Dual of modulation spaces

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Abstract. The aim of this paper is the study of the dual of modulation spaces $M^{p,q}(\mathbf{R}^d)$ for $0 < p, q < \infty$.

1. Introduction

We have constructed the modulation spaces $M^{p,q}(\mathbf{R}^d)$ in [2] for general $0 < p, q \leq \infty$, which coincide with the usual modulation spaces when $1 \leq p, q \leq \infty$, and studied their basic properties. The aim of this paper is the study of the dual of $M^{p,q}(\mathbf{R}^d)$ for $0 < p, q < \infty$. When $1 \leq p, q < \infty$, the fact that $M^{p',q'}(\mathbf{R}^d)$ is the dual of $M^{p,q}(\mathbf{R}^d)$ is already known, where $1/p + 1/p' = 1/q + 1/q' = 1$. (See Feichtinger [1].) So in this paper we are concerned with the dual, in particular when $p < 1$ or $q < 1$. Motivated by the fact that the modulation spaces have similar properties to that of the Besov spaces (Proposition 2.2), we employ H. Triebel's method [3] to study the dual. But gained results are similar to the sequence spaces l^p rather than the Besov spaces $B_{p,q}^s(\mathbf{R}^d)$.

2. Preliminaries

2.1. Basic definition. The following notations will be used throughout this article. Let $\mathcal{S}(\mathbf{R}^d)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbf{R}^d$ and $\mathcal{S}'(\mathbf{R}^d)$ be the topological dual of $\mathcal{S}(\mathbf{R}^d)$. The Fourier transform is $\hat{f}(\omega) = \int f(t)e^{-2\pi i\omega \cdot t} dt$, and the inverse Fourier transform is $\check{f}(t) = \hat{f}(-t)$. We define for $0 < p < \infty$

$$\|f\|_{L^p} = \left(\int_{\mathbf{R}^d} |f(x)|^p dx \right)^{\frac{1}{p}}$$

and $\|f\|_{L^\infty} = \text{ess. sup}_{x \in \mathbf{R}^d} |f(x)|$. We use the pairing $\langle f, g \rangle$ between $f \in \mathcal{S}'(\mathbf{R}^d)$ and $g \in \mathcal{S}(\mathbf{R}^d)$, in a manner consistent with the inner product $\langle f, g \rangle = \int f(t) \overline{g(t)} dt$ on $L^2(\mathbf{R}^d)$. For a function f on $\mathbf{R}^d$, the translation and the modulation operators are defined by

$$T_x f(t) = f(t - x), \quad \text{and} \quad M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) \quad (x, \omega \in \mathbf{R}^d),$$

respectively. And $\tilde{f}$ is the function defined by $\tilde{f}(x) := \overline{f(-x)}$.

2.2. Modulation spaces and basic properties. We recall the definition of the modulation spaces.

First for $\alpha > 0$ we define $\Phi^\alpha(\mathbf{R}^d)$ to be the space of all $g \in \mathcal{S}(\mathbf{R}^d)$ satisfying

$$\text{supp } \hat{g} \subset \{\xi \mid |\xi| \leq 1\}, \quad \text{and} \quad \sum_{k \in \mathbf{Z}^d} \hat{g}(\xi - \alpha k) \equiv 1, \quad \forall \xi \in \mathbf{R}^d.$$

In the following, we choose a sufficiently small $\alpha > 0$ so that the function space $\Phi^\alpha(\mathbf{R}^d)$ is not empty. With this, we have defined the modulation spaces as follows:

Definition 2.1. Given a $g \in \Phi^\alpha(\mathbf{R}^d)$, and $0 < p, q \leq \infty$, we define the modulation space $M^{p,q}(\mathbf{R}^d)$ to be the space of all tempered distributions $f \in \mathcal{S}'(\mathbf{R}^d)$ such that the quasi-norm

$$\|f\|_{M^{p,q}} := \left(\int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} |f * (M_\omega g)(x)|^p dx \right)^{\frac{q}{p}} d\omega \right)^{\frac{1}{q}}$$

is finite, with obvious modifications if p or $q = \infty$.

