $A, B \subset L^1(\mu)$ be two convex, μ -closed sets, with one norm-bounded. Then $A + B$ is convex and norm closed.*

Proof. Suppose for instance that A is norm bounded.

Let $a_n + b_n \rightarrow x$; from **Theorem 4** there exist $(a_{n_k})_k, a_o \in A$ such that

$$\frac{1}{k} \sum_{p=1}^k a_{n_p} \rightarrow a_o \quad \mu\text{-a.e.}$$

On the other side $a_{n_k} + b_{n_k} \rightarrow x$ and therefore the arithmetic means also converge strongly to x . Therefore

$$\frac{1}{k} \sum_{p=1}^k (a_{n_p} + b_{n_p}) \rightarrow x \quad \mu\text{-a.e.};$$

hence

$$\frac{1}{k} \sum_{p=1}^k b_{n_p} \rightarrow x - a_o \quad \mu\text{-a.e.}$$

and since B is μ -closed, $x - a_o \in B$, which in turn gives $x \in A + B$ as declared. $\square$

In order to obtain the μ -closedness of the sum we need more

Proposition 3. *Let $A, B \subset L^1(\mu)$ be two convex, μ -closed, norm-bounded sets. Then $A + B$ is convex and μ -closed.*

Proof. Let $a_n + b_n \rightarrow x$; since $(a_n + b_n)_n$ is bounded, again from **Theorem 4** there exist subsequences $(a_{n_k})_k, (b_{n_k})_k$ such that

$$(i) \quad a_{n_k} + b_{n_k} \rightarrow x \quad \mu\text{-a.e.} \quad (\text{and therefore } \frac{1}{k} \sum_{p=1}^k (a_{n_k} + b_{n_k}) \rightarrow x \quad \mu\text{-a.e.});$$

$$(ii) \quad \frac{1}{k} \sum_{p=1}^k a_{n_k} \rightarrow a_o \in A \quad \mu\text{-a.e.};$$

$$(iii) \quad \frac{1}{k} \sum_{p=1}^k b_{n_k} \rightarrow b_o \in B \quad \mu\text{-a.e.}$$

Then $x = a_o + b_o$ that is $x \in A + B$. $\square$

The μ -closedness of the sum can be proven also replacing the μ -closedness of one summand by weak compactness.

Proposition 4. *If μ is finite, $A, B \subset X$ are such that:*

- (i) *A is convex, μ -closed and norm bounded;*
- (ii) *B is weakly compact.*

Then $A + B$ is μ closed.

Proof. Let $(b_n)_n \subset B$; since B is bounded, from **Theorem 4** there exists a subsequence $(b_{n_k})_k$ weakly converging to some $b_o \in B$ and such that the sequence $(\beta_n)_n$ of its arithmetic means converge μ -almost everywhere to some $b' \in X$.

By means of Egoroff Theorem $b_o = b$ μ -a.e.

Then the same proof of **Proposition 3** applies. $\square$

Theorem 7. *If H is a non empty proper subset of X , which is μ -closed and convex, then H^c is unbounded.*

Proof. Without loss of generality we can assume that $0 \notin H$.

Indeed, if $x_o \in X \setminus H$, and $F = x_o - H$ then $0 \notin F$ and F is convex and μ -closed.

Since F^c is norm bounded iff H^c is norm bounded (for $F^c = x_o - H^c$), if the assertion holds for F^c then it holds for H^c .

We assume by contradiction that H^c is norm bounded, and we shall prove that this yields that every μ -closed, bounded and convex subset of X would be totally bounded, and therefore compact. But, by means of **Example 1** or **Example 2** this consequence is false.

Assuming H^c norm-bounded, there exists $R > 0$ such that $H^c \subset RB_1$. Let $\varepsilon > 0$ be fixed, and define $B = \frac{\varepsilon}{R} H^c$. Then B is norm bounded and $0 \in B$. Furthermore $B^c = \frac{\varepsilon}{R} H$ is μ -closed and convex

Let C be any μ -closed, bounded and convex subset of X .

Claim 1. *There exist finitely many $z_1, \dots, z_n \in C$ such that*

$$C \subset \bigcup_{i=1}^n (z_i + B).$$

Proof. First we show that $\bigcap_{z \in C} [C \cap (z + B^c)] = \emptyset$.

