A note on maximal operator on $\ell^{\{p_n\}}$ and $L^{p(x)}(\mathbb{R})$

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Abstract. We consider a discrete analogue of Hardy-Littlewood maximal operator on the generalized Lebesgue space $\ell^{\{p_n\}}$ of sequences defined on $\mathbb{Z}$. It is known a necessary and sufficient condition $\mathcal{P}$ which guarantees an existence of a real number $p > 1$ such that the norms in the space $\ell^{\{p_n\}}$ and in the classical space ℓ^p are equivalent. Of course, this condition immediately implies the boundedness of maximal operator on $\ell^{\{p_n\}}$ and, moreover, $\lim_{|n| \rightarrow \infty} p_n = p$. We construct two examples of sequences $\{p_n\}$ satisfying $\lim_{|n| \rightarrow \infty} p_n = p$ in this paper. In the first example the maximal operator is unbounded on $\ell^{\{p_n\}}$ and the sequence $\{p_n\}$ from the second example does not satisfy $\mathcal{P}$ but the maximal operator is bounded. Moreover, it is known a sufficient integral condition to a behavior of a function $p(x)$ at infinity which guarantees the boundedness of the maximal operator on $L^{p(\cdot)}(\mathbb{R}^n)$. As a main result of this paper we construct a function $p(x)$ which does not satisfy this integral condition nevertheless the maximal operator is bounded.

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1. Introduction

The generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and its discrete analogy $\ell^{\{p_n\}}$ are investigated more and more in recent years. We refer to [17] for the establishment of the fundamental properties of these spaces, to [7] for some properties of the norm of $L^{p(x)}(\Omega)$ and to [22, 23] for some motivation where non-linear system of partial differential equation with coefficients of variable rate of growth are studied.

In [8] and [20] a discrete version $\ell^{\{p_n\}}$ of $L^{p(x)}$ is introduced and some interesting properties of these spaces are proved. Especially, a necessary and sufficient condition to bounded sequences $\{p_n\}$ and $\{q_n\}$ is found to guarantee the equivalency of norms in spaces $\ell^{\{p_n\}}$ and $\ell^{\{q_n\}}$. In [9, 10] the imbedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, an analogy of Sobolev imbedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, is proved. In [11] a sufficient condition for compactness of the imbedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is found.

For further information on properties of $L^{p(x)}$ spaces and singular operators spaces see [3, 4, 5, 6, 13, 14, 15, 16].

In [2] L.Diening proved that the maximal operator is bounded on $L^{p(x)}(\Omega)$ for a bounded Ω provided $-|p(x) - p(y)| \ln |x - y| \leq C$ for $|x - y| \leq \frac{1}{2}$, $x, y \in \Omega$. In [19] the authors showed that the maximal operator can be unbounded if there exists $x \in \Omega$ such that $\lim_{y \rightarrow x} |p(x) - p(y)| \ln |x - y| = -\infty$.

The partial generalization of this problem to the domain $\mathbb{R}^n$ is done in [1] where a sufficient condition for boundedness of maximal operator on $L^{p(x)}(\mathbb{R}^n)$ is found for a certain class of functions $p(x)$. A more general sufficient condition is found in [21] for a wider class of functions $p(x)$.

As a natural motivation for the investigation of the behavior of maximal operator on $L^{p(x)}(\mathbb{R})$ can serve a discrete version of maximal operator on $\ell^{\{p_n\}}$. We will consider in the next a set of all real sequences $\{a_n\}_{n \in \mathbb{Z}} := \{a_n\}$ defined on $\mathbb{Z}$. Given a real number $p \geq 1$ denote by $\mathcal{B}_p$ the set of all "exponent sequences" $\{p_n\}$, $1 \leq p_n$ and $\lim_{|n| \rightarrow \infty} p_n = p$.

It is known (see [21]) a necessary and sufficient condition $\mathcal{P}$ for sequence $\{p_n\}$ to guarantee an existence of a real number $r \geq 1$ such that norms in $\ell^{\{p_n\}}$ and in ℓ^r are equivalent. Of course, if a sequence $\{p_n\}$ satisfies $\mathcal{P}$ then by classical results on maximal operator on ℓ^r we immediately obtain the boundedness of maximal operator on $\ell^{\{p_n\}}$.

On the other hand we construct in this paper in Section 5 an example of $\{p_n\} \in \mathcal{B}_r \setminus \mathcal{P}$ such that the maximal operator is unbounded on $\ell^{\{p_n\}}$.

In connection with this example the natural problem appears. Does there exist a sequence $\{p_n\} \in \mathcal{B}_r \setminus \mathcal{P}$ such that the maximal operator is bounded on $\ell^{\{p_n\}}$? In Section 5 an answer to this question is given and it is positive.

We construct a $\{p_n\} \in \mathcal{B}_r \setminus \mathcal{P}$ such that the maximal operator is bounded on $\ell^{\{p_n\}}$.

In [8] an example of a Lipschitz function $p(x)$ defined on $\mathbb{R}$ is given such that M is unbounded on $L^{p(x)}(\mathbb{R})$. But p has no limit at infinity.

In [21] there is found a sufficient condition $\mathcal{P}$ to the behavior of a function p at infinity to guarantee the boundedness of M on $L^{p(x)}(\mathbb{R})$. The main result of this paper is contained in Section 6 where we construct a function $p(x)$ which does not satisfy the condition $\mathcal{P}$ but M is bounded on $L^{p(x)}(\mathbb{R})$.

2. Preliminaries

In this section the basic notations, definitions and lemmas are given. We consider in this paper mappings defined on integers. Given $\mathbf{a} : \mathbb{Z} \rightarrow \mathbb{R}$ we speak about a sequence $\mathbf{a} = \{a_n\}_{n \in \mathbb{Z}}$.

Let $p \geq 1$. Denote

$$\begin{aligned} \mathcal{M} &= \{\mathbf{a}; a_n \in \mathbb{R}\}, & \mathcal{M}_+ &= \{\mathbf{a}; a_n \geq 0\} \\ \mathcal{B} &= \{\mathbf{p} \in \mathcal{M}; \text{there is } K \geq 1 \text{ such that } 1 \leq p_n \leq K\}, \\ \mathcal{B}_p &= \{\mathbf{p} \in \mathcal{B}; \lim_{|n| \rightarrow \infty} p_n = p\}. \end{aligned}$$

Definition 2.1. Let $\mathbf{p} \in \mathcal{B}$. Denote for $\mathbf{a} \in \mathcal{M}$ the Luxemburg norm by

$$\|\mathbf{a}\|_{\mathbf{p}} = \inf \left\{ \lambda > 0; \sum_{n \in \mathbb{Z}} \left| \frac{a_n}{\lambda} \right|^{p_n} \leq 1 \right\}.$$

Define a space $\ell^{\mathbf{p}}$ by

$$\ell^{\mathbf{p}} = \{\mathbf{a} \in \mathcal{M}; \|\mathbf{a}\|_{\mathbf{p}} < \infty\}.$$

Recall that for a constant sequence $\mathbf{p}$ such that $p_n = p$ we obtain the well-known classical space ℓ^p equipped with the norm

$$\|\mathbf{a}\| = \left(\sum_{n \in \mathbb{Z}} |a_n|^p \right)^{1/p}.$$

In [8] (see Lemma 2.6) the following lemma is proved.

Lemma 2.2. Let $\mathbf{p} \in \mathcal{B}$. Then

$$\ell^{\mathbf{p}} = \left\{ \mathbf{a} \in \mathcal{M}; \sum_{n \in \mathbb{Z}} |a_n|^{p_n} < \infty \right\}.$$

In what follows we will use the idea of one-sides maximal operators which was used for instance in [18]. Let us define a discrete version of maximal operator and its one-sides modifications.

Definition 2.3. Given $\mathbf{a} \in \mathcal{M}$ we define operators M^- , M^+ and M by

$$\begin{aligned}(M^-\mathbf{a})_n &= \sup_{k \leq n} \frac{1}{n-k+1} \sum_{j=k}^n |a_j|, \\ (M^+\mathbf{a})_n &= \sup_{n \leq k} \frac{1}{k-n+1} \sum_{j=n}^k |a_j| \\ (M\mathbf{a})_n &= \sup_{k \leq n \leq l} \frac{1}{l-k+1} \sum_{j=k}^l |a_j|.\end{aligned}$$

Lemma 2.4. *The inequality*

$$(M\mathbf{a})_j \leq 2 \left((M^+\mathbf{a})_j + (M^-\mathbf{a})_j \right)$$

holds for each $j \in \mathbb{Z}$ and $\mathbf{a} \in \mathcal{M}$.

Proof. Let $\mathbf{a} \in \mathcal{M}$ and fix $j \in \mathbb{Z}$. Find integers $m, n \geq 0$ such that

$$(M\mathbf{a})_j \leq \frac{2}{m+n+1} \sum_{s=j-m}^{j+n} |a_s|.$$

Now, we can write

$$\begin{aligned}(2.1) \quad (M\mathbf{a})_j &\leq \frac{2}{m+n+1} \sum_{s=j-m}^{j+n} |a_s| \\ &= \frac{2(m+1)}{m+n+1} \frac{1}{m+1} \sum_{s=j-m}^j |a_s| + \frac{2(n+1)}{m+n+1} \frac{1}{n+1} \sum_{s=j}^{j+n} |a_s| \\ &\leq 2 \left(\frac{1}{m+1} \sum_{s=j-m}^j |a_s| + \frac{1}{n+1} \sum_{s=j}^{j+n} |a_s| \right) \\ &\leq 2 \left((M^-\mathbf{a})_j + (M^+\mathbf{a})_j \right)\end{aligned}$$

which proves the lemma. $\square$

Remind that the Hardy-Littlewood maximal operator is defined by

$$\mathfrak{M}f(x) = \sup_{a < x < b} \frac{1}{b-a} \int_a^b |f(x)| dx$$

for all $f \in L^1_{loc}(\mathbb{R})$. Define one-side maximal operators by

$$\mathfrak{M}^+ f(x) = \sup_{x < b} \frac{1}{b-x} \int_x^b |f(x)| dx, \quad \mathfrak{M}^- f(x) = \sup_{a < x} \frac{1}{x-a} \int_a^x |f(x)| dx.$$

Recall a classical theorem (see [12], Theorem 21.76 (i)).

Lemma 2.5. *Let $p > 1$. Then the inequality*

$$\left(\int_{\mathbb{R}} (\mathfrak{M}^+ f(x))^p dx \right)^{1/p} \leq \frac{p}{p-1} \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{1/p}$$

holds for all $f \in L^p(\mathbb{R})$. The same inequality holds for $\mathfrak{M}^-$.

Lemma 2.6. *Let $p > 1$. Then the inequality*

$$\left(\int_{\mathbb{R}} (\mathfrak{M} f(x))^p dx \right)^{1/p} \leq \frac{2p}{p-1} \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{1/p}$$

holds for all $f \in L^p(\mathbb{R})$.

Proof. Clearly,

$$\begin{aligned} \mathfrak{M} f(x) &= \sup_{a < x < b} \frac{1}{b-a} \int_a^b |f(x)| dx \\ &= \sup_{a < x < b} \frac{1}{b-a} \left(\int_a^x |f(x)| dx + \int_x^b |f(x)| dx \right) \\ &\leq \sup_{a < x < b} \frac{1}{b-a} \int_a^x |f(x)| dx + \sup_{a < x < b} \frac{1}{b-a} \int_x^b |f(x)| dx \\ &\leq \sup_{a < x} \frac{1}{x-a} \int_a^x |f(x)| dx + \sup_{x < b} \frac{1}{b-x} \int_x^b |f(x)| dx \\ &= \mathfrak{M}^+ f(x) + \mathfrak{M}^- f(x). \end{aligned}$$

Thus, we have by Lemma 2.5

$$\begin{aligned} \left(\int_{\mathbb{R}} (\mathfrak{M} f(x))^p dx \right)^{1/p} &\leq \left(\int_{\mathbb{R}} (\mathfrak{M}^+ f(x))^p dx \right)^{1/p} + \left(\int_{\mathbb{R}} (\mathfrak{M}^- f(x))^p dx \right)^{1/p} \\ &\leq \frac{2p}{p-1} \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{1/p} \end{aligned}$$

which finishes the proof. $\square$

In what follows we will use the discrete analogue of the previous lemma for modular inequality.

Lemma 2.7. *Let $1 < p < q < \infty$. Then there exists $C_{p,q} > 0$ such that for each real number r , $p \leq r \leq q$, and for any $\mathbf{a} \in \mathcal{M}_+$ the inequality*

$$\sum_{j \in \mathbb{Z}} (M\mathbf{a})_j^r \leq C_{p,q} \sum_{j \in \mathbb{Z}} a_j^r$$

holds.