We state basic properties of modulation spaces, which will play an important role in this article. (See [2].)

Proposition 2.2. *Let $0 < p, q \leq \infty$ and $g \in \Phi^\alpha(\mathbf{R}^d)$. Then*

$$(a) \quad \left(\sum_{k \in \mathbf{Z}^d} \left(\int_{\mathbf{R}^d} |f * (M_{\alpha k} g)(x)|^p dx \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}$$

is an equivalent quasi-norm on $M^{p,q}(\mathbf{R}^d)$ with modifications if p or $q = \infty$.

(b) *Different test functions $g_1, g_2 \in \Phi^\alpha(\mathbf{R}^d)$ define the same space and equivalent quasi-norms on $M^{p,q}(\mathbf{R}^d)$.*

(c) *Let $0 < p_0 \leq p_1 \leq \infty$ and $0 < q_0 \leq q_1 \leq \infty$. Then*

$$M^{p_0, q_0}(\mathbf{R}^d) \subset M^{p_1, q_1}(\mathbf{R}^d).$$

(d) *We have the continuous embeddings*

$$\mathcal{S}(\mathbf{R}^d) \subset M^{p,q}(\mathbf{R}^d) \subset \mathcal{S}'(\mathbf{R}^d)$$

for $0 < p, q \leq \infty$.

(e) *$M^{p,q}(\mathbf{R}^d)$ is a quasi-Banach space if $0 < p, q \leq \infty$ (Banach space if $1 \leq p, q \leq \infty$).*

(f) *If $0 < p, q < \infty$, then $\mathcal{S}(\mathbf{R}^d)$ is dense in $M^{p,q}(\mathbf{R}^d)$.*

These facts have been derived from the following.

Let $0 < p \leq \infty$, and Γ be a compact subset of $\mathbf{R}^d$. Then L_Γ^p is defined by

$$L_\Gamma^p = \{f \in \mathcal{S}'(\mathbf{R}^d) \mid \exists \xi_0 \in \mathbf{R}^d, \text{supp } \hat{f} \subset \xi_0 + \Gamma, \|f\|_{L^p} < \infty\}.$$

Lemma 2.3. *Let Γ be a compact subset of $\mathbf{R}^d$ and let $0 < p \leq q \leq \infty$. Then there exists a positive constant C (which depends only on the diameter of Γ and p) such that*

$$\|f\|_{L^q} \leq C \|f\|_{L^p}$$

holds for all $f \in L_\Gamma^p$.

Lemma 2.4. *Let $0 < p \leq 1$ and Γ, Γ' be compact subsets of $\mathbf{R}^d$. Then there exists a positive constant C (which depends only on the diameters of Γ, Γ' and p) such that*

$$\left\| |f| * |g| \right\|_{L^p} \leq C \|f\|_{L^p} \|g\|_{L^p}$$

holds for all $f \in L_\Gamma^p$ and all $g \in L_{\Gamma'}^p$.

In the sequel, we shall not distinguish between equivalent quasi-norms of a given quasi-normed space.

2.3. Summary of the results. We now formulate our results.

(i) For $0 < p, q \leq 1$, we have

$$(M^{p,q}(\mathbf{R}^d))' = M^{p',q'}(\mathbf{R}^d) = M^{\infty,\infty}(\mathbf{R}^d),$$

(ii) for $0 < p < 1, 1 < q < \infty$,

$$M^{p',q'}(\mathbf{R}^d) \subset (M^{p,q}(\mathbf{R}^d))' \subset M^{\infty,\infty}(\mathbf{R}^d),$$

(iii) and for $1 \leq p < \infty, 0 < q < 1$,

$$(M^{p,q}(\mathbf{R}^d))' = M^{p',q'}(\mathbf{R}^d) = M^{p',\infty}(\mathbf{R}^d).$$

Here the symbol $(M^{p,q}(\mathbf{R}^d))'$ is used to denote all bounded linear functionals on $M^{p,q}(\mathbf{R}^d)$ and p' is a conjugate exponent of p which is determined in the usual way by $\frac{1}{p} + \frac{1}{p'} = 1$ if $1 \leq p < \infty$, and we put $p' = \infty$ if $0 < p < 1$. q' is determined in the same way.