In fact, if $x \in \bigcap_{z \in C} [C \cap (z + B^c)]$ exist, then $x - C \subset B^c$. In particular then $0 = x - x \in B^c$, while we have $0 \in B$; contradiction.

Since each $C \cap (z + B^c)$ is μ -closed, convex and norm bounded, **Theorem 2** implies that there exist $z_1, \dots, z_n \in C$ such that $\bigcap_{i=1}^n [C \cap (z_i + B^c)] = \emptyset$.

Then $C \subset \bigcup_{i=1}^n (z_i + B)$. Indeed, if $x \in C$ then $x \notin z_{i_o} + B^c$ for some $i_o \in \{1, \dots, n\}$ and hence $x - z_{i_o} \notin B^c$ which in turn implies that $x - z_{i_o} \in B$ namely, $x \in z_{i_o} + B$.

Now, as $B = \frac{\varepsilon}{R} H^c \subset \varepsilon B_1$, by **Claim 1** we find

$$C \subset \bigcup_{i=1}^n (z_i + \varepsilon B_1). \quad \square$$

5. Extreme points of the sum of two sets

We shall now investigate how extremal points of A and B relate to extremal points of $A + B$. The following two results can be proven analogously to [5] (Theorem 1, p. 5).

Proposition 5. *Let A and B be convex. If $\text{Ext}(A + B) \neq \emptyset$ then for every $P \in \text{Ext}(A + B)$ there exist a unique $x \in A$ and a unique $y \in B$ such that $P = x + y$; moreover $x \in \text{Ext } A$, and $y \in \text{Ext } B$ (and thus $\text{Ext } A$ and $\text{Ext } B$ are non empty).*

Proposition 6. *Let A and B be convex. Suppose that $P \in \text{Ext } A + \text{Ext } B$ can be uniquely written as the sum $P = a + b$. Then $P \in \text{Ext}(A + B)$*

We recall the following definition from [5]

Definition 2. Given two sets A and B admitting extremal points, we denote by $\text{Ext}_B A$ the set of extremal points x of A for which there exists $y \in \text{Ext } B$ such that $x + y \in \text{Ext}(A + B)$.

Theorem 8. *Let A and B be two non empty convex subsets of $L^1(\mu)$ such that*

- (i) A is closed, norm bounded and bounded from below;
- (ii) B is compact and bounded from below;
- (iii) the D_A -faces of A are μ -closed.

Then

$$A = \overline{\text{co}} \text{Ext}_B A.$$

Proof. Observe that $A + B$ is closed, convex and bounded.

Also, from the compactness of B , we find that $D_{A+B} = D_A$.

The inclusion $D_A \subset D_{A+B}$ is an immediate consequence of the compactness of B . Conversely, if $x^* \in D_{A+B}$ and $a_o \in A, b_o \in B$ are such that $x^*(a_o + b_o) \leq x^*(a + b)$ for every $a \in A$ and $b \in B$, then

$$x^*(a_o) = x^*(a_o + b_o) - x^*(b_o) \leq x^*(a + b_o) - x^*(b_o) = x^*(a)$$

for each $a \in A$ whence $x^* \in D_A$.

Moreover, the same easy computation yields that for every $x^* \in D_A$

$$(5) \quad F_{A+B}(x^*) = F_A(x^*) + F_B(x^*).$$

Since $F_B(x^*)$ is compact, it follows from **Proposition 3** that $F_{A+B}(x^*)$ is μ -closed; moreover it is convex, and by (i) and (ii) norm bounded and bounded from below. Therefore from **Theorem 5** $\text{Ext} F_{A+B}(x^*) \neq \emptyset$.

Then we can apply **Proposition 1** to $A + B$ and conclude that

$$(6) \quad A + B = \overline{\text{co}} \text{Ext}(A + B).$$

We can now repeat the proof of Theorem I.1.2 in [5].

Define $C = \overline{\text{co}} \text{Ext}_B A$.

Then $\text{Ext}(A + B) \subset C + B \subset A + B$.