Proof. Consider first the operator M^+ . Let $\mathbf{a} \in \mathcal{M}_+$. Set $f(x) = \sum_{k \in \mathbb{Z}} a_k \chi_{(k, k+1)}(x)$. Fix $k \in \mathbb{Z}$ and $x \in (k, k+1)$. Then for any $m \geq 0$

$$\begin{aligned} \frac{1}{m+1} \sum_{j=k}^{k+m} a_j &= \frac{1}{m+1} \sum_{j=k}^{k+m} \int_j^{j+1} a_j \chi_{(j, j+1)}(x) dx \\ &= \frac{1}{m+1} \sum_{j=k}^{k+m} \int_j^{j+1} f(x) dx \\ &= \frac{1}{m+1} \int_k^{k+m+1} f(x) dx \\ &\leq \mathfrak{M}f(x). \end{aligned}$$

Thus,

$$(M^+\mathbf{a})_k = \sup_{m \geq 0} \frac{1}{m+1} \sum_{j=k}^{k+m} a_j \leq \mathfrak{M}f(x) \quad \text{for all } x \in (k, k+1)$$

and so,

$$(M^+\mathbf{a})_k^r \leq \int_k^{k+1} (\mathfrak{M}f(x))^r dx.$$

Consequently, we have by Lemma 2.6

$$\begin{aligned} \sum_{k \in \mathbb{Z}} (M^+\mathbf{a})_k^r &\leq \sum_{k \in \mathbb{Z}} \int_k^{k+1} (\mathfrak{M}f(x))^r dx \\ &= \int_{\mathbb{R}} (\mathfrak{M}f(x))^r dx \\ &\leq \left(\frac{2r}{r-1} \right)^r \int_{\mathbb{R}} (f(x))^r dx \\ &= \left(\frac{2r}{r-1} \right)^r \sum_{k \in \mathbb{Z}} \int_k^{k+1} a_k^r \chi_{(k, k+1)}(x) dx \\ &= \left(\frac{2r}{r-1} \right)^r \sum_{k \in \mathbb{Z}} a_k^r. \end{aligned}$$

Analogously we can prove an estimate

$$\sum_{k \in \mathbb{Z}} (M^- \mathbf{a})_k^r \leq \left(\frac{2r}{r-1} \right)^r \sum_{k \in \mathbb{Z}} a_k^r.$$

Using (2.1) we have

$$\begin{aligned} \sum_{k \in \mathbb{Z}} (M \mathbf{a})_k^r &\leq 2^{2r-1} \left(\sum_{k \in \mathbb{Z}} (M^+ \mathbf{a})_k^r + \sum_{k \in \mathbb{Z}} (M^- \mathbf{a})_k^r \right) \\ &\leq 2^{2r} \left(\frac{2r}{r-1} \right)^r \sum_{k \in \mathbb{Z}} a_k^r. \end{aligned}$$

Setting

$$C_{p,q} = \max_{p \leq r \leq q} 2^{2r} \left(\frac{2r}{r-1} \right)^r$$

we obtain the desired inequality. $\square$

Definition 2.8. Let $\mathbf{s} \in \mathcal{M}$. Say that $\mathbf{s}$ satisfies a condition $\mathcal{P}$ (write $\mathbf{s} \in \mathcal{P}$) if there exists a real number $c > 0$ such that

$$\sum_{\{n \in \mathbb{Z}; s_n > 0\}} s_n c^{1/s_n} < \infty.$$

Definition 2.9. Let $p > 1$. Say that $\mathbf{p} \in \mathcal{B}$ satisfies a condition $\mathcal{P}_p$ (write $\mathbf{p} \in \mathcal{P}_p$) if $\mathbf{p} - p := \{p_n - p\}_{n \in \mathbb{Z}} \in \mathcal{P}$, i.e. there exists a real number $c > 0$ such that

$$\sum_{\{n \in \mathbb{Z}; |p_n - p| > 0\}} |p_n - p| c^{1/|p_n - p|} < \infty.$$

The following basic lemma is proved in [21] (see Theorem 4.3).

Lemma 2.10. Let $\mathbf{p}, \mathbf{q} \in \mathcal{B}$ be given sequences. Then $|\mathbf{p} - \mathbf{q}| \in \mathcal{P}$ if and only if the norms in $\ell^{\mathbf{p}}$ and $\ell^{\mathbf{q}}$ are equivalent. Especially, assuming $|\mathbf{p} - \mathbf{q}| \in \mathcal{P}$ there is a constant $C > 0$ such that the implication

$$\sum_{j \in \mathbb{Z}} b_j^{q_j} \leq 1 \Rightarrow \sum_{j \in \mathbb{Z}} b_j^{p_j} \leq C$$

holds.

As an easy consequence we obtain the following two lemmas.

Lemma 2.11. Let $\mathbf{p} \in \mathcal{B}$ and $p > 1$. Then the norms in $\ell^{\mathbf{p}}$ and ℓ^p are equivalent if and only if $\mathbf{p} \in \mathcal{P}_p$.

Lemma 2.12. *Let $\mathbf{p}, \mathbf{q} \in \mathcal{B}$ be given sequences, $q_n \geq p_n$. Let C be the constant from Lemma 2.10. Then*

$$\sum_{j \in \mathbb{Z}} b_j^{q_j} \leq B \Rightarrow \sum_{j \in \mathbb{Z}} b_j^{p_j} \leq \max \left\{ 1, \sup_{j \in \mathbb{Z}} B^{\frac{p_j}{q_j}} \right\} C.$$

As an easy consequence of Lemma 2.7 and Lemma 2.11 we obtain the following lemma.

Lemma 2.13. *Let $p > 1$, $\mathbf{p} \in \mathcal{P}_p$. Then M is bounded on $\ell^{\mathbf{p}}$.*

Lemma 2.14. *Let $p > 1$. Then $\mathcal{P}_p \subset \mathcal{B}_p$.*

Proof. Let

$$\sum_{\{n \in \mathbb{Z}; |p_n - p| > 0\}} |p_n - p| c^{1/|p_n - p|} < \infty$$

for some positive c . Without lose of generality we can suppose $c \leq 1$. Assume that $\mathbf{p} \notin \mathcal{B}_p$. Then there exists $\varepsilon > 0$ and $\{n_k\}_{k=1}^{\infty}$ such that $|p_{n_k} - p| > \varepsilon$ for each k . Since $c \leq 1$ we have $c^{\frac{1}{|p_{n_k} - p|}} \geq c^{\frac{1}{\varepsilon}}$ which implies

$$\begin{aligned} \infty &> \sum_{\{n \in \mathbb{Z}; |p_n - p| > 0\}} |p_n - p| c^{1/|p_n - p|} \\ &\geq \sum_{k=1}^{\infty} |p_{n_k} - p| c^{1/|p_{n_k} - p|} \\ &\geq \sum_{k=1}^{\infty} \varepsilon c^{\frac{1}{\varepsilon}} \\ &= \infty \end{aligned}$$

and gives a contradiction. $\square$

Definition 2.15. *Say that the operator M is bounded if there exists $C > 0$ such that*

$$\sum_{n \in \mathbb{Z}} a_n^{p_n} \leq 1 \Rightarrow \sum_{n \in \mathbb{Z}} (M\mathbf{a})_n^{p_n} \leq C$$

holds for all $\mathbf{a} \in \mathcal{M}_+$. Analogously we would define the boundedness of M^+ and M^- .

In the next we will consider a special class of exponent sequences $\mathbf{p}$. Given a non-decreasing sequence of positive integers $\{K_n\}_{n=1}^{\infty}$ we adopt a notation

$$(2.2) \quad A_0 = 0, \quad A_n = \sum_{j=1}^n K_j$$

and define for $n \geq 1$ sets

$$\mathcal{K}_n = \{j; 1 + A_{n-1} \leq j \leq A_n\}.$$

Definition 2.16. Let $p > 1$. Say that $\mathfrak{p} \in \mathcal{B}_p$ satisfies a condition $\mathcal{S}_p$ (write $p \in \mathcal{S}_p$) if there exists a non-decreasing sequence of positive integers $\{K_n\}_{n=1}^\infty$ and sequence of real numbers $\{r_n\}_{n=0}^\infty$, $r_n \searrow 0$, such that

$$(2.3) \quad p_j = \begin{cases} p + r_n & \text{if } j \in \mathcal{K}_n \quad n = 1, 2, \dots \\ p + r_0 & \text{if } j = 0 \\ p - j & \text{if } j < 0. \end{cases}$$

It is easy to see the following lemma.

Lemma 2.17. The inclusion $\mathcal{S}_p \subset \mathcal{B}_p$ holds.

3. Unboundedness of M

Proposition 3.1. Let $p > 1$. Let $\mathfrak{p} \in \mathcal{S}_p$ and $\{K_n\}_{n=1}^\infty$, $\{r_n\}_{n=1}^\infty$ be corresponding sequences. Assume that a sequence $\{K_n^{r_n - r_{n+1}}\}_{n=1}^\infty$ is unbounded. Then M is unbounded on $\ell^{\mathfrak{p}}$.

Proof. Set $\varepsilon_n = 1 - \frac{p+r_{n+1}}{p+r_n}$. Since $K_n^{r_n - r_{n+1}}$ is unbounded then $K_n^{\varepsilon_n}$ is unbounded and consequently, there is a sequence $\{b_n\}_{n=1}^\infty$ such that

$$(3.1) \quad \sum_{n=1}^\infty b_n^{1-\varepsilon_n} \leq 1 \quad \text{and} \quad \sum_{n=1}^\infty K_n^{\varepsilon_n} b_n^{1-\varepsilon_n} = \infty.$$

Set $c_n = \left(\frac{b_n}{K_n}\right)^{\frac{1}{p+r_n}}$ and define $\mathfrak{a} \in \mathcal{M}$ by

$$a_j = \begin{cases} c_n & \text{if } j \in \mathcal{K}_n, \\ 0 & \text{if } j \leq 0. \end{cases}$$

Then

$$\begin{aligned} \sum_{j \in \mathbb{Z}} a_j^{p_j} &= \sum_{n=1}^\infty \sum_{j \in \mathcal{K}_n} a_j^{p+r_n} \\ &= \sum_{n=1}^\infty K_n c_n^{p+r_n} = \sum_{n=1}^\infty b_n \\ &\leq \sum_{n=1}^\infty b_n^{1-\varepsilon_n} \leq 1. \end{aligned}$$

It is easy to see that we have for $j \in \mathcal{K}_{n+1}$

$$(M^- \mathbf{a})_j \geq \frac{K_n c_n}{K_n + j - A_n}$$

which implies

$$\begin{aligned}
\sum_{j \in \mathbb{Z}} (M \mathbf{a})_j^{p_j} &\geq \sum_{j \in \mathbb{Z}} (M^- \mathbf{a})_j^{p_j} \\
&\geq \sum_{n=1}^{\infty} (M^- \mathbf{a})_j^{p_j} \\
&= \sum_{n=1}^{\infty} \sum_{j \in \mathcal{K}_n} (M^- \mathbf{a})_j^{p_j} \\
&\geq \sum_{n=1}^{\infty} (K_n c_n)^{p+r_{n+1}} \sum_{j=1+A_n}^{A_{n+1}} \frac{1}{(K_n + j - A_n)^{p+r_{n+1}}} \\
&= \sum_{n=1}^{\infty} (K_n c_n)^{p+r_{n+1}} \sum_{s=1}^{K_{n+1}} \frac{1}{(K_n + s)^{p+r_{n+1}}} \\
&\geq \sum_{n=1}^{\infty} (K_n c_n)^{p+r_{n+1}} \sum_{s=1}^{K_n} \frac{1}{(K_n + s)^{p+r_{n+1}}} \\
&\geq \sum_{n=1}^{\infty} (K_n c_n)^{p+r_{n+1}} \sum_{s=1}^{K_n} \frac{1}{(2K_n)^{p+r_{n+1}}} \\
&\geq \frac{1}{2^{p+r_0}} \sum_{n=1}^{\infty} K_n c_n^{p+r_{n+1}} \\
&= \frac{1}{2^{p+r_0}} \sum_{n=1}^{\infty} K_n \left(\frac{b_n}{K_n} \right)^{\frac{p+r_{n+1}}{p+r_n}} \\
&= \frac{1}{2^{p+r_0}} \sum_{n=1}^{\infty} K_n^{\varepsilon_n} b_n^{1-\varepsilon_n} \\
&= \infty. \quad \square
\end{aligned}$$

4. Boundedness of M

Lemma 4.1. *Assume that $\mathbf{p}$ given by (2.3) satisfies the following conditions: There exists a constant $D \geq 1$ such that*

(i) for a sequence $\mathfrak{q}$ given by

$$q_j = \begin{cases} p + r_n & \text{if } j \in \mathcal{K}_{n+1} \\ p + r_0 & \text{if } j = 0 \\ q_{-j} & \text{if } j < 0 \end{cases} \quad n = 0, 1, 2, \dots$$

we have $\mathfrak{q} - \mathfrak{p} \in \mathcal{P}$, i.e. there is $c > 0$ such that

$$\sum_{n=1}^{\infty} (r_n - r_{n+1}) c^{\frac{1}{r_n - r_{n+1}}} \leq D;$$

(ii) the inequality

$$K_m^{1 - \frac{1}{p+r_m}} \leq D K_n^{1 - \frac{1}{p+r_n}}$$

holds for all $n, m \in \mathbb{N}$, $m \leq n$.

(iii) the inequality

$$K_n^{r_n - r_{n+1}} \leq D.$$

holds for each $n \in \mathbb{N}$;

(iv) the inequality

$$n K_n^{1 - \frac{1}{p+r_{n+1}}} \leq D K_{n+1}^{1 - \frac{1}{p+r_{n+1}}}$$

holds for each $n \in \mathbb{N}$;

Then there exists $C > 0$ such that the implication

$$\sum_{j \in \mathbb{Z}} a_j^{p_j} \leq 1 \Rightarrow \sum_{j \in \mathbb{Z}} (M^- \mathfrak{a})_j^{p_j} \leq C$$

is satisfied for all $\mathfrak{a} \in \mathcal{M}_+$ with $a_n = 0$ for $n \leq 0$.

