Bloch spaces of holomorphic functions in the polydisk

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Abstract. This work is an introduction to anisotropic spaces of holomorphic functions, which have ω -weight and are generalizations of Bloch spaces to a polydisc. We prove that these classes form an algebra and are invariant with respect to monomial multiplication. Some theorems on projection and diagonal mapping are proved. We establish a description of $(A^p(\omega))^*$ (or $(H^p(\omega))^*$) via the Bloch classes for all $0 < p \leq 1$.

1. Introduction

The aim of this paper is to extend the Bloch spaces to a polydisk so that the well-known properties of the Bloch spaces of one variable remain true. Moreover, our generalization gives results which are new also for function of one variable. We are interested, for instance, in theorems on projection and the description of $(A^p(\alpha))^*$ via Bloch spaces. One can consider other generalisations of the Bloch space for polydisk (see for example [1]).

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In the paper of F. Shamoyan [2], the spaces $A^p(\alpha)$ have been generalized and investigated as $H^p(\omega)$ spaces in a polydisk. The Bloch spaces play the same role for $A^p(\alpha)$ as the BMOA spaces for the Hardy theory. Hence, it is very important to generalize the ω weighted Bloch spaces to a polydisk.

In Section 1, such classes are defined and an auxiliary lemma is proved. In Section 2, some theorems describing the properties of the introduced Bloch spaces are proved. In particular, we prove that they are invariant with respect to monomial multiplication and in the some case they are a Lipschitz class. In Section 3, we consider bounded operators in these classes and diagonal mappings. Section 4 is devoted to a description of $(A^p(\omega))^*$ via the introduced Bloch spaces.

2. Preliminaries and basic constructions

Let

$$U^n = \{z = (z_1, \dots, z_n) \in C^n, |z_j| < 1, 1 \leq j \leq n\}$$

be the unit polydisk in the n -dimensional complex plane C^n and let

$$T^n = \{z = (z_1, \dots, z_n) \in C^n, |z_i| = 1, 1 \leq i \leq n\}$$

be its torus. We denote by $H(U^n)$ the set of holomorphic functions in U^n and by $H^\infty(U^n)$ the set of bounded holomorphic functions in U^n .

Let S be the class of all non-negative measurable functions ω on $(0, 1)$, for which there exist positive numbers $M_\omega, q_\omega, m_\omega, (m_\omega, q_\omega \in (0, 1))$, such that

$$m_\omega \leq \frac{\omega(\lambda r)}{\omega(r)} \leq M_\omega,$$

for all $r \in (0, 1)$ and $\lambda \in [q_\omega, 1]$. For properties of functions from S , see [4]. Using the results of [4], one can prove that

$$(1) \quad \omega(x) = \exp \left\{ \eta(x) + \int_x^1 \frac{\varepsilon(u)}{u} du \right\}, \quad t \in (0, 1),$$

where $\varepsilon(x), \eta(x)$ are measurable bounded functions and if we set

$$-\alpha_\omega = \frac{\log m_\omega}{\log q_\omega^{-1}}; \quad \beta_\omega = \frac{\log M_\omega}{\log q_\omega^{-1}}.$$

then

$$-\alpha_\omega \leq \varepsilon(u) \leq \beta_\omega.$$

Next we assume that $\eta(x) = 0$, $x \in (0, 1)$. We put $\omega(1 - |z|) = \prod_{j=1}^n \omega_j(1 - |z_j|)$ and $(1 - \bar{z}z) = \prod_{j=1}^n (1 - \bar{z}_j z_j)$.

Definition 2.1. Let $f \in H(U^n)$, $\omega = (\omega_1, \dots, \omega_n)$, $\omega_j \in S$, $1 \leq j \leq n$, $0 < p \leq 1$. A function f belongs to the space $A^p(\omega)$ (or $H^p(\omega)$) if

$$\|f\|_{A^p(\omega)}^p = \int_{U^n} |f(z)|^p \omega(1 - |z|) dm_{2n}(z) < +\infty,$$

where $dm_{2n}(z)$ is the $2n$ -dimensional Lebesgue measure on U^n (see [2]).

In particular, if $\omega(t) = t^\alpha$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j > -1$, $1 \leq j \leq n$, then we have $A^p(\omega) = A^p(\alpha)$ (see [5]).