Suppose $0 < p, q < 1$. Then for the sequence spaces, $(l^p)' = l^\infty$, that is all dual spaces are equal. But for the Besov spaces, Triebel [3] has proved that $(B_{p,q}^s(\mathbf{R}^d))' = B_{\infty,\infty}^{-s+d(\frac{1}{p}-1)}$. In this sense, the modulation spaces are similar to the sequence spaces rather than the Besov spaces for the dual. At present, we don't know whether or not $(M^{p,q}(\mathbf{R}^d))' = M^{p',q'}(\mathbf{R}^d)$ when $0 < p < 1 < q < \infty$.

2.4. Technical lemmas. Our theory deeply depends on the following transformation from a function f on $\mathbf{R}^d$ into a family of functions on $\mathbf{R}^d$,

$$f \mapsto \{f * (M_{\alpha k}g)(x)\}_{k \in \mathbf{Z}^d}.$$

It is natural to ask whether it is possible to recover f . The following key lemma in this paper asserts that this transformation is invertible on $\mathcal{S}'(\mathbf{R}^d)$ and also implies the duality on $M^{p,q}(\mathbf{R}^d)$.

Lemma 2.5. (*The Inversion Formula*) Let $f \in \mathcal{S}'(\mathbf{R}^d)$, $\psi \in \mathcal{S}(\mathbf{R}^d)$ and $g \in \Phi^\alpha(\mathbf{R}^d)$. Then there exists a positive constant N which depends only on the size of $\text{supp } \hat{g}$, $\alpha > 0$ and the dimension d , such that

$$\langle f, \psi \rangle = \sum_{k \in \mathbf{Z}^d} \sum_{|r| \leq N} \langle f * (M_{\alpha k}g), \psi * (M_{\alpha(k+r)}g) \rangle.$$

Proof. First note that there exists a constant N (depending only on the size of $\text{supp } \hat{g}$, $\alpha > 0$ and dimension d) such that $\widehat{T_{\alpha k}g} = \sum_{|r| \leq N} T_{\alpha(k+r)}\hat{g}$.

$\widehat{T_{\alpha k} \widehat{g}}$ for all $k \in \mathbf{Z}^d$. Since $\widehat{\psi} = \lim_{n \rightarrow \infty} \sum_{|k| \leq n} \widehat{T_{\alpha k} \widehat{g}} \cdot \widehat{\psi}$ in $\mathcal{S}(\mathbf{R}^d)$, we have

$$\begin{aligned}
\langle f, \psi \rangle &= \langle \widehat{f}, \widehat{\psi} \rangle \\
&= \langle \widehat{f}, \lim_{n \rightarrow \infty} \sum_{|k| \leq n} \widehat{T_{\alpha k} \widehat{g}} \cdot \widehat{\psi} \rangle \\
&= \lim_{n \rightarrow \infty} \sum_{|k| \leq n} \langle \widehat{f}, \widehat{T_{\alpha k} \widehat{g}} \cdot \widehat{\psi} \rangle \\
&= \sum_{k \in \mathbf{Z}^d} \langle \widehat{f}, \sum_{|r| \leq N} T_{\alpha(k+r)} \widehat{g} \cdot \widehat{T_{\alpha k} \widehat{g}} \cdot \widehat{\psi} \rangle \\
&= \sum_{k \in \mathbf{Z}^d} \sum_{|r| \leq N} \langle \widehat{f} \cdot T_{\alpha k} \widehat{g}, T_{\alpha(k+r)} \widehat{g} \cdot \widehat{\psi} \rangle \\
&= \sum_{k \in \mathbf{Z}^d} \sum_{|r| \leq N} \langle f * (M_{\alpha k} g), \psi * (M_{\alpha(k+r)} g) \rangle. \quad \square
\end{aligned}$$