Proof. Let

$$(4.1) \quad \sum_{j \in \mathbb{Z}} a_j^{p_j} = \sum_{j=1}^{\infty} a_j^{p_j} = \sum_{n=1}^{\infty} \sum_{j \in \mathcal{K}_n} a_j^{p+r_n} \leq 1.$$

Fix $n \in \mathbb{N}$ and $j \in \mathcal{K}_{n+1}$. Estimate $(M^- \mathfrak{a})_j$. We can easily write

$$\mathfrak{a} = \sum_{m=1}^{\infty} \mathfrak{a}^{(m)} \quad \text{where } \mathfrak{a}^{(m)} = \mathfrak{a} \chi_{\mathcal{K}_m}.$$

Since M^- is sub-linear we obtain

$$(M^- \mathbf{a})_j \leq \sum_{m=1}^{\infty} (M^- \mathbf{a}^{(m)})_j.$$

Due to the fact $(M^- \mathbf{a}^{(m)})_j = 0$ for $m > n + 1$ we have for each $n \in \mathbb{N}$

(with a convention $\sum_{m=1}^0 = 0$)

$$(4.2) \quad (M^- \mathbf{a})_j \leq \sum_{m=1}^{n-1} (M^- \mathbf{a}^{(m)})_j + (M^- \mathbf{a}^{(n)})_j + (M^- \mathbf{a}^{(n+1)})_j.$$

This gives

$$\begin{aligned} & \sum_{j \in \mathbb{Z}} (M^- \mathbf{a})_j^{p_j} \\ &= \sum_{j=1}^{\infty} (M^- \mathbf{a})_j^{p_j} \\ &= \sum_{n=1}^{\infty} \sum_{j \in \mathcal{K}_n} (M^- \mathbf{a})_j^{p+r_n} \\ &= \sum_{j \in \mathcal{K}_1} (M^- \mathbf{a})_j^{p+r_1} + \sum_{n=2}^{\infty} \sum_{j \in \mathcal{K}_n} (M^- \mathbf{a})_j^{p+r_n} \\ &= \sum_{j \in \mathcal{K}_1} (M^- \mathbf{a})_j^{p+r_1} + \sum_{n=1}^{\infty} \sum_{j \in \mathcal{K}_{n+1}} (M^- \mathbf{a})_j^{p+r_{n+1}} \\ &\leq \sum_{j \in \mathcal{K}_1} (M^- \mathbf{a})_j^{p+r_1} + \sum_{n=1}^{\infty} \sum_{j \in \mathcal{K}_{n+1}} \left\{ \sum_{m=1}^{n-1} (M^- \mathbf{a}^{(m)})_j + (M^- \mathbf{a}^{(n)})_j \right. \\ &\quad \left. + (M^- \mathbf{a}^{(n+1)})_j \right\}^{p+r_{n+1}} \\ &\leq \sum_{j \in \mathcal{K}_1} (M^- \mathbf{a})_j^{p+r_1} + 3^{p+r_1} \sum_{n=1}^{\infty} \left\{ \sum_{j \in \mathcal{K}_{n+1}} \left(\sum_{m=1}^{n-1} (M^- \mathbf{a}^{(m)})_j \right)^{p+r_{n+1}} \right. \\ &\quad \left. + \sum_{j \in \mathcal{K}_{n+1}} (M^- \mathbf{a}^{(n)})_j^{p+r_{n+1}} + \sum_{j \in \mathcal{K}_{n+1}} (M^- \mathbf{a}^{(n+1)})_j^{p+r_{n+1}} \right\} \\ &= \sum_{j \in \mathcal{K}_1} (M^- \mathbf{a})_j^{p+r_1} + 3^{p+r_1} \sum_{n=1}^{\infty} \sum_{j \in \mathcal{K}_{n+1}} \left(\sum_{m=1}^{n-1} (M^- \mathbf{a}^{(m)})_j \right)^{p+r_{n+1}} \\ &\quad + 3^{p+r_1} \sum_{n=1}^{\infty} \sum_{j \in \mathcal{K}_{n+1}} (M^- \mathbf{a}^{(n)})_j^{p+r_{n+1}} + 3^{p+r_1} \sum_{n=2}^{\infty} \sum_{j \in \mathcal{K}_n} (M^- \mathbf{a}^{(n)})_j^{p+r_n}. \end{aligned}$$

Since $\sum_{j \in \mathcal{K}_1} (M^- \mathbf{a})_j^{p+r_1} = \sum_{j \in \mathcal{K}_1} (M^- \mathbf{a}^{(1)})_j^{p+r_1}$ we obtain

$$\begin{aligned}
 (4.3) \quad \sum_{j \in \mathbb{Z}} (M^- \mathbf{a})_j^{p_j} &\leq 3^{p+r_1} \sum_{n=1}^{\infty} \sum_{j \in \mathcal{K}_{n+1}} \left(\sum_{m=1}^{n-1} (M^- \mathbf{a}^{(m)})_j \right)^{p+r_{n+1}} \\
 &\quad + 3^{p+r_1} \sum_{n=1}^{\infty} \sum_{j \in \mathcal{K}_{n+1}} (M^- \mathbf{a}^{(n)})_j^{p+r_{n+1}} \\
 &\quad + 3^{p+r_1} \sum_{n=1}^{\infty} \sum_{j \in \mathcal{K}_n} (M^- \mathbf{a}^{(n)})_j^{p+r_n} \\
 &:= 3^{p+r_1} (I_1 + I_2 + I_3).
 \end{aligned}$$

Estimate first I_1 . Using the convention $\sum_{m=1}^0 = 0$ we can write

$$\begin{aligned}
 I_1 &= \sum_{n=1}^{\infty} \sum_{j \in \mathcal{K}_{n+1}} \left(\sum_{m=1}^{n-1} (M^- \mathbf{a}^{(m)})_j \right)^{p+r_{n+1}} \\
 &= \sum_{n=2}^{\infty} \sum_{j \in \mathcal{K}_{n+1}} \left(\sum_{m=1}^{n-1} (M^- \mathbf{a}^{(m)})_j \right)^{p+r_{n+1}}
 \end{aligned}$$

and we have for $1 \leq m < n$ and $j \in \mathcal{K}_{n+1}$

$$\begin{aligned}
 (M^- \mathbf{a}^{(m)})_j &\leq \frac{1}{K_{m+1} + K_{m+2} + \cdots + K_n + j - A_n} \sum_{s \in \mathcal{K}_m} a_s \\
 &\leq \frac{1}{K_n + j - A_n} \sum_{s \in \mathcal{K}_m} a_s.
 \end{aligned}$$

Moreover,

$$\begin{aligned}
 \sum_{j \in \mathcal{K}_{n+1}} \frac{1}{(K_n + j - A_n)^{p+r_{n+1}}} &= \sum_{s=1}^{K_{n+1}} \frac{1}{(K_n + s)^{p+r_{n+1}}} \\
 &\leq \sum_{s=1}^{\infty} (K_n + s)^{-(p+r_{n+1})} \\
 &\leq \int_{K_n}^{\infty} x^{-(p+r_{n+1})} dx \\
 &= \frac{1}{p+r_{n+1}-1} K_n^{-p-r_{n+1}+1} \\
 &\leq \frac{1}{p-1} K_n^{-p-r_{n+1}+1}
 \end{aligned}$$

and so,

$$\begin{aligned}
I_1 &= \sum_{n=2}^{\infty} \sum_{j \in \mathcal{K}_{n+1}} \left(\sum_{m=1}^{n-1} (M^{-\mathbf{a}^{(m)}})_j \right)^{p+r_{n+1}} \\
&\leq \sum_{n=2}^{\infty} \sum_{j \in \mathcal{K}_{n+1}} \left(\sum_{m=1}^{n-1} \left(\frac{1}{K_n + j - A_n} \sum_{s \in \mathcal{K}_m} a_s \right) \right)^{p+r_{n+1}} \\
&= \sum_{n=2}^{\infty} \sum_{j \in \mathcal{K}_{n+1}} \frac{1}{(K_n + j - A_n)^{p+r_{n+1}}} \left(\sum_{m=1}^{n-1} \sum_{s \in \mathcal{K}_m} a_s \right)^{p+r_{n+1}} \\
&\leq \frac{1}{p-1} \sum_{n=2}^{\infty} \frac{1}{K_n^{p+r_{n+1}-1}} \left(\sum_{m=1}^{n-1} \sum_{s \in \mathcal{K}_m} a_s \right)^{p+r_{n+1}}.
\end{aligned}$$

Using the Hölder inequality we obtain

$$\begin{aligned}
I_1 &\leq \frac{1}{p-1} \sum_{n=2}^{\infty} \frac{1}{K_n^{p+r_{n+1}-1}} \left(\sum_{m=1}^{n-1} \left(\sum_{s \in \mathcal{K}_m} 1 \right)^{\frac{p+r_m-1}{p+r_m}} \left(\sum_{s \in \mathcal{K}_m} a_s^{p+r_m} \right)^{\frac{1}{p+r_m}} \right)^{p+r_{n+1}} \\
&= \frac{1}{p-1} \sum_{n=2}^{\infty} \frac{1}{K_n^{p+r_{n+1}-1}} \left(\sum_{m=1}^{n-1} K_m^{\frac{p+r_m-1}{p+r_m}} \left(\sum_{s \in \mathcal{K}_m} a_s^{p+r_m} \right)^{\frac{1}{p+r_m}} \right)^{p+r_{n+1}}.
\end{aligned}$$

Using (4.1) we have $\sum_{s \in \mathcal{K}_m} a_s^{p+r_m} \leq \sum_{m=1}^{\infty} \sum_{s \in \mathcal{K}_m} a_s^{p+r_m} \leq 1$ which gives

$$(4.4) \quad I_1 \leq \frac{1}{p-1} \sum_{n=2}^{\infty} \frac{1}{K_n^{p+r_{n+1}-1}} \left(\sum_{m=1}^{n-1} K_m^{1-\frac{1}{p+r_m}} \right)^{p+r_{n+1}}.$$

Now, if $n \geq 3$ we have by (ii) an inequality $K_m^{1-\frac{1}{p+r_m}} \leq DK_{n-2}^{1-\frac{1}{p+r_{n-2}}}$ for $1 \leq m \leq n-2$ which gives

$$\begin{aligned}
(4.5) \quad J &:= \sum_{m=1}^{n-1} K_m^{1-\frac{1}{p+r_m}} \\
&= K_{n-1}^{1-\frac{1}{p+r_{n-1}}} + \sum_{m=1}^{n-2} K_m^{1-\frac{1}{p+r_m}} \\
&\leq K_{n-1}^{1-\frac{1}{p+r_{n-1}}} + D \sum_{m=1}^{n-2} K_{n-2}^{1-\frac{1}{p+r_{n-2}}} \\
&= K_{n-1}^{1-\frac{1}{p+r_{n-1}}} + D(n-2)K_{n-2}^{1-\frac{1}{p+r_{n-2}}}.
\end{aligned}$$

By (iii) we have $K_{n-2}^{r_{n-2}-r_{n-1}} \leq D$ and so,

$$K_{n-2}^{\frac{(p+r_{n-2})-(p+r_{n-1})}{(p+r_{n-1})(p+r_{n-2})}} \leq D^{\frac{1}{(p+r_{n-1})(p+r_{n-2})}} \leq D$$

which gives

$$K_{n-2}^{\frac{1}{p+r_{n-1}} - \frac{1}{p+r_{n-2}}} \leq D$$

and finally,

$$K_{n-2}^{1-\frac{1}{p+r_{n-2}}} \leq DK_{n-2}^{1-\frac{1}{p+r_{n-1}}}.$$

This shows with (4.5) and (iv)

$$\begin{aligned} J &\leq K_{n-1}^{1-\frac{1}{p+r_{n-1}}} + (n-2)DK_{n-2}^{1-\frac{1}{p+r_{n-1}}} \\ &\leq K_{n-1}^{1-\frac{1}{p+r_{n-1}}} + D^2K_{n-1}^{1-\frac{1}{p+r_{n-1}}} \\ &= (1+D^2)K_{n-1}^{1-\frac{1}{p+r_{n-1}}}. \end{aligned}$$

If $n = 2$ then $\sum_{m=1}^{n-1} K_m^{1-\frac{1}{p+r_m}} = K_1^{1-\frac{1}{p+r_1}} = K_{n-1}^{1-\frac{1}{p+r_{n-1}}}$. In both cases we obtain

$$\sum_{m=1}^{n-1} K_m^{1-\frac{1}{p+r_m}} \leq (1+D^2)K_{n-1}^{1-\frac{1}{p+r_{n-1}}}$$

which gives with (4.4)

$$\begin{aligned} (4.6) \quad I_1 &\leq \frac{1}{p-1} \sum_{n=2}^{\infty} \frac{1}{K_n^{p+r_{n+1}-1}} \left((1+D^2)K_{n-1}^{1-\frac{1}{p+r_{n-1}}} \right)^{p+r_{n+1}} \\ &\leq \frac{(1+D^2)^{p+r_1}}{p-1} \sum_{n=2}^{\infty} \frac{K_{n-1}^{p+r_{n+1}-\frac{p+r_{n+1}}{p+r_{n-1}}}}{K_n^{p+r_{n+1}-1}} \\ &= \frac{(1+D^2)^{p+r_1}}{p-1} \sum_{n=2}^{\infty} \left(\frac{K_{n-1}}{K_n} \right)^{p+r_{n+1}-1} K_{n-1}^{\frac{r_{n-1}-r_{n+1}}{p+r_{n-1}}}. \end{aligned}$$

Since K_n is non-decreasing sequence of positive integers we have by (iii)

$$\begin{aligned} (4.7) \quad K_{n-1}^{\frac{r_{n-1}-r_{n+1}}{p+r_{n-1}}} &\leq K_{n-1}^{r_{n-1}-r_{n+1}} \\ &\leq K_{n-1}^{r_{n-1}-r_n} K_{n-1}^{r_n-r_{n+1}} \\ &\leq K_{n-1}^{r_{n-1}-r_n} K_n^{r_n-r_{n+1}} \\ &\leq D^2. \end{aligned}$$

Moreover, we obtain by (iv)

$$\left(\frac{K_{n-1}}{K_n} \right)^{1 - \frac{1}{p+r_n}} \leq \frac{D}{n-1}$$

and so,

$$\left(\frac{K_{n-1}}{K_n} \right)^{p+r_n-1} \leq \frac{D^{p+r_n}}{(n-1)^{p+r_n}} \leq \frac{D^{p+r_1}}{(n-1)^{p+r_n}}.$$