In fact, let $P \in \text{Ext}(A + B)$. By **Proposition 6** there exist a unique $x \in \text{Ext} A$ and a unique $y \in \text{Ext} B$ such that $P = x + y$. Moreover, by its very definition, $x \in \text{Ext}_B A$. Hence $P \in \text{Ext}_B A + B \subset C + B$.

Since $\text{Ext}_B A \subset \text{Ext} A \subset A$, and A is convex and closed, then $C \subset A$, whence the second inclusion follows. $\square$

Also the following version holds

Theorem 9. *If μ is finite, and A and B are two non empty convex subsets of $L^1(\mu)$ such that*

- (i) *A is closed, norm bounded and bounded from below;*
- (ii) *B is weakly compact and bounded from below;*
- (iii) *the D_A -faces of A are μ -closed.*

Then $A = \overline{\text{co}} \text{Ext}_B A$, and $\text{Ext}_A B$ is dense in $\text{Ext} B$ (whence $B = \overline{\text{co}} \text{Ext}_A B$).

Proof. We can obtain (5) in a completely analogous way, and then, replacing **Proposition 3** with **Proposition 4** conclude the proof of the first part of the statement as above.

To prove the second part consider $C = \overline{\text{co}} \text{Ext}_A B$, and observe that as in **Theorem 8** we can prove that $B = C$. Then, as in [5] Theorem I.2.2,

since $M = \text{Ext}_A B$ is such that $C = \overline{\text{co}} M = B$ is weakly compact, from Milman's Theorem (see [6], §25.1 (7)),

$$\text{Ext } B = \text{Ext } C \subset \text{clExt}_A B. \quad \square$$

We can replace the lower order boundedness in **Theorem 8** in the following sense

Theorem 10. *Let A and B be two non empty convex subsets of $L^1(\mu)$ such that*

- (i) *A is closed, norm bounded and bounded from below;*
- (ii) *B is compact and strictly convex;*
- (iii) *the D_A -faces of A are μ -closed.*

Then

$$A = \overline{\text{co}} \text{Ext}_B A.$$

Proof. All what needs to be adjusted to this new situation is how one gets (6) for we do not know whether $F_{A+B}(x^*)$ is bounded from below.

Clearly $F_B(x^*)$ is convex and compact, and so $\text{Ext} F_B(x^*) \neq \emptyset$, by the usual Krein-Milman Theorem, while $\text{Ext} F_A(x^*) \neq \emptyset$ in force of (i), (iii) and **Theorem 5**.

We shall now show that for every $x^* \in D_A$, $\text{Ext} F_{A+B}(x^*) \neq \emptyset$. This will derive from the inclusion

$$\text{Ext } F_A(x^*) + \text{Ext } F_B(x^*) \subset \text{Ext } F_{A+B}(x^*).$$

Let $x \in \text{Ext } F_A(x^*)$, $y \in \text{Ext } F_B(x^*)$, and assume by contradiction, that $x + y \notin \text{Ext } F_{A+B}(x^*)$.

Then there should exist $a_1, a_2 \in F_A(x^*)$ and $b_1, b_2 \in F_B(x^*)$ such that $a_i + b_i \in F_{A+B}(x^*)$ for $i = 1, 2$ and

$$x + y = \frac{a_1 + a_2 + b_1 + b_2}{2} = \frac{a_1 + a_2}{2} + \frac{b_1 + b_2}{2}.$$

But, since $F_B(x^*)$ is a singleton, $b_1 = b_2 = y$ and so, since $x \in \text{Ext} F_A(x^*)$, $a_1 = a_2 = x$. Hence, from **Proposition 6** $(x + y) \in \text{Ext}_{A+B}(x^*)$.

It should be noted that the general inclusion $\text{Ext}(A + B) \subset \text{Ext } A + \text{Ext } B$ proven in **Proposition 5** applied to (5) gives also the inclusion of $\text{Ext} F_{A+B}(x^*)$ in $\text{Ext} F_A(x^*) + \text{Ext} F_B(x^*)$, and therefore

$$\text{Ext} F_{A+B}(x^*) = \text{Ext} F_A(x^*) + \text{Ext} F_B(x^*). \quad \square$$

Also in this case we can replace the compactness of B with weak compactness provided μ is finite.

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