Lemma 2.6. *Let $f \in \mathcal{S}'(\mathbf{R}^d)$ and $\eta \in \mathcal{S}(\mathbf{R}^d)$. Then the convolution $f * \eta$ exists in $\mathcal{O}_M$ and can be represented in the form*

$$\langle f * \eta(x), \psi(x) \rangle = \langle f(x), \tilde{\eta} * \psi(x) \rangle, \quad \psi \in \mathcal{S}(\mathbf{R}^d).$$

Lemma 2.7. *For $g \in \mathcal{S}(\mathbf{R}^d)$, $k \in \mathbf{Z}^d$, $\alpha > 0$ and $\xi \in \mathbf{R}^d$ we have*

$$(\widehat{T_{\alpha k} \widehat{g}})(\xi) = (\widetilde{M_{\alpha k} g})^\wedge(\xi) = (M_{\alpha k} \tilde{g})^\wedge(\xi).$$

3. Dual spaces

In this section we study the space $(M^{p,q}(\mathbf{R}^d))^*$ of bounded anti-linear functionals on $M^{p,q}(\mathbf{R}^d)$, equivalently, study the dual of $M^{p,q}(\mathbf{R}^d)$. We mean here, a functional l on a quasi-Banach space B is anti-linear if for any $\phi, \psi \in B$ and $\alpha \in \mathbf{C}$, we have $l(\phi + \psi) = l(\phi) + l(\psi)$ and $l(\alpha\phi) = \bar{\alpha}l(\phi)$. Throughout this section, g denotes a function in $\Phi^\alpha(\mathbf{R}^d)$. If $1 \leq p < \infty$, then the conjugate exponent p' of p is determined in the usual way by $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, we put $p' = \infty$ if $0 < p < 1$.

3.1. The case $1 \leq p < \infty$.

Theorem 3.1. *Let $1 \leq p < \infty$ and $0 < q \leq 1$.*

(a) *Suppose $f \in M^{p',\infty}(\mathbf{R}^d)$. Then the anti-linear functional l given by*

$$(1) \quad l(\psi) = \sum_{k \in \mathbf{Z}^d} \sum_{|r| \leq N} \langle f * (M_{\alpha k} g), \psi * (M_{\alpha(k+r)} g) \rangle,$$

initially defined on the dense subspace $\mathcal{S}(\mathbf{R}^d)$ of $M^{p,q}(\mathbf{R}^d)$, has a unique bounded extension to $M^{p,q}(\mathbf{R}^d)$ and satisfies

$$\|l\|_{(M^{p,q})^*} \leq C \|f\|_{M^{p',\infty}}.$$

(b) Conversely, every bounded anti-linear functional l on $M^{p,q}(\mathbf{R}^d)$ can be realized as above, with an $f \in M^{p',\infty}(\mathbf{R}^d)$, and with

$$\|f\|_{M^{p',\infty}} \leq C' \|l\|_{(M^{p,q})^*}.$$

Proof. (a) Let $f \in M^{p',\infty}(\mathbf{R}^d)$, $\psi \in \mathcal{S}(\mathbf{R}^d)$. Then by Hölder's inequality and continuous embedding for sequence spaces, $l^q \subset l^1$ if $q \leq 1$, we have

$$\begin{aligned} |l(\psi)| &\leq \sum_{|r| \leq N} \sum_{k \in \mathbf{Z}^d} \|f * (M_{\alpha k} g)(x)\|_{L^{p'}} \|\psi * (M_{\alpha(k+r)} g)(x)\|_{L^p} \\ &\leq C \| \|f * (M_{\alpha k} g)\|_{L^{p'}} \|_{l^\infty} \| \|\psi * (M_{\alpha k} g)\|_{L^p} \|_{l^q} \\ &= C \|f\|_{M^{p',\infty}} \|\psi\|_{M^{p,q}}. \end{aligned}$$