Since $r_n \searrow 0$ we have $\frac{p+r_{n+1}-1}{p+r_n-1} \rightarrow 0$ and sonsequently,

$$\begin{aligned} \sum_{n=2}^{\infty} \left(\frac{K_{n-1}}{K_n} \right)^{p+r_{n+1}-1} &= \sum_{n=2}^{\infty} \left(\frac{K_{n-1}}{K_n} \right)^{(p+r_n-1) \frac{p+r_{n+1}-1}{p+r_n-1}} \\ &\leq D^{p+r_1} \sum_{n=2}^{\infty} (n-1)^{(p+r_n-1) \frac{p+r_{n+1}-1}{p+r_n-1}} \\ &:= \tilde{D} < \infty \end{aligned}$$

which gives with (4.6) and (4.7)

$$(4.8) \quad I_1 \leq \frac{(1+D^2)^{p+r_1}}{p-1} D^2 \tilde{D}.$$

Estimate I_3 . Fix for a moment n . Denote $q = p + r_n$. Since $\mathbf{a}^{(n)} \in \ell^{p+r_n} = \ell^q$ we have by Theorem 2.7 with a constant $C_p := C_{p,p+r_0}$

$$\begin{aligned} \sum_{j \in \mathcal{K}_n} (M\mathbf{a}^{(n)})_j^{p+r_n} &= \sum_{j \in \mathcal{K}_n} (M\mathbf{a}^{(n)})_j^q \\ &\leq \sum_{j \in \mathbb{Z}} (M\mathbf{a}^{(n)})_j^q \leq C_p \sum_{j \in \mathbb{Z}} (\mathbf{a}^{(n)})_j^q \\ &= C_p \sum_{j \in \mathcal{K}_n} (\mathbf{a}^{(n)})_j^{p+r_n} \end{aligned}$$

which gives

$$\begin{aligned} (4.9) \quad I_3 &= \sum_{n=1}^{\infty} \sum_{j \in \mathcal{K}_n} (M^{-}\mathbf{a}^{(n)})_j^{p+r_n} \\ &\leq \sum_{n=1}^{\infty} \sum_{j \in \mathcal{K}_n} (M\mathbf{a}^{(n)})_j^{p+r_n} = \sum_{n=1}^{\infty} C_p \sum_{j \in \mathcal{K}_n} (\mathbf{a}^{(n)})_j^{p+r_n} \\ &= C_p \sum_{j \in \mathbb{Z}} a_j^{p_j} \leq C_p < \infty. \end{aligned}$$

Let us estimate I_2 . Define a sequence $\mathbf{b} \in \mathcal{M}$ by

$$b_j = \begin{cases} (M^- \mathbf{a}^{(n)})_j & \text{if } j \in \mathcal{K}_{n+1} \\ 0 & \text{if } j \leq A_1. \end{cases} \quad n = 1, 2, \dots$$

Fix for a moment n and set $q = p + r_n$. Since $\mathbf{a}^{(n)} \in \ell^{p+r_n} = \ell^q$ with constant exponent sequence we obtain by Theorem 2.7 (again with the constant $C_p := C_{p,p+r_0}$)

$$\begin{aligned} \sum_{j \in \mathcal{K}_{n+1}} b_j^{p+r_n} &= \sum_{j \in \mathcal{K}_{n+1}} ((M^- \mathbf{a}^{(n)})_j)^{p+r_n} \\ &\leq \sum_{j \in \mathcal{K}_{n+1}} ((M^- \mathbf{a}^{(n)})_j)^q \\ &\leq \sum_{j \in \mathbb{Z}} ((M \mathbf{a}^{(n)})_j)^q \leq C_p \sum_{j \in \mathbb{Z}} (\mathbf{a}^{(n)})_j^q \\ &= C_p \sum_{j \in \mathcal{K}_n} (\mathbf{a}^{(n)})_j^{p+r_n} \\ &= C_p \sum_{j \in \mathcal{K}_n} a_j^{p+r_n} \end{aligned}$$

and so, using the definition of $\mathbf{q}$ from (i) we have by (4.1)

$$\sum_{j \in \mathbb{Z}} b_j^{q_j} = \sum_{n=1}^{\infty} \sum_{j \in \mathcal{K}_{n+1}} b_j^{p+r_n} \leq C_p \sum_{n=1}^{\infty} \sum_{j \in \mathcal{K}_n} a_j^{p+r_n} = C_p \sum_{j \in \mathbb{Z}} a_j^{p_j} \leq C_p.$$

By (i) we have $\mathbf{q} - \mathbf{p} \in \mathcal{P}$ and so, by Lemma 2.12 we obtain

$$\sum_{j \in \mathbb{Z}} b_j^{p_j} \leq \max \left\{ 1, \sup_{j \in \mathbb{Z}} C_p^{\frac{p_j}{q_j}} \right\} C$$

which gives

$$\begin{aligned} (4.10) \quad I_2 &= \sum_{n=1}^{\infty} \sum_{j \in \mathcal{K}_{n+1}} (M^- \mathbf{a}^{(n)})_j^{p+r_{n+1}} \\ &= \sum_{n=1}^{\infty} \sum_{j \in \mathcal{K}_{n+1}} b_j^{p+r_{n+1}} \\ &= \sum_{j \in \mathbb{Z}} b_j^{p_j} \leq \max \left\{ 1, \sup_{j \in \mathbb{Z}} C_p^{\frac{p_j}{q_j}} \right\} C. \end{aligned}$$

Finally, using (4.8), (4.9) and (4.10) we have

$$\begin{aligned} \sum_{j \in \mathbb{Z}} (M^- \mathbf{a})_j^{p_j} &\leq 3^{p+r_1} (I_1 + I_2 + I_3) \\ &\leq 3^{p+r_1} \left(\frac{(1+D^2)^{p+r_1}}{p-1} D^2 \tilde{D} + C_p + \max \left\{ 1, \sup_{j \in \mathbb{Z}} C_p^{\frac{p_j}{q_j}} \right\} C \right) \end{aligned}$$

which finishes the proof. $\square$

Lemma 4.2. *Let a real number D satisfy the assumption (iv) from Lemma 4.1. Then there is a constant $C > 0$ such that the implication*

$$\sum_{j=1}^{\infty} a_j^{p_j} \leq 1 \Rightarrow \sum_{j=1}^{\infty} (M^+ \mathbf{a})_j^{p_j} \leq C$$

holds for all $\mathbf{a} \in \mathcal{M}_+$ with $a_j = 0$ for $j \leq 0$ and the implication

$$\sum_{j=-\infty}^{-1} a_j^{p_j} \leq 1 \Rightarrow \sum_{j=-\infty}^{-1} (M^- \mathbf{a})_j^{p_j} \leq C$$

holds for all $\mathbf{a} \in \mathcal{M}_+$ with $a_j = 0$ for $j \geq 0$.

Proof. Since $\mathbf{p}$ is symmetric it suffices to prove only the first implication. Assume

$$(4.11) \quad \sum_{j=1}^{\infty} a_j^{p_j} \leq 1.$$

Fix $n \in \mathbb{N}$ and $j \in \mathcal{K}_n$. Denote $\mathcal{L}_k = \bigcup_{s=k}^{\infty} \mathcal{K}_s$ for any $k \in \mathbb{N}$. Then

$\mathbf{a} = \sum_{s=1}^{n+1} \mathbf{a} \chi_{\mathcal{K}_s} + \mathbf{a} \chi_{\mathcal{L}_{n+2}}$. Since $(M^+ \mathbf{a} \chi_{\mathcal{K}_s})_j = 0$ for $s < n$ we have by the sub-linearity of M^+

$$(4.12) \quad (M^+ \mathbf{a})_j \leq (M^+ \mathbf{a} \chi_{\mathcal{K}_n})_j + (M^+ \mathbf{a} \chi_{\mathcal{K}_{n+1}})_j + (M^+ \mathbf{a} \chi_{\mathcal{L}_{n+2}})_j.$$

Estimate $(M^+ \mathbf{a} \chi_{\mathcal{L}_{n+2}})_j$. Clearly,

$$(M^+ \mathbf{a} \chi_{\mathcal{L}_{n+2}})_j \leq \sup_{k > A_{n+1}} \frac{1}{k-j+1} \sum_{j=1+A_{n+1}}^k a_j$$

Since $\sum_{j=1}^{\infty} a_j \leq 1$ we have $a_j \leq 1$ and so, $a_j^{p+r_n} \leq a_j^{p_j}$. By Jensen's inequality we obtain

$$\begin{aligned} (M^+ \mathbf{a} \chi_{\mathcal{L}_{n+2}})_j^{p+r_n} &\leq \left(\sup_{k > A_{n+1}} \frac{1}{k-j+1} \sum_{j=1+A_{n+1}}^k a_j \right)^{p+r_n} \\ &\leq \sup_{k > A_{n+1}} \frac{1}{k-j+1} \sum_{j=1+A_{n+1}}^k a_j^{p+r_n} \end{aligned}$$

$$\begin{aligned}
&\leq \sup_{k > A_{n+1}} \frac{1}{k-j+1} \sum_{j=1+A_{n+1}}^k a_j^{p_j} \\
&\leq \sup_{k > A_{n+1}} \frac{1}{K_{n+1}} \sum_{j=1+A_{n+1}}^k a_j^{p_j} \\
&\leq \frac{1}{K_{n+1}}.
\end{aligned}$$

Thus, by (4.12) we have

$$\begin{aligned}
(4.13) \quad &\sum_{j \in \mathcal{K}_n} (M^+ \mathbf{a})_j^{p+r_n} \\
&\leq 3^{p+r_n-1} \left\{ \sum_{j \in \mathcal{K}_n} (M^+ \mathbf{a} \chi_{\mathcal{K}_n})_j^{p+r_n} + \sum_{j \in \mathcal{K}_n} (M^+ \mathbf{a} \chi_{\mathcal{K}_{n+1}})_j^{p+r_n} + \sum_{j \in \mathcal{K}_n} \frac{1}{K_{n+1}} \right\} \\
&= 3^{p+r_n-1} (I_1 + I_2 + I_3).
\end{aligned}$$

Estimate I_1 and I_2 . By Lemma 2.7 with a constant $C_p := C_{p,p+r_0}$ we obtain

$$\begin{aligned}
(4.14) \quad I_1 &= \sum_{j \in \mathcal{K}_n} (M^+ (\mathbf{a} \chi_{\mathcal{K}_n}))_j^{p+r_n} \\
&\leq \sum_{j \in \mathbb{Z}} (M (\mathbf{a} \chi_{\mathcal{K}_n}))_j^{p+r_n} \\
&\leq C_p \sum_{j \in \mathbb{Z}} (\mathbf{a} \chi_{\mathcal{K}_n})_j^{p+r_n} \\
&= C_p \sum_{j \in \mathcal{K}_n} a_j^{p+r_n}
\end{aligned}$$

and analogously,

$$\begin{aligned}
(4.15) \quad I_2 &= \sum_{j \in \mathcal{K}_n} (M^+ (\mathbf{a} \chi_{\mathcal{K}_{n+1}}))_j^{p+r_n} \\
&\leq \sum_{j \in \mathbb{Z}} (M (\mathbf{a} \chi_{\mathcal{K}_{n+1}}))_j^{p+r_n} \\
&\leq C_p \sum_{j \in \mathbb{Z}} (\mathbf{a} \chi_{\mathcal{K}_{n+1}})_j^{p+r_n} \\
&= C_p \sum_{j \in \mathcal{K}_{n+1}} a_j^{p+r_n}.
\end{aligned}$$

Since $a_j \leq 1$ and $r_n \geq r_{n+1}$ for $n \geq 0$ we have $a_j^{p+r_n} \leq a_j^{p+r_{n+1}}$ which gives with (4.15)

$$I_2 \leq C_p \sum_{j \in \mathcal{K}_{n+1}} a_j^{p+r_n} \leq C_p \sum_{j \in \mathcal{K}_{n+1}} a_j^{p+r_{n+1}}.$$

Then

$$\begin{aligned} & \sum_{j \in \mathcal{K}_n} (M^+ \mathbf{a})_j^{p+r_n} \\ & \leq 3^{p+r_n-1} \left\{ C_p \sum_{j \in \mathcal{K}_n} a_j^{p+r_n} + C_p \sum_{j \in \mathcal{K}_{n+1}} a_j^{p+r_{n+1}} + \frac{K_n}{K_n + 1} \right\} \\ & \leq 3^{p+r_0-1} \left\{ C_p \sum_{j \in \mathcal{K}_n} a_j^{p+r_n} + C_p \sum_{j \in \mathcal{K}_{n+1}} a_j^{p+r_{n+1}} + \frac{K_n}{K_{n+1}} \right\} \end{aligned}$$

which yields immediately with (4.14) and (4.13)

(4.16)

$$\begin{aligned} \sum_{j=1}^{\infty} (M^+ \mathbf{a})_j^{p_j} &= \sum_{n=1}^{\infty} \sum_{j \in \mathcal{K}_n} (M^+ \mathbf{a})_j^{p_j} \\ &\leq 3^{p+r_1-1} C_p \sum_{n=1}^{\infty} \sum_{j \in \mathcal{K}_n} a_j^{p+r_n} + 3^{p+r_0-1} C_p \sum_{n=1}^{\infty} \sum_{j \in \mathcal{K}_{n+1}} a_j^{p+r_{n+1}} \\ &\quad + 3^{p+r_1-1} \sum_{n=1}^{\infty} \frac{K_n}{K_{n+1}} \\ &\leq 3^{p+r_1-1} \max\{1, C_p\} \left(2 \sum_{j=1}^{\infty} a_j^{p_j} + \sum_{n=1}^{\infty} \frac{K_n}{K_{n+1}} \right) \\ &\leq 3^{p+r_1-1} \max\{1, C_p\} \left(2 + \sum_{n=1}^{\infty} \frac{K_n}{K_{n+1}} \right). \end{aligned}$$

Now, by the assumption $nK_n^{1-\frac{1}{p+r_{n+1}}} \leq DK_{n+1}^{1-\frac{1}{p+r_{n+1}}}$ we have

$$\left(\frac{K_n}{K_{n+1}} \right)^{1-\frac{1}{p+r_{n+1}}} \leq \frac{D}{n}$$

which gives

$$\left(\frac{K_n}{K_{n+1}} \right) \leq \left(\frac{D}{n} \right)^{\frac{p+r_{n+1}}{p+r_{n+1}-1}} \leq \frac{D^{\frac{p}{p-1}}}{n^{\frac{p+r_1}{p+r_1-1}}}.$$

Since $\frac{p+r_1}{p+r_1-1} > 1$ the series $\sum_{n=1}^{\infty} \frac{K_n}{K_{n+1}}$ converges and so, by (4.16) we obtain

$$\sum_{j=1}^{\infty} (M^+ \mathbf{a})_j^{p_j} \leq 3^{p+r_1-1} \max\{1, C_p\} (2 + \sum_{n=1}^{\infty} \frac{K_n}{K_{n+1}}) := C$$

which proves the lemma. $\square$

Lemma 4.3. *Let $\mathbf{p}$ satisfy the assumptions (i)-(iv) from Lemma 4.1. Then there is $C > 0$ such that the implication*

$$\sum_{j=1}^{\infty} a_j^{p_j} \leq 1 \Rightarrow \sum_{j=-\infty}^{-1} (M^+ \mathbf{a})_j^{p_j} \leq C$$

holds for all $\mathbf{a} \in \mathcal{M}_+$, $a_j = 0$ for $j \leq 0$ and the implication

$$\sum_{j=-\infty}^{-1} a_j^{p_j} \leq 1 \Rightarrow \sum_{j=1}^{\infty} (M^- \mathbf{a})_j^{p_j} \leq C$$

holds for all $\mathbf{a} \in \mathcal{M}_+$, $a_j = 0$ for $j \geq 0$.