For $A^p(\omega)$ we assume that $0 < \beta_{\omega_j} < 1$, $1 \leq j \leq n$.

Definition 2.2. Let $\beta = (\beta_1, \dots, \beta_n)$, $\beta_j > -1$, $1 \leq j \leq n$. For a holomorphic function $f(z) = \sum_{|k|=0}^{\infty} a_k z^k$, $z \in U^n$,

(i) we define the fractional differentiation D^β in the following way:

$$D^\beta f(z) = \sum_{|k|=0}^{\infty} \prod_{j=1}^n \frac{\Gamma(\beta_j + 1 + k_j)}{\Gamma(\beta_j + 1)\Gamma(k_j + 1)} a_k z^k, \quad k = (k_1, \dots, k_n), \quad z \in U^n,$$

$$D^\beta f(\bar{z}) = \sum_{|k|=0}^{\infty} \prod_{j=1}^n \frac{\Gamma(\beta_j + 1 + k_j)}{\Gamma(\beta_j + 1)\Gamma(k_j + 1)} a_k \bar{z}^k, \quad k = (k_1, \dots, k_n), \quad z \in U^n.$$

(ii) the inverse operator $D^{-\beta}$ we define in the standard sense: $D^{-\beta} D^\beta f(z) = f(z)$.

In particular,

$$Df(z) = \frac{\partial^n (f(z)z)}{\partial z_1 \dots \partial z_n}.$$

The following properties of D are valid:

Lemma 2.3. (1) If

$$h(z) = \int_{U^n} \frac{\phi(\zeta) dm_{2n}(\zeta)}{(1 - \bar{\zeta}z)^\alpha}, \quad z \in U^n,$$

then

$$|D^\beta h(z)| \leq C(\beta, \pi) \int_{U^n} \frac{|\phi(\zeta)| dm_{2n}(\zeta)}{|1 - \bar{\zeta}z|^{\beta+\alpha}}$$

(2) If $Df(z) = Dg(z)$, then $f(z) = g(z)$, z in U^n .

Definition 2.4. Let $f \in H(U^n)$, $\omega = (\omega_1, \dots, \omega_n)$, $\omega_j \in S$, $1 \leq j \leq n$. A function f belongs to the Bloch space $B_\omega^n \equiv B_\omega$ if

$$(2) \quad M_f = \sup_{z \in U^n} \left\{ \frac{(1 - |z|^2)}{\omega(1 - |z|^2)} |Df(z)| \right\} < +\infty.$$

It is easy to see that B_ω is a Banach space with respect to the norm

$$\|f\|_{B_\omega} = |f(0)| + M_f.$$

For B_ω we assume that $0 < \alpha_{\omega_j} < 1$, $1 \leq j \leq n$.

Proposition 2.5. If $n = 1$, $\omega(t) = t^{1-s}$, then we deal the with known Bloch space of one variable (see [3], [6]).

The proof is evident.

Let $L_\omega^\infty = L_\omega^\infty(U^n)$ be the class of measurable functions on U^n , for which

$$\|f\|_{L_\omega^\infty} = \sup_{z \in U^n} \{|f(z)|\omega(1 - |z|^2)\} < +\infty.$$

To prove the main results, we need the following lemmas.

Lemma 2.6. Let $n = 1$, $\omega \in S$, $a+1-\beta_\omega > 0$, $b > 1$ and $b-a-2 > \alpha_\omega$. Then

$$\int_U \frac{(1 - |w|^2)^a \omega(1 - |w|^2)}{|1 - z\bar{w}|^b} dm_2(w) \leq \frac{C\omega(1 - |z|^2)}{(1 - |z|^2)^{b-a-2}}, \quad C = C(\omega, a, b)$$

Proof. We have

$$\int_{-\pi}^{\pi} \frac{d\theta}{|1 - r\rho e^{i\theta}|^b} \leq \frac{C(b)}{(1 - r\rho)^{b-1}}, \quad b > 1.$$