Thus $l(\psi)$ defined by (1) converges and defines a bounded anti-linear functional l on $\mathcal{S}(\mathbf{R}^d)$ in the quasi-norm in $M^{p,q}(\mathbf{R}^d)$. Then since $\mathcal{S}(\mathbf{R}^d)$ is dense in $M^{p,q}(\mathbf{R}^d)$ and $M^{p,q}(\mathbf{R}^d)$ is a quasi-Banach space, we have desired result.

(b) We first note that $(M^{p,q}(\mathbf{R}^d))^*$ is embedded in $\mathcal{S}'(\mathbf{R}^d)$, since $\mathcal{S}(\mathbf{R}^d)$ is a dense subset of $M^{p,q}(\mathbf{R}^d)$. Therefore, setting $\langle l, \psi \rangle := l(\psi)$, we obtain (1) with $f = l$ by virtue of Lemma 2.5.

Let $1 \leq p < \infty$, $0 < q \leq 1$ and $l \in (M^{p,q}(\mathbf{R}^d))^*$. We show that for each $k \in \mathbf{Z}^d$, $l * (M_{\alpha k} g)(x)$ is a well-defined element of $L^{p'}(\mathbf{R}^d)$. With $\psi \in \mathcal{S}(\mathbf{R}^d)$, we estimate

$$|\langle l * (M_{\alpha k} g), \psi \rangle| = |\langle l, \widetilde{(M_{\alpha k} g) * \psi} \rangle| \leq \|l\|_{(M^{p,q})^*} \|\widetilde{(M_{\alpha k} g) * \psi}\|_{M^{p,q}}.$$

By Lemma 2.7, the following relation holds true:

$$\text{supp } (\widetilde{M_{\alpha k} g})^\wedge \cap \bigcup_{n \in \mathbf{Z}^d} \text{supp } (M_{\alpha n} g)^\wedge = \bigcup_{|r| \leq N} \text{supp } (\widetilde{M_{\alpha k} g})^\wedge \cap \text{supp } (M_{\alpha(k+r)} g)^\wedge.$$

From this and Young's inequality, we have

$$\begin{aligned}
||(\widetilde{M_{\alpha k}g}) * \psi||_{M^{p,q}}^q &= \sum_{n \in \mathbf{Z}^d} \left(\int_{\mathbf{R}^d} |(\widetilde{M_{\alpha k}g}) * \psi * M_{\alpha n}g(x)|^p dx \right)^{\frac{q}{p}} \\
&= \sum_{|r| \leq N} \left(\int_{\mathbf{R}^d} |(\widetilde{M_{\alpha k}g}) * \psi * M_{\alpha(k+r)}g(x)|^p dx \right)^{\frac{q}{p}} \\
&\leq ||\psi||_{L^p}^q \sum_{|r| \leq N} ||(M_{\alpha k}\tilde{g}) * (M_{\alpha(k+r)}g)||_{L^1}^q = C||\psi||_{L^p}^q.
\end{aligned}$$

Thus for each $k \in \mathbf{Z}^d$, $l * (M_{\alpha k}g)(x)$ is a well-defined element of $L^{p'}(\mathbf{R}^d)$. Taking supremum over $k \in \mathbf{Z}^d$, we obtain the estimate

$$||l||_{M^{p',q'}} = \sup_{k \in \mathbf{Z}^d} ||l * (M_{\alpha k}g)||_{L^{p'}} < \infty. \quad \square$$