Proof. Since $\mathbf{p}$ is symmetric it suffices to prove only the first part of the lemma. Assume $\mathbf{a} \in \mathcal{M}_+$, $\sum_{j=1}^{\infty} a_j^{p_j} \leq 1$ and $a_j = 0$ for $j \leq 0$. Fix an integer $j < 0$. Find an integer $k > 0$ such that

$$(M^+ \mathbf{a})_j \leq \frac{2}{k-j+1} \sum_{s=1}^k a_s.$$

If $k \geq -j$ then evidently $-j$ lies between 1 and k and so,

$$\frac{2}{k-j+1} \sum_{s=1}^k a_s \leq \frac{2}{k} \sum_{s=1}^k a_s \leq 2(M\mathbf{a})_{-j}.$$

If $k < -j$ then

$$\frac{2}{k-j+1} \sum_{s=1}^k a_s \leq \frac{2}{-j} \sum_{s=1}^k a_s \leq \frac{2}{-j} \leq 2(M\mathbf{a})_{-j}.$$

In both cases we obtain

$$(M^+ \mathbf{a})_j \leq 2(M\mathbf{a})_{-j}.$$

Denote by C_1 the constant C from Lemma 4.1 and by C_2 the constant C from Lemma 4.2. Now, using Lemma 2.4, Lemma 4.1, Lemma 4.2 and the symmetry of $\mathbf{p}$ we obtain

$$\sum_{j=-\infty}^{-1} (M^+ \mathbf{a})_j^{p_j} \leq \sum_{j=-\infty}^{-1} 2^{p_j} (M\mathbf{a})_{-j}^{p_j}$$

$$\begin{aligned}
&\leq \sum_{j=-\infty}^{-1} 2^{p_j} \left(2((M^- \mathbf{a})_{-j} + (M^+ \mathbf{a})_{-j}) \right)^{p_j} \\
&\leq \sum_{j=-\infty}^{-1} 4^{p_j} \left((M^- \mathbf{a})_{-j} + (M^+ \mathbf{a})_{-j} \right)^{p_j} \\
&\leq 4^{p+r_0} \sum_{j=1}^{\infty} \left((M^- \mathbf{a})_j + (M^+ \mathbf{a})_j \right)^{p_j} \\
&\leq 4^{p+r_0} \sum_{j=1}^{\infty} 2^{p_j-1} \left((M^- \mathbf{a})_j^{p_j} + (M^+ \mathbf{a})_j^{p_j} \right) \\
&\leq 4^{p+r_0} 2^{p+r_0-1} (C_1 + C_2) := C
\end{aligned}$$

which finishes the proof. $\square$

Lemma 4.4. *Let $\mathbf{p}$ satisfy the assumptions (i)-(iv) from Lemma 4.1. Then there exists $C > 0$ such that implications*

$$\sum_{j \in \mathbb{Z}} a_j^{p_j} \leq 1 \Rightarrow \sum_{j \in \mathbb{Z}} (M^- a)_j^{p_j} \leq C$$

and

$$\sum_{j \in \mathbb{Z}} a_j^{p_j} \leq 1 \Rightarrow \sum_{j \in \mathbb{Z}} (M^+ a)_j^{p_j} \leq C$$

hold for all $\mathbf{a} \in \mathcal{M}$.

Proof. Let C_1, C_2, C_3 denote the constants C from Lemmas 4.1, 4.2 and 4.3. Since $\mathbf{p}$ is symmetric it suffices to prove only the first implication. Assume $\sum_{j \in \mathbb{Z}} a_j^{p_j} \leq 1$. Denote $\mathbf{a}_+ = \mathbf{a} \chi_{\{1,2,\dots\}}$, $\mathbf{a}_- = \mathbf{a} \chi_{\{-1,-2,\dots\}}$ and $\mathbf{a}_0 = \mathbf{a} \chi_{\{0\}}$. Clearly, $\mathbf{a} = \mathbf{a}_- + \mathbf{a}_0 + \mathbf{a}_+$ and due to the sub-linearity of M^- we obtain

$$(M^- \mathbf{a})_j \leq (M^- (\mathbf{a}_-))_j + (M^- (\mathbf{a}_0))_j + (M^- (\mathbf{a}_+))_j.$$

Thus,

(4.17)

$$\begin{aligned}
\sum_{j \in \mathbb{Z}} (M^- \mathbf{a})_j^{p_j} &\leq \sum_{j \in \mathbb{Z}} 3^{p_j-1} \left((M^- (\mathbf{a}_-))_j^{p_j} + (M^- (\mathbf{a}_0))_j^{p_j} + (M^- (\mathbf{a}_+))_j^{p_j} \right) \\
&\leq 3^{p+r_0-1} \left(\sum_{j \in \mathbb{Z}} (M^- (\mathbf{a}_-))_j^{p_j} + \sum_{j \in \mathbb{Z}} (M^- (\mathbf{a}_0))_j^{p_j} \right. \\
&\quad \left. + \sum_{j \in \mathbb{Z}} (M^- (\mathbf{a}_+))_j^{p_j} \right) \\
&= 3^{p+r_0-1} (I_1 + I_2 + I_3).
\end{aligned}$$

Estimate I_3 . Since $(\mathbf{a}_+)_j = 0$ for $j \leq 0$ we have $(M^-(\mathbf{a}_+))_j = 0$ for $j \leq 0$ and so,

$$I_3 = \sum_{j \in \mathbb{Z}} (M^-(\mathbf{a}_+))_j^{p_j} = \sum_{j=1}^{\infty} (M^-(\mathbf{a}_+))_j^{p_j}.$$

Clearly, $\sum_{j=1}^{\infty} a_j^{p_j} = \sum_{j \in \mathbb{Z}} (\mathbf{a}_+)_j^{p_j} \leq \sum_{j \in \mathbb{Z}} (\mathbf{a})_j^{p_j} \leq 1$ and by Lemma 4.1 we obtain

$$(4.18) \quad I_3 = \sum_{j \in \mathbb{Z}} (M^-(\mathbf{a}_+))_j^{p_j} \leq C_1.$$

Estimate I_1 . Clearly,

$$(4.19) \quad I_1 = \sum_{j=-\infty}^{-1} (M^-(\mathbf{a}_-))_j^{p_j} + (M^-(\mathbf{a}_-))_0^{p_0} + \sum_{j=1}^{\infty} (M^-(\mathbf{a}_-))_j^{p_j}.$$

Since $\sum_{j=-\infty}^{-1} a_j^{p_j} \leq 1$ then $a_j \leq 1$ for all j and consequently, $\frac{1}{-k+1} \sum_{s=k}^0 a_s \leq 1$ which immediately yields $(M^-(\mathbf{a}_-))_0 \leq 1$ and so, using (4.19), Lemma 4.2 and Lemma 4.3 we have

$$(4.20) \quad I_1 \leq C_2 + 1 + C_3.$$

Estimate I_2 . Clearly,

$$I_2 = \sum_{j \in \mathbb{Z}} (M^-(\mathbf{a}_0))_j^{p_j} = \sum_{j=0}^{\infty} (M^-(\mathbf{a}_0))_j^{p_j}.$$

Fix for a moment $j \geq 0$. Then using the fact $a_0 \leq 1$ we have $(M^-(\mathbf{a}_0))_j = \frac{a_0}{1+j} \leq \frac{1}{1+j}$. Then

$$(4.21) \quad I_2 \leq \sum_{j=0}^{\infty} \frac{a_0^{p_j}}{(1+j)^{p_j}} \leq \sum_{j=0}^{\infty} \frac{1}{(1+j)^{p_j}} \leq \sum_{j=0}^{\infty} \frac{1}{(1+j)^p}.$$

Thus, (4.18), (4.18), (4.20) and (4.21) give

$$\sum_{j \in \mathbb{Z}} (M^-(\mathbf{a}))_j^{p_j} \leq 3^{p+r_0-1} \left(C_1 + C_2 + 1 + C_3 + \sum_{j=0}^{\infty} \frac{1}{(1+j)^p} \right) := C$$

which finishes the proof. $\square$

Proposition 4.5. *Let $\mathbf{p}$ satisfy the assumptions (i)-(iv) from Lemma 4.1. Then M is bounded on $\ell^{\mathbf{p}}$.*

Proof. The proof is an immediate consequence of Lemma 2.4 and Lemma 4.4. $\square$

5. Examples

Assume in this section $\alpha, \beta > 0$ and $L > 1$. Let $[\cdot]$ denote the integer part as usually. Set $K_n = [L^{n^\alpha}]$ and $r_n = \frac{1}{n^\beta}$ for $n \in \mathbb{N}$ and $r_0 = 1$. Let $\mathbf{p}$ be given by (2.3). Evidently,

$$(5.1) \quad \lim_{|n| \rightarrow \infty} p_n = p$$

and so, $\mathbf{p} \in \mathcal{B}_p$.

Proposition 5.1. *Let $\alpha \leq \beta$. Then M is bounded on $\ell^{\mathbf{p}}$.*

Proof. Choose $c = \frac{1}{2L}$ in Definition 2.9 and we have

$$\begin{aligned} \sum_{n \in \mathbb{Z}; |p_n - p| > 0} |p_n - p| c^{\frac{1}{|p_n - p|}} &= 2 \sum_{n=1}^{\infty} K_n r_n c^{\frac{1}{r_n}} \\ &= 2 \sum_{n=1}^{\infty} [L^{n^\alpha}] \frac{1}{n^\beta} \left(\frac{1}{2L} \right)^{n^\beta} \\ &\leq 2 \sum_{n=1}^{\infty} L^{n^\alpha} \frac{1}{n^\beta 2^{n^\beta}} \left(\frac{1}{L} \right)^{n^\beta} \\ &\leq 2 \sum_{n=1}^{\infty} \frac{L^{n^\alpha - n^\beta}}{n^\beta} \frac{1}{2^{n^\beta}}. \end{aligned}$$

According to the assumption we have $n^\alpha \leq n^\beta$ and so, $L^{n^\alpha - n^\beta} \leq 1$. Since $\beta > 0$ we obtain

$$\sum_{n \in \mathbb{Z}; |p_n - p| > 0} |p_n - p| c^{\frac{1}{|p_n - p|}} \leq 2 \sum_{n=1}^{\infty} \frac{1}{2^{n^\beta}} < \infty$$

which proves $\mathbf{p} \in \mathcal{P}_p$ and consequently, by Lemma 2.13, M is bounded. $\square$

We shall in the next frequently use the following easy estimate which can be obtained using Lagrange mean value theorem. For $\beta > 0$ and $n \in \mathbb{N}$ we have

$$(5.2) \quad \frac{\beta}{(n+1)^{1+\beta}} \leq \frac{1}{n^\beta} - \frac{1}{(n+1)^\beta} \leq \frac{\beta}{n^{1+\beta}}.$$

Proposition 5.2. *Let $\alpha > 1 + \beta$. Then M is not bounded on ℓ^p .*

Proof. Since $L^{n^\alpha} \nearrow \infty$ we can choose a real L_1 , $1 < L_1 < L$, and we have

$$[L^{n^\alpha}] \geq L^{n^\alpha} - 1 \geq L_1^{n^\alpha}.$$

for sufficient large n . Then

$$K_n^{r_n - r_{n+1}} = [L^{n^\alpha}]^{\frac{1}{n^\beta} - \frac{1}{(n+1)^\beta}} \geq L_1^{n^\alpha \frac{\beta}{(n+1)^{1+\beta}}}.$$

By the assumption $\alpha > 1 + \beta$ we have $\frac{n^\alpha}{(n+1)^\beta} \nearrow \infty$ and consequently, $K_n^{r_n - r_{n+1}}$ is unbounded. The assertion of our lemma now follows from Proposition 3.1. $\square$

Lemma 5.3. *Let $\alpha \leq 1 + \beta$. Then $\mathfrak{p}$ satisfies the assumption (i) from Lemma 4.1.*

Proof. Prove $\mathfrak{q} - \mathfrak{p} \in \mathcal{P}$. Choose $c = L^{-\beta}$ in Definition 2.9. Clearly,

$$\begin{aligned} A &:= \sum_{n \in \mathbb{Z}; |q_n - p_n| > 0} |q_n - p_n| c^{\frac{1}{|q_n - p_n|}} \\ &= 2 \sum_{n=1}^{\infty} K_n(r_{n-1} - r_n) L^{\frac{\beta}{r_{n-1} - r_n}} \\ &= \underbrace{2K_1(r_0 - r_1) L^{\frac{\beta}{r_0 - r_1}}}_B + 2 \sum_{n=2}^{\infty} [L^{n^\alpha}] \left(\frac{1}{(n-1)^\beta} - \frac{1}{n^\beta} \right) L^{\frac{\beta}{\frac{1}{n-1}^\beta - \frac{1}{n}^\beta}}. \end{aligned}$$

By (5.2) we have

$$A \leq B + 2 \sum_{n=2}^{\infty} L^{n^\alpha} \frac{\beta}{(n-1)^{1+\beta}} L^{-\frac{1}{n^{1+\beta}}} = B + 2\beta \sum_{n=2}^{\infty} \frac{L^{n^\alpha - n^{1+\beta}}}{(n-1)^{1+\beta}}.$$

By the assumption $\alpha > 1 + \beta$ we have $L^{n^\alpha - n^{1+\beta}} \leq 1$ and so,

$$A \leq B + 2\beta \sum_{n=2}^{\infty} \frac{1}{(n-1)^{1+\beta}} < \infty$$

which finishes the proof.

In the next we will use another fact. Since $L^{n^\alpha} \nearrow \infty$ we have $\lim_{n \rightarrow \infty} [L^{n^\alpha}] L^{-n^\alpha} = 1$ and so, there exists $\omega > 1$ such that

$$(5.3) \quad L^{n^\alpha} \leq \omega [L^{n^\alpha}]$$

holds for all $n \geq 1$. $\square$

Lemma 5.4. *Let $\beta \leq \alpha$. Then the property assumption (ii) from Lemma 4.1 is satisfied.*

Proof. Assume $m, n \in \mathbb{N}$, $m \leq n$. Then the easy inequality $\frac{A+1}{B+1} \leq \frac{A}{B}$ for $A \geq B > 0$ gives

$$\frac{pn^\beta + 1}{pm^\beta + 1} \leq \frac{pn^\beta}{pm^\beta} = \left(\frac{n}{m}\right)^\beta \leq \left(\frac{n}{m}\right)^\alpha.$$

Since $\frac{(p-1)m^\beta + 1}{(p-1)n^\beta + 1} \leq 1$ we obtain

$$\frac{(p-1)m^\beta + 1}{(p-1)n^\beta + 1} \frac{pn^\beta + 1}{pm^\beta + 1} \leq \left(\frac{n}{m}\right)^\alpha$$

which can be rewritten as

$$\frac{1 - \frac{m^\beta}{pm^\beta + 1}}{1 - \frac{n^\beta}{pn^\beta + 1}} \leq \left(\frac{n}{m}\right)^\alpha$$

and so,

$$m^\alpha \left(1 - \frac{1}{p + \frac{1}{m^\beta}}\right) \leq n^\alpha \left(1 - \frac{1}{p + \frac{1}{n^\beta}}\right).$$

□

Since $L > 1$ we obtain

$$L^{m^\alpha \left(1 - \frac{1}{p + \frac{1}{m^\beta}}\right)} \leq L^{n^\alpha \left(1 - \frac{1}{p + \frac{1}{n^\beta}}\right)}.$$

and using (5.3) we have

$$\begin{aligned} K_m^{1 - \frac{1}{p+r_m}} &= [L^{m^\alpha}]^{1 - \frac{1}{p + \frac{1}{m^\beta}}} \\ &\leq L^{m^\alpha \left(1 - \frac{1}{p + \frac{1}{m^\beta}}\right)} \\ &\leq L^{n^\alpha \left(1 - \frac{1}{p + \frac{1}{n^\beta}}\right)} \\ &\leq \left(\omega[L^{n^\alpha}]\right)^{1 - \frac{1}{p + \frac{1}{n^\beta}}} \\ &\leq \omega[L^{n^\alpha}]^{1 - \frac{1}{p + \frac{1}{n^\beta}}} \\ &= \omega K_n^{1 - \frac{1}{p+r_n}} \end{aligned}$$

which finishes the proof.

Lemma 5.5. *Let $\alpha \leq 1 + \beta$. Then the assumption (iii) from Lemma 4.1 is satisfied.*

Proof. By (5.2) we can write

$$K_n^{r_n - r_{n-1}} = [L^{n^\alpha}]^{\left(\frac{1}{n^\beta} - \frac{1}{(n+1)^\beta}\right)} \leq L^{n^\alpha \frac{\beta}{n^{1+\beta}}} = (L^\beta)^{n^{\alpha-\beta-1}} \leq L^\beta$$

which proves the lemma. □

Lemma 5.6. *Let $\alpha > 1$. Then the assumption (iv) from Lemma 4.1 is satisfied.*

Proof. Set $S = L^{1-\frac{1}{p}}$. Since $\frac{K_n}{K_{n+1}} \leq 1$ and $S > 1$ we have by (5.3)

$$\begin{aligned} n \left(\frac{K_n}{K_{n+1}} \right)^{1-\frac{1}{p+r_{n+1}}} &\leq n \left(\frac{K_n}{K_{n+1}} \right)^{1-\frac{1}{p}} \\ &= n \left(\frac{[L^{n^\alpha}]}{[L^{(n+1)^\alpha}]} \right)^{1-1/p} \\ &\leq n \left(\frac{\omega L^{n^\alpha}}{L^{(n+1)^\alpha}} \right)^{1-1/p} \\ &= \omega^{1-1/p} n S^{n^\alpha - (n+1)^\alpha}. \end{aligned}$$

By Lagrange mean value theorem we have $(n+1)^\alpha - n^\alpha \geq \alpha n^{\alpha-1}$. Since $\alpha > 1$ we obtain $n S^{n^\alpha - (n+1)^\alpha} \leq n S^{-\alpha n^{\alpha-1}} \rightarrow 0$ for $n \rightarrow \infty$. Thus, $n \left(\frac{K_n}{K_{n+1}} \right)^{1-\frac{1}{p+r_{n+1}}}$ is bounded which proves the lemma. $\square$

Proposition 5.7. *Let $\beta < \alpha \leq 1 + \beta$. Then M is bounded on ℓ^p .*

Proof. The assumptions (i), (ii), (iii) and (iv) from Lemma 4.1 are satisfied by Lemmas 5.3, 5.4, 5.5 and 5.6. By Proposition 4.5 M is bounded. $\square$

Lemma 5.8. *Let $\beta = 1$. Then M is bounded on ℓ^p if and only if $0 < \alpha \leq 2$.*

Proof. Recall that $0 < \alpha < \infty$. If $0 < \alpha \leq 1$ then M is bounded by Proposition 5.1. If $1 < \alpha \leq 2$ then M is bounded by Proposition 5.7. Finally, if $2 < \alpha$ then M is unbounded by Proposition (5.2) which finishes the proof. $\square$

Theorem 5.9. *There exists $\mathfrak{p} \in \mathcal{B}_p \setminus \mathcal{P}_p$ such that M is unbounded on ℓ^p .*

Proof. Choose $\alpha > 2$ and $\beta = 1$. By (5.1) $\mathfrak{p} \in \mathcal{B}_p$. According to Lemma 5.8 M is not bounded and of course, by Lemma 2.13 $\mathfrak{p} \notin \mathcal{P}_p$. $\square$

Theorem 5.10. *There exists $\mathfrak{p} \in \mathcal{B}_p \setminus \mathcal{P}_p$ such that M is bounded on ℓ^p .*

Proof. Choose $\beta = 1$ and $1 < \alpha \leq 2$. By Lemma 5.8 M is bounded. By (5.1) we immediately have $\mathfrak{p} \in \mathcal{B}_p$. It remains to prove $\mathfrak{p} \notin \mathcal{P}_p$. Let $c > 0$ be an arbitrary. Then

$$\sum_{n \in \mathbb{Z}; |p_n - p| > 0} |p_n - p| c^{\frac{1}{|p_n - p|}} = 2 \sum_{n=1}^{\infty} K_n r_n c^{\frac{1}{r_n}} = 2 \sum_{n=1}^{\infty} [L^{n^\alpha}] \frac{1}{n} c^n = \infty.$$

Thus, $\mathfrak{p} \notin \mathcal{P}_p$. $\square$

6. Boundedness of the maximal operator on $L^{p(\cdot)}(\mathbb{R})$ for a special function $p(x)$

We consider in this section the Hardy-Littlewood maximal operator on functions and so, we must distinguish it from the maximal operator M on sequences. We will denote the maximal operator on functions by the symbol $\mathfrak{M}$ as in Section 2.

Definition 6.1. Let $f \in L^1_{loc}(\mathbb{R}^n)$. Set

$$\mathfrak{M}f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q f(t) dt.$$

Denote by $\mathfrak{B}(\mathbb{R}^n)$ a set of all bounded measurable functions $p(\cdot)$, $1 \leq \text{ess inf}\{p(t); t \in \mathbb{R}\}$.

Recall some definitions and basic assertions from [2] and [21].

Definition 6.2. Let $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$. Say that $p(\cdot) \in \mathcal{L}$ if there is a constant $K > 0$ such that

$$|p(x) - p(y)| \leq \frac{K}{-\ln|x - y|}$$

for $x, y \in \mathbb{R}^n$, $0 < |x - y| \leq \frac{1}{2}$.

Definition 6.3. Let $\varepsilon(\cdot)$ be a measurable function. Denote $\mathbb{P}(\varepsilon(\cdot)) = \{x \in \mathbb{R}^n; \varepsilon(x) > 0\}$. Say that $\varepsilon(\cdot) \in \mathcal{P}$ if there exists a positive real number c such that

$$(6.1) \quad \int_{\mathbb{P}(\varepsilon(\cdot))} \varepsilon(x) c^{1/\varepsilon(x)} dx < \infty.$$

Definition 6.4. Let $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$. Say that $p(\cdot) \in \mathcal{N}$ if there is a real number $p_\infty > 1$ such that $|p(\cdot) - p_\infty| \in \mathcal{P}$.

The following theorem is the main result in [21].

Theorem 6.5. Let $p(\cdot) \in \mathcal{L} \cap \mathcal{N}$, $1 < \text{ess inf}\{p(t); t \in \mathbb{R}^n\}$. Then $\mathfrak{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Proof. See Theorem 2.14 from [21]. □

It is not difficult to calculate the following important example.

Example 6.6. Let $1 < \inf p(\cdot) \leq \sup p(\cdot) < \infty$, p be a real number, $p > 1$, and $p(\cdot) \in \mathcal{L}$.

(i) Assume that there are two real numbers $r \geq e$, $L > 0$ such that

$$|p(x) - p| \leq \frac{L}{\ln|x|} \quad \text{for all } |x| \geq r.$$

Then $p(\cdot) \in \mathcal{N}$ and so, $\mathfrak{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ by Theorem 6.5.

(ii) Assume that we have for some $0 < \alpha < 1$

$$|p(x) - p| = \frac{1}{\ln^\alpha |x|} \quad \text{for all } |x| \geq e.$$

Then $p(\cdot) \notin \mathcal{N}$.

Definition 6.7. Let $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$. Say that $p(\cdot) \in \mathcal{A}$ if there exists a real number $p_\infty > 1$ such that

$$|\{x \in \mathbb{R}^n; |p(x) - p_\infty| \geq \delta\} \cap B_r^c| \rightarrow 0 \quad \text{provided } r \rightarrow \infty$$

for each $\delta > 0$.

Lemma 6.8. The inclusion $\mathcal{N} \subset \mathcal{A}$ holds.

Proof. See Lemma 3.2 from [21]. □

Lemma 6.9. Let $\mathbf{p} \in \mathcal{B}$, $1 < p_* \leq p_n \leq p^* < \infty$ and $M : \ell^{\mathbf{p}} \rightarrow \ell^{\mathbf{p}}$ be bounded. Then the implication

$$\sum_{n \in \mathbb{Z}} a_n^{\max\{p_n, p_{n+1}\}} \leq 1 \Rightarrow \sum_{n \in \mathbb{Z}} a_n^{\min\{p_n, p_{n+1}\}} < \infty$$

holds holds for all $\mathbf{a} \in \mathcal{M}_+$.

Proof. Set $\mathbb{Z}_1 = \{n \in \mathbb{Z}; p_n > p_{n+1}\}$ and $\mathbb{Z}_2 = \{n \in \mathbb{Z}; p_n \leq p_{n+1}\}$. Let $\mathbf{a} \in \mathcal{M}_+$. Then

$$\sum_{n \in \mathbb{Z}} a_n^{\max\{p_n, p_{n+1}\}} = \sum_{n \in \mathbb{Z}_1} a_n^{p_n} + \sum_{n \in \mathbb{Z}_2} a_n^{p_{n+1}}.$$

Define sequences $\mathbf{b}, \mathbf{c} \in \mathcal{M}_+$ by

$$b_n = \begin{cases} a_n & \text{if } n \in \mathbb{Z}_1, \\ 0 & \text{if } n \in \mathbb{Z}_2; \end{cases} \quad c_n = \begin{cases} a_n & \text{if } n \in \mathbb{Z}_2, \\ 0 & \text{if } n \in \mathbb{Z}_1. \end{cases}$$

Thus, $\sum_{n \in \mathbb{Z}} b_n^{p_n} \leq 1$ and $\sum_{n \in \mathbb{Z}} c_n^{p_n} \leq 1$. Since M is bounded on $\ell^{\mathbf{p}}$ we obtain

$$\begin{aligned} \infty &> \sum_{n \in \mathbb{Z}} (M\mathbf{b})_n^{p_n} \\ &\geq \sum_{n \in \mathbb{Z}} \left(\frac{b_{n-1} + b_n}{2} \right)^{p_n} \\ &\geq 2^{-p^*} \sum_{n \in \mathbb{Z}} b_{n-1}^{p_n} \\ &= 2^{-p^*} \sum_{n \in \mathbb{Z}} b_n^{p_{n+1}} \\ &= 2^{-p^*} \sum_{n \in \mathbb{Z}_1} a_n^{p_{n+1}}. \end{aligned}$$

Analogously,

$$\begin{aligned}
\infty &> \sum_{n \in \mathbb{Z}} (M\mathfrak{c})_n^{p_n} \\
&\geq \sum_{n \in \mathbb{Z}} \left(\frac{c_n + c_{n+1}}{2} \right)^{p_n} \\
&\geq 2^{-p^*} \sum_{n \in \mathbb{Z}} c_n^{p_n} \\
&= 2^{-p^*} \sum_{n \in \mathbb{Z}_2} a_n^{p_n}.
\end{aligned}$$

Thus, $\sum_{n \in \mathbb{Z}_1} a_n^{p_{n+1}} < \infty$, $\sum_{n \in \mathbb{Z}_2} a_n^{p_n} < \infty$ which finishes the proof. $\square$

Lemma 6.10. *Let $1 < p_* \leq p_n \leq p^* < \infty$ and $M : \ell^p \rightarrow \ell^p$ be bounded. Let r, s be non-negative integers, $r \leq s$. Set $\mathbb{Z}_n(r, s) = \{k \in \mathbb{Z}; n - r \leq k \leq n + s\}$ for $n \in \mathbb{Z}$. Then the implication*

$$\sum_{n \in \mathbb{Z}} a_n^{\max \mathbb{Z}_n(r, s)} \leq 1 \Rightarrow \sum_{n \in \mathbb{Z}} a_n^{\min \mathbb{Z}_n(r, s)} < \infty$$

holds.