3.2. The case $0 < p < 1$.

Theorem 3.2. *Let $0 < p < 1$ and $0 < q < \infty$.*

(a) *Suppose $f \in M^{\infty,q'}(\mathbf{R}^d)$. Then the anti-linear functional l given by*

$$(2) \quad l(\psi) = \sum_{k \in \mathbf{Z}^d} \sum_{|r| \leq N} \langle f * (M_{\alpha k}g), \psi * (M_{\alpha(k+r)}g) \rangle,$$

initially defined on the dense subspace $\mathcal{S}(\mathbf{R}^d)$ of $M^{p,q}(\mathbf{R}^d)$, has a unique bounded extension to $M^{p,q}(\mathbf{R}^d)$ and satisfies

$$||l||_{(M^{p,q})^*} \leq C||f||_{M^{p',\infty}}.$$

(b) *Every bounded anti-linear functional l on $M^{p,q}(\mathbf{R}^d)$ can be realized as above, with an $f \in M^{\infty,\infty}(\mathbf{R}^d)$, and with*

$$||f||_{M^{\infty,\infty}} \leq C'||l||_{(M^{p,q})^*}.$$

Proof. (a). Let $f \in M^{\infty,q'}(\mathbf{R}^d)$. Then using Lemma 2.3, we have

$$\begin{aligned}
|l(\psi)| &\leq \sum_{k \in \mathbf{Z}^d} \sum_{|r| \leq N} |\langle f * (M_{\alpha k}g), \psi * (M_{\alpha(k+r)}g) \rangle| \\
&\leq \sum_{k \in \mathbf{Z}^d} \sum_{|r| \leq N} ||f * (M_{\alpha k}g)(x)||_{L^\infty} ||\psi * (M_{\alpha(k+r)}g)(x)||_{L^1} \\
&\leq c \sum_{|r| \leq N} \sum_{k \in \mathbf{Z}^d} ||f * (M_{\alpha k}g)(x)||_{L^\infty} ||\psi * (M_{\alpha(k+r)}g)(x)||_{L^p}
\end{aligned}$$

$$\leq C \|f\|_{M^{\infty, q'}} \|\psi\|_{M^{p, q}}.$$

In the last inequality, we use Hölder's inequality with $k \in \mathbf{Z}^d$ if $q \geq 1$, and continuous embedding $l^q \subset l^1$ if $q \leq 1$.

(b). Let $l \in (M^{p, q}(\mathbf{R}^d))^*$. Then using the definition of a convolution, we have

$$(3) \quad |l*(M_{\alpha k}g)(x_0)| = |\langle l, \overline{M_{\alpha k}g(x_0 - \cdot)} \rangle| \leq c \|l\|_{(M^{p, q})^*} \|M_{\alpha k}g(x_0 - \cdot)\|_{M^{p, q}}.$$

holds for all $x_0 \in \mathbf{R}^d$ and $k \in \mathbf{Z}^d$. Moreover,

$$\begin{aligned} \|M_{\alpha k}g(x_0 - \cdot)\|_{M^{p, q}}^q &= \sum_{n \in \mathbf{Z}^d} \left(\int |M_{\alpha k}(x_0 - \cdot) * (M_{\alpha n}g)(x)|^p dx \right)^{\frac{q}{p}} \\ &= \sum_{|r| \leq N} \|M_{\alpha k}g(x_0 - \cdot) * M_{\alpha(k+r)}g(x)\|_{L^p}^q. \end{aligned}$$

It follows from by Lemma 2.4 that the last sum is bounded by

$$C \sum_{|r| \leq N} \|M_{\alpha k}g(x_0 - \cdot)\|_{L^p}^q \|M_{\alpha(k+r)}g(x)\|_{L^p}^q,$$

where C is independent of x_0 and k . Combining this estimate with (3) and taking supremum over $x_0 \in \mathbf{R}^d$ and $k \in \mathbf{Z}^d$, we get the desired result. $\square$

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