Proof. The assertion is trivial for $r = s = 0$. Let us prove the lemma for $r = 0$ and $s > 0$. Use the mathematical induction. For $s = 1$ our implication follows from Lemma 6.9. Assume that the assertion is true for $s > 1$. Let

$$(6.2) \quad \sum_{n \in \mathbb{Z}} a_n^{\max \mathbb{Z}_n(0, s+1)} \leq 1.$$

Set $\mathbb{Z}_1 = \{n \in \mathbb{Z}; \min \mathbb{Z}_n(0, s+1) = \min \mathbb{Z}_n(0, s)\}$, $\mathbb{Z}_2 = \{n \in \mathbb{Z}; \min \mathbb{Z}_n(0, s+1) > p_{n+s+1}\}$. Clearly,

$$(6.3) \quad \sum_{n \in \mathbb{Z}} a_n^{\min \mathbb{Z}_n(0, s+1)} = \sum_{n \in \mathbb{Z}_1} a_n^{\min \mathbb{Z}_n(0, s)} + \sum_{n \in \mathbb{Z}_2} a_n^{p_{n+s+1}} := I_1 + I_2.$$

Let us estimate I_1 . Set

$$b_n = \begin{cases} a_n & \text{if } n \in \mathbb{Z}_1, \\ 0 & \text{if } n \notin \mathbb{Z}_1. \end{cases}$$

Then

$$\sum_{n \in \mathbb{Z}} b_n^{\max \mathbb{Z}_n(0, s)} = \sum_{n \in \mathbb{Z}_1} a_n^{\max \mathbb{Z}_n(0, s)} \leq 1$$

and by the induction assumption we have

$$(6.4) \quad I_1 = \sum_{n \in \mathbb{Z}_1} a_n^{\min \mathbb{Z}_n(0, s)} = \sum_{n \in \mathbb{Z}} b_n^{\min \mathbb{Z}_n(0, s)} < \infty.$$

Estimate I_2 . Set

$$b_n = \begin{cases} a_n & \text{if } n \in \mathbb{Z}_2, \\ 0 & \text{if } n \notin \mathbb{Z}_2. \end{cases}$$

Since $\max \mathbb{Z}_n(0, s+1) = \max \mathbb{Z}_n(0, s)$ for $n \in \mathbb{Z}_2$ we have by (6.2)

$$\sum_{n \in \mathbb{Z}} b_n^{\max \mathbb{Z}_n(0, s)} = \sum_{n \in \mathbb{Z}_2} a_n^{\max \mathbb{Z}_n(0, s)} = \sum_{n \in \mathbb{Z}_2} a_n^{\max \mathbb{Z}_n(0, s+1)} \leq 1$$

and by the induction assumption we obtain

$$\sum_{n \in \mathbb{Z}} b_n^{p_n} \leq \sum_{n \in \mathbb{Z}} b_n^{\max \mathbb{Z}_n(0, s)} = \sum_{n \in \mathbb{Z}_2} a_n^{\max \mathbb{Z}_n(0, s+1)} := D < \infty.$$

Thus,

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \left(\frac{b_n}{\max\{1, D\}^{1/p_*}} \right)^{p_n} &\leq \sum_{n \in \mathbb{Z}} \left(\frac{b_n}{\max\{1, D\}^{1/p_n}} \right)^{p_n} \\ &= \frac{1}{\max\{1, D\}} \sum_{n \in \mathbb{Z}} b_n^{p_n} \leq 1. \end{aligned}$$

It shows that a sequence $\mathfrak{c}$ given by $c_n = \max\{1, D\}^{-1/p_*} b_n$ satisfies $\sum_{n \in \mathbb{Z}} c_n^{p_n} \leq 1$. Since M is bounded on $\ell^{\mathfrak{p}}$ we obtain

$$\begin{aligned} &\infty > \sum_{n \in \mathbb{Z}} (M\mathfrak{c})_n^{p_n} \\ &\geq \sum_{n \in \mathbb{Z}} \left(\frac{c_{n-s-1} + c_{n-s} + \cdots + c_n}{s+2} \right)^{p_n} \\ &\geq (s+2)^{-p^*} \sum_{n \in \mathbb{Z}} c_{n-s-1}^{p_n} = (s+2)^{-p^*} \sum_{n \in \mathbb{Z}} c_n^{p_{n+s+1}} \\ &\geq (s+2)^{-p^*} \max\{1, D\}^{-\frac{p^*}{p_*}} \sum_{n \in \mathbb{Z}} b_n^{p_{n+s+1}} \\ &= (s+2)^{-p^*} \max\{1, D\}^{-\frac{p^*}{p_*}} \sum_{n \in \mathbb{Z}_2} a_n^{p_{n+s+1}}. \end{aligned}$$

Thus, $I_2 < \infty$ which proves with (6.3) and (6.4) the assertion for $s+1$.

Now, using the substitution $c_n = a_{-n}$ we can easily prove the assertion for all r which finishes the proof. $\square$

In what follows we fix a sequence $\mathfrak{p} \in \mathcal{B}$, $1 < p_* \leq p_n \leq p^* < \infty$, and a continuous function $p(\cdot)$ defined on $\mathbb{R}$ which satisfy for each $n \in \mathbb{Z}$

$$(6.5) \quad \min\{p_n, p_{n+1}\} \leq p(t) \leq \max\{p_n, p_{n+1}\} \text{ for } n \leq t \leq n+1, \\ p(n) = p_n.$$

We will prove that $\mathfrak{M}$ is bounded on $L^{p(x)}(\mathbb{R})$ provided M is bounded on $\ell^{\mathfrak{p}}$ and $p(\cdot) \in \mathcal{L}$.

In [21] (see Lemma 2.7) is proved the following lemma.

Lemma 6.11. *Let $p(\cdot) \in \mathcal{L}$, $1 < \inf_{t \in \mathbb{R}} p(t) \leq p(x) \leq \sup_{t \in \mathbb{R}} p(t) < \infty$. Assume that a function f satisfies $f(t) = 0$ or $|f(t)| \geq 1$ for all $t \in \mathbb{R}$ and $\int_{\mathbb{R}} |f(t)|^{p(t)} dt \leq 1$. Then*

$$\int_{\mathbb{R}} |\mathfrak{M}f(x)|^{p(x)} dx < \infty.$$

Denote for $f \in L_{1,loc}(\mathbb{R})$

$$\mathfrak{M}_1 f(x) = \sup_{x \in I, |I| \leq 3} \frac{1}{|I|} \int_I |f(t)| dt, \quad \mathfrak{M}_2 f(x) = \sup_{x \in I, |I| \geq 3} \frac{1}{|I|} \int_I |f(t)| dt$$

where I denotes an interval.

Lemma 6.12. *Let M is bounded on ℓ^p and $|f(t)| \leq 1$ for $t \in \mathbb{R}$, $\int_{\mathbb{R}} |f(t)|^{p(t)} dt \leq 1$. Then*

$$\int_{\mathbb{R}} |\mathfrak{M}_2 f(t)|^{p(t)} dt < \infty.$$

Proof. Let $\mathbf{a}$ be a sequence given by $a_n = \int_n^{n+1} |f(t)| dt$. Clearly, $a_n \leq 1$ for all $n \in \mathbb{Z}$. Since $|f(t)| \leq 1$ we obtain by Hölder inequality

$$\begin{aligned} (6.6) \quad \sum_{n \in \mathbb{Z}} a_n^{\max\{p_n, p_{n+1}\}} &\leq \sum_{n \in \mathbb{Z}} \int_n^{n+1} |f(t)|^{\max\{p_n, p_{n+1}\}} dt \\ &= \sum_{n \in \mathbb{Z}} \int_n^{n+1} |f(t)|^{p(t)} dt \\ &= \int_{\mathbb{R}} |f(t)|^{p(t)} dt \\ &\leq 1. \end{aligned}$$

By Lemma 6.9 we have

$$(6.7) \quad \sum_{n \in \mathbb{Z}} a_n^{p_n} \leq \sum_{n \in \mathbb{Z}} a_n^{\min\{p_n, p_{n+1}\}} < \infty$$

and since M is bounded on ℓ^p we obtain

$$(6.8) \quad \sum_{n \in \mathbb{Z}} (M\mathbf{a})_n^{p_n} < \infty.$$

Fix $x \in \mathbb{R}$. Estimate $\mathfrak{M}_2 f(x)$. Denote $[x]$ the integer part of x as usually. By the definition of $\mathfrak{M}_2$ we can choose $r \leq x \leq s$ such that $s - r \geq 3$ and

$$\mathfrak{M}_2 f(x) \leq \frac{2}{s-r} \int_r^s |f(t)| dt.$$

Then

$$\begin{aligned}
\mathfrak{M}_2 f(x) &\leq \frac{2}{[s] - [r] - 1} \int_{[r]}^{[s]+1} |f(t)| dt \\
&= \frac{2}{[s] - [r] - 1} \sum_{k=[r]}^{[s]} \int_k^{k+1} |f(t)| dt \\
&= \frac{2([s] - [r] + 1)}{[s] - [r] - 1} \frac{1}{[s] - [r] + 1} \sum_{k=[r]}^{[s]} a_k \\
&\leq 6(M\mathfrak{a})_{[x]}.
\end{aligned}$$

Since $|f(t)| \leq 1$ for each real t and consequently, $|\mathfrak{M}_2 f(t)| \leq 1$ for each real t , we have

$$\begin{aligned}
\int_n^{n+1} |\mathfrak{M}_2 f(t)|^{p(t)} dt &\leq \int_n^{n+1} |\mathfrak{M}_2 f(t)|^{\min\{p_n, p_{n+1}\}} dt \\
&\leq 6^{p^*} \int_n^{n+1} (M\mathfrak{a})_n^{\min\{p_n, p_{n+1}\}} dt \\
&= 6^{p^*} (M\mathfrak{a})_n^{\min\{p_n, p_{n+1}\}}.
\end{aligned}$$

By (6.8) we obtain

$$\sum_{n \in \mathbb{Z}} (M\mathfrak{a})_n^{\max\{p_n, p_{n+1}\}} \leq \sum_{n \in \mathbb{Z}} (M\mathfrak{a})_n^{p_n} < \infty$$

and by (6.7) we have

$$\sum_{n \in \mathbb{Z}} (M\mathfrak{a})_n^{\min\{p_n, p_{n+1}\}} < \infty$$

which gives

$$\begin{aligned}
\int_{\mathbb{R}} |\mathfrak{M}_2 f(t)|^{p(t)} dt &= \sum_{n \in \mathbb{Z}} \int_n^{n+1} |\mathfrak{M}_2 f(t)|^{p(t)} dt \\
&\leq 6^{p^*} \sum_{n \in \mathbb{Z}} (M\mathfrak{a})_n^{\min\{p_n, p_{n+1}\}} \\
&< \infty
\end{aligned}$$

and finishes the proof. $\square$

Lemma 6.13. *Let $p(\cdot) \in \mathcal{L}$, $d > 0$. Then there exists a constant $K(d, p(\cdot))$ depending only on d and $p(\cdot)$ such that the implication*

$$\int_{\mathbb{R}} |f(t)|^{p(t)} dt \leq 1 \Rightarrow \int_I |\mathfrak{M} f(t)|^{p(t)} dt \leq K(d, p(\cdot))$$

holds for all intervals $I \subset \mathbb{R}$, $|I| \leq d$.

Proof. In [2] the following assertion is proved. If $p(\cdot) \in \mathcal{P}$, $|I| < \infty$ then there is a constant $C(p(\cdot), I)$ such that the implication

$$\int_{\mathbb{R}} |f(t)|^{p(t)} dt \leq 1 \Rightarrow \int_I |\mathfrak{M}f(t)|^{p(t)} dt \leq C(p(\cdot), I).$$

An easy calculation of $C(p(\cdot), I)$ shows that $C(p(\cdot), I)$ depends in fact on the function $p(\cdot)$ and only on the measure of $|I|$ which finishes the proof. $\square$

Denote $p_{m,I} = \inf_{x \in I} p(x)$ and $p_{M,I} = \sup_{x \in I} p(x)$. As an easy consequence of the previous lemma we have the following lemma.

Lemma 6.14. *Let $p(\cdot) \in \mathcal{L}$, $d > 0$, $0 \leq \alpha \leq 1$ and $K(d, p(\cdot))$ be taken from Lemma 6.13. Then for each interval I , $|I| \leq d$ the implication*

$$\int_{\mathbb{R}} |f(t)|^{p(t)} dt \leq \alpha \Rightarrow \int_I |\mathfrak{M}f(t)|^{p(t)} dt \leq K(d, p(\cdot)) \alpha^{\frac{p_{m,I}}{p_{M,I}}}$$

holds.

Proof. Take $0 < \alpha \leq 1$. Set

$$g(t) = \alpha^{-\frac{1}{p_{M,I}}} f(t) \chi_I(t).$$

Clearly,

$$\begin{aligned} \int_{\mathbb{R}} |g(t)|^{p(t)} dt &= \int_I |\alpha^{-\frac{1}{p_{M,I}}} f(t)|^{p(t)} dt \\ &\leq \int_I |\alpha^{-\frac{1}{p(t)}} f(t)|^{p(t)} dt \\ &= \frac{1}{\alpha} \int_{\mathbb{R}} |f(t)|^{p(t)} dt \\ &\leq 1. \end{aligned}$$

By Lemma 6.13 we have

$$\begin{aligned} \alpha^{-\frac{p_{m,I}}{p_{M,I}}} \int_I |\mathfrak{M}(f \chi_I)(t)|^{p(t)} dt &\leq \int_I \alpha^{-\frac{p(t)}{p_{M,I}}} |\mathfrak{M}(f \chi_I)(t)|^{p(t)} dt \\ &= \int_I |\alpha^{-\frac{1}{p_{M,I}}} \mathfrak{M}(f \chi_I)(t)|^{p(t)} dt \\ &= \int_I |\mathfrak{M}g(t)|^{p(t)} dt \\ &\leq K(d, p) \end{aligned}$$

which finishes the proof. $\square$

Lemma 6.15. *Let $p(\cdot) \in \mathcal{L}$ and $M : \ell^p \rightarrow \ell^p$ be bounded and $\int_{\mathbb{R}} |f(t)|^{p(t)} dt \leq 1$. Then*

$$\int_{\mathbb{R}} |\mathfrak{M}_1 f(t)|^{p(t)} dt < \infty.$$

Proof. Clearly, it suffices to prove the implication

$$\int_{\mathbb{R}} |f(t)|^{p(t)} dt \leq \frac{1}{7} \Rightarrow \int_{\mathbb{R}} |\mathfrak{M}_1 f(t)|^{p(t)} dt < \infty.$$

Set $\alpha_n = \int_{n-3}^{n+4} |f(t)|^{p(t)} dt$. It is easy to see

$$\begin{aligned} (6.9) \quad \sum_{n \in \mathbb{Z}} \alpha_n &= \sum_{n \in \mathbb{Z}} \int_{n-3}^{n+4} |f(t)|^{p(t)} dt \\ &= 7 \sum_{n \in \mathbb{Z}} \int_n^{n+1} |f(t)|^{p(t)} dt \\ &= 7 \int_{\mathbb{R}} |f(t)|^{p(t)} dt \\ &\leq 1 \end{aligned}$$

Then $\alpha_n \leq 1$ for each n and by Lemma 6.14 we obtain

$$(6.10) \quad \int_{n-3}^{n+4} \mathfrak{M}(f\chi_{[n-3, n+4]})(t)^{p(t)} dt \leq K(7, p(\cdot)) \alpha_n^{\frac{p_{m, [n-3, n+4]}}{p_{M, [n-3, n+4]}}}.$$

Fix $n \in \mathbb{Z}$ and $t \in [n, n+1]$. Since $\mathfrak{M}_1 f(t)$ depends only on values of $f(t)$ on the interval $[n-3, n+4]$ we have

$$\int_n^{n+1} |\mathfrak{M}_1 f(t)|^{p(t)} dt = \int_n^{n+1} |\mathfrak{M}_1(f\chi_{[n-3, n+4]})(t)|^{p(t)} dt.$$

Thus,

$$\begin{aligned} (6.11) \quad \int_{\mathbb{R}} |\mathfrak{M}_1 f(t)|^{p(t)} dt &= \sum_{n \in \mathbb{Z}} \int_n^{n+1} |\mathfrak{M}_1(f\chi_{[n-3, n+4]})(t)|^{p(t)} dt \\ &\leq \sum_{n \in \mathbb{Z}} \int_n^{n+1} |\mathfrak{M}(f\chi_{[n-3, n+4]})(t)|^{p(t)} dt \\ &\leq \sum_{n \in \mathbb{Z}} K(7, p(\cdot)) \alpha_n^{\frac{p_{m, [n-3, n+4]}}{p_{M, [n-3, n+4]}}}. \end{aligned}$$

Set $p_{m,n} = \min\{p_{n-3}, p_{n-2}, \dots, p_{n+4}\}$, $p_{M,n} = \max\{p_{n-3}, p_{n-2}, \dots, p_{n+4}\}$. Clearly, by (6.5) $p_{m, [n-3, n+4]} = p_{m,n}$ and $p_{M, [n-3, n+4]} = p_{M,n}$. Define β_n by $\alpha_n = \beta_n^{p_{M,n}}$. By (6.9) we have $\sum_{n \in \mathbb{Z}} \beta_n^{p_{M,n}} \leq 1$ and by Lemma 6.10 we obtain $\sum_{n \in \mathbb{Z}} \beta_n^{p_{m,n}} < \infty$ which gives with (6.11)

$$\int_{\mathbb{R}} |\mathfrak{M}_1 f(t)|^{p(t)} dt \leq \sum_{n \in \mathbb{N}} K(7, p(\cdot)) \alpha_n^{\frac{p_{m,n}}{p_{M,n}}} = K(7, p(\cdot)) \sum_{n \in \mathbb{N}} \beta_n^{p_{m,n}} < \infty$$

and finishes the proof. $\square$

Theorem 6.16. *Let $p(\cdot) \in \mathcal{L}$ and assume that $M : \ell^p \rightarrow \ell^p$ is bounded. Then $\mathfrak{M} : L^{p(\cdot)}(\mathbb{R}) \rightarrow L^{p(\cdot)}(\mathbb{R})$ is bounded, too.*

Proof. It suffices to prove the implication

$$\int_{\mathbb{R}} |f(t)|^{p(t)} dt \leq 1 \Rightarrow \int_{\mathbb{R}} \mathfrak{M}|f(t)|^{p(t)} dt < \infty.$$

Assume $\int_{\mathbb{R}} |f(t)|^{p(t)} dt \leq 1$. Write $f(t) = f\chi_{\{t; |f(t)| > 1\}}(t) + f\chi_{\{t; |f(t)| \leq 1\}}(t) := f_1(t) + f_2(t)$. Clearly, $\mathfrak{M}f(t) \leq \mathfrak{M}f_1(t) + \mathfrak{M}f_2(t)$ and so,

$$\begin{aligned} \int_{\mathbb{R}} |\mathfrak{M}f(t)|^{p(t)} dt &\leq 2^{p^*} \left(\int_{\mathbb{R}} |\mathfrak{M}_1 f(t)|^{p(t)} dt + \int_{\mathbb{R}} |\mathfrak{M}_2 f(t)|^{p(t)} dt \right) \\ &= 2^{p^*} (I_1 + I_2). \end{aligned}$$

By Lemma 6.11 we have $I_1 < \infty$. Let us estimate I_2 . Since $\mathfrak{M}f_2(t) \leq \max\{\mathfrak{M}_1 f_2(t), \mathfrak{M}_2 f_2(t)\}$ it suffices to estimate

$$I_{21} := \int_{\mathbb{R}} |\mathfrak{M}_1 f_2(t)|^{p(t)} dt, \quad I_{22} := \int_{\mathbb{R}} |\mathfrak{M}_2 f_2(t)|^{p(t)} dt.$$

But $I_{21} < \infty$ by Lemma 6.15 and $I_{22} < \infty$ by Lemma 6.12 which finishes the proof. $\square$

Theorem 6.17. *There is a function $p(\cdot) \in \mathcal{L} \cap (\mathcal{A} \setminus \mathcal{N})$ such that $\mathfrak{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R})$.*

Proof. Choose real numbers α, L with $1 < \alpha \leq 2$, $L > 1$. Set $K_n = [L^{n^\alpha}]$ and $r_n = \frac{1}{n}$ for $n \in \mathbb{N}$, $r_0 = 1$. Let the sequence $\mathbf{p}$ be given by (2.3) and the function p be given by

$$(6.12) \quad p(t) = \sum_{n \in \mathbb{Z}} (p_{n+1}(t-n) + p_n(n+1-t)) \chi_{[n, n+1]}(t).$$

Since $p(\cdot)$ is bounded and linear on each interval $[n, n+1]$ the function $p(\cdot)$ is Lipschitz and consequently, $p(\cdot) \in \mathcal{L}$. Evidently, $p(\cdot)$ satisfies (6.5). The operator M is bounded on ℓ^p by Lemma 5.8 and so, $\mathfrak{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R})$ by Theorem 6.16. It remains to prove $p(\cdot) \in \mathcal{L} \cap (\mathcal{A} \setminus \mathcal{N})$. Clearly, $p(\cdot) \in \mathcal{L}$ by the assumption. Moreover, by Definition 2.16 $\lim_{|n| \rightarrow \infty} p_n = p + \lim_{|n| \rightarrow \infty} r_n = p + \lim_{|n| \rightarrow \infty} \frac{1}{n} = p$ and thus, $p(\cdot) \in \mathcal{A}$. Let us prove $p(\cdot) \notin \mathcal{N}$. Choose an arbitrary c , $0 < c < 1$. Estimate

$$\begin{aligned} \int_{\mathbb{R}} |p(t) - p| c^{\frac{1}{|p(t)-p|}} dt &\geq \sum_{n=1}^{\infty} \int_{1+A_{n-1}}^{A_n} r_n c^{\frac{1}{r_n}} dt = \sum_{n=1}^{\infty} (K_n - 1) \frac{c^n}{n} \\ &= \sum_{n=1}^{\infty} [L^{n^\alpha}] \frac{c^n}{n} - \sum_{n=1}^{\infty} \frac{c^n}{n} \geq \sum_{n=1}^{\infty} \frac{L^{n^\alpha} c^n}{n} - 2 \sum_{n=1}^{\infty} \frac{c^n}{n}. \end{aligned}$$

Since $c < 1$ we have $\sum_{n=1}^{\infty} \frac{c^n}{n} < \infty$. Since $L > 1$ and $\alpha > 1$ we obtain $\sum_{n=1}^{\infty} \frac{L^{n^\alpha} c^n}{n} = \infty$ which gives $p(\cdot) \notin \mathcal{N}$ and finishes the proof. $\square$

It would be undoubtedly interesting to estimate the asymptotic behavior of $p(\cdot)$ at infinity given by (6.12) and compare it with Example 6.6.

Lemma 6.18. *Let $\alpha > 1$, $L > 1$, $K_n = [L^{n^\alpha}]$, $r_n = \frac{1}{n}$ and $p(\cdot)$ be given by (6.12). Then there are two numbers $c_1, c_2 > 0$ such that inequalities*

$$\frac{c_1}{(\ln |x|)^{1/\alpha}} \leq |p(x) - p| \leq \frac{c_2}{(\ln |x|)^{1/\alpha}}$$

hold for $|x| \geq e$.

Proof. It suffices to find $c > 1, c_1, c_2 > 0$ such that

$$(6.13) \quad \frac{c_1}{(\ln x)^{1/\alpha}} \leq p(x) - p \leq \frac{c_2}{(\ln x)^{1/\alpha}} \quad \text{for all } x > c.$$

Take $m \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n > m$ the following inequalities hold:

$$(6.14) \quad \ln n \leq \alpha(\ln L)n^{\alpha-1};$$

$$(6.15) \quad L^{n^\alpha} \leq 2[L^{n^\alpha}];$$

$$(6.16) \quad 2\ln 2 \leq (\ln L)n^\alpha \quad \text{and} \quad \ln 2 \leq (\ln L)(n+1)^\alpha.$$

Fix $n > m$. By (2.2) and (6.15) we have

$$(6.17) \quad L^{n^\alpha} \leq 2[L^{n^\alpha}] = 2K_n \leq 2 \sum_{j=1}^n K_j = 2A_n.$$

By the Lagrange mean value theorem we obtain

$$\alpha n^{\alpha-1} \leq (n+1)^\alpha - n^\alpha \leq \alpha(n+1)^{\alpha-1}$$

which gives with (6.14)

$$\ln n \leq \alpha(\ln L)n^{\alpha-1} \leq (\ln L)((n+1)^\alpha - n^\alpha)$$

and so, $nL^{n^\alpha} \leq L^{(n+1)^\alpha}$. As a consequence we obtain

$$(6.18) \quad \begin{aligned} A_{n+1} &= K_1 + K_2 + \cdots + K_n + K_{n+1} \leq nK_n + K_{n+1} \\ &\leq nL^{n^\alpha} + L^{(n+1)^\alpha} \leq 2L^{(n+1)^\alpha}. \end{aligned}$$

Choose $x \in [A_n, A_{n+1}]$. By (2.3) and (6.12) we have

$$(6.19) \quad \frac{1}{n+1} \leq p(x) - p \leq \frac{1}{n}.$$

Moreover, due to (6.17) and (6.18) we obtain

$$\frac{1}{2}L^{n^\alpha} \leq x \leq 2L^{(n+1)^\alpha}.$$

These inequalities can be rewritten as

$$n^\alpha \ln L - \ln 2 \leq \ln x \leq (n+1)^\alpha \ln L + \ln 2$$

Consequently, by (6.16) we have

$$\frac{1}{2} n^\alpha \ln L \leq \ln x \leq 2(n+1)^\alpha \ln L$$

which yields

$$\left(\frac{1}{2 \ln L}\right)^{1/\alpha} \frac{1}{n+1} \leq \frac{1}{(\ln x)^{1/\alpha}} \leq \left(\frac{2}{\ln L}\right)^{1/\alpha} \frac{1}{n}.$$

Since $\frac{1}{n} \leq \frac{2}{n+1}$ we have by (6.19)

$$\begin{aligned} \frac{1}{2} \left(\frac{1}{2 \ln L}\right)^{1/\alpha} (p(x) - p) &\leq \frac{1}{2} \left(\frac{1}{2 \ln L}\right)^{1/\alpha} \frac{1}{n} \leq \frac{1}{(\ln x)^{1/\alpha}} \\ &\leq 2 \left(\frac{2}{\ln L}\right)^{1/\alpha} \frac{1}{n+1} \leq 2 \left(\frac{2}{\ln L}\right)^{1/\alpha} (p(x) - p). \end{aligned}$$

We have proved for $n > m$ and $x \in [A_n, A_{n+1}]$

$$\frac{\frac{1}{2} \left(\frac{\ln L}{2}\right)^{1/\alpha}}{(\ln x)^{1/\alpha}} \leq p(x) - p \leq \frac{2(2 \ln L)^{1/\alpha}}{(\ln x)^{1/\alpha}}.$$

Now, the easy fact $[A_n, \infty] = \bigcup_{n>m} [A_n, A_{n+1}]$ proves (6.13) with $c = A_{m+1}$, $c_1 = \frac{1}{2} \left(\frac{\ln L}{2}\right)^{1/\alpha}$ and $c_2 = 2(2 \ln L)^{1/\alpha}$. $\square$

Remark 6.19. Let $\alpha > 0$, $L > 1$, $K_n = [L^{n^\alpha}]$, $r_n = \frac{1}{n}$ and $p(\cdot)$ be given by (6.12).

If $0 < \alpha \leq 1$ then we have $|p(x) - p| \leq \frac{c}{\ln|x|}$ which gives $p(\cdot) \in \mathcal{N}$ by Example 6.6 (i) and so, $\mathfrak{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R})$ again by Example 6.6 (i).

If $1 < \alpha \leq 2$ then the function $p(\cdot) - p$ decreases to 0 comparably with $\frac{1}{\ln|x|^{1/\alpha}}$ and so, $p(\cdot) \notin \mathcal{N}$ by Example 6.6 (ii) but $\mathfrak{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R})$ by Lemma 6.17.

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