Then

$$\begin{aligned} & \int_U \frac{(1 - |w|^2)^a \omega(1 - |w|^2)}{|1 - z\bar{w}|^b} dm_2(w) \\ & \leq C(b) \int_0^1 \frac{(1 - \rho^2)^a \omega(1 - \rho^2) \rho d\rho d\theta}{(1 - r\rho)^{b-1}} \\ & \leq 2^{b-1} C(b) \int_0^1 \frac{(1 - t)^a \omega(1 - t) dt}{(1 - r^2 t)^{b-1}} \\ & = C_1(b) \int_0^1 \frac{u^a \omega(u) du}{(1 - r^2 + ur^2)^{b-1}} \end{aligned}$$

$$\begin{aligned}
&= C_1(b) \left\{ \int_0^{1-r^2} \frac{u^a \omega(u) du}{(1-r^2+ur^2)^{b-1}} + \int_{1-r^2}^1 \frac{u^a \omega(u) du}{(1-r^2+ur^2)^{b-1}} \right\} \\
&= C_1(\alpha)(I_1 + I_2).
\end{aligned}$$

First we estimate the integral I_1 :

$$\begin{aligned}
I_1 &\leq \int_0^{1-r^2} \frac{u^a \omega(u)}{(1-r^2)^{b-1}} du \\
&= \frac{1}{(1-r^2)^{b-1}} \int_0^{1-r^2} u^a \omega(u) du \\
&= \frac{(a+1)^{-1}}{(1-r^2)^{b-1}} \left[(1-r^2)^{a+1} \omega(1-r^2) + \int_0^{1-r^2} u^a \omega(u) \varepsilon(u) du \right].
\end{aligned}$$

It follows that

$$(a+1) \int_0^{1-r^2} u^a \omega(u) du = (1-r^2)^{a+1} \omega(1-r^2) + \int_0^{1-r^2} u^a \omega(u) \varepsilon(u) du,$$

and

$$\int_0^{1-r^2} (a+1-\varepsilon(u)) u^a \omega(u) du = (1-r^2)^{a+1} \omega(1-r^2).$$

On the other hand, we have the inequality

$$a+1-\beta_\omega \leq a+1-\varepsilon(u)$$

which yields

$$(a+1-\beta_\omega) \int_0^{1-r^2} u^a \omega(u) du \leq (1-r^2)^{a+1} \omega(1-r^2)$$

or

$$(3) \quad I_1 \leq C_1(\omega) \frac{\omega(1-r^2)}{(1-r^2)^{b-a-2}}.$$

We now estimate I_2 . For $r^2 \geq 1/2$, we have

$$\begin{aligned}
I_2 &\leq 2^{b-1} \int_{1-r^2}^1 \frac{\omega(u)}{u^{b-a-1}} \\
&= \frac{2^{b-1}}{a-b+2} \int_{1-r^2}^1 \omega(u) du^{a-b+2} \\
&= \frac{2^{b-1}}{b-a-2} \left[\frac{\omega(1-r^2)}{(1-r^2)^{b-a-2}} + \omega(1) + \int_{1-r^2}^1 \frac{\omega(u) \varepsilon(u)}{u^{b-a-1}} du \right].
\end{aligned}$$

Then

$$\int_{1-r^2}^1 \frac{\omega(u)}{u^{b-a-1}} du = \frac{\omega(1-r^2)}{(1-r^2)^{b-a-2}} - \omega(1) - \int_{1-r^2}^1 \frac{\omega(u)\varepsilon(u)}{u^{b-a-1}} du,$$

and it follows that

$$\begin{aligned} \int_{1-r^2}^1 \left(1 + \frac{\varepsilon(u)}{b-a-2}\right) \frac{\omega(u)}{u^{b-a-1}} du &= \frac{\omega(1-r^2)}{(1-r^2)^{b-a-2}} - \omega(1) \\ &\leq \frac{\omega(1-r^2)}{(1-r^2)^{b-a-2}}. \end{aligned}$$

Then the inequality

$$1 + \frac{\varepsilon(u)}{b-a-2} \geq 1 - \frac{\alpha_\omega}{b-a-2} > 0$$

gives us

$$(4) \quad \int_{1-r^2}^1 \frac{\omega(u)}{u^{b-a-1}} du \leq C_2(\omega) \frac{\omega(1-r^2)}{(1-r^2)^{b-a-2}}.$$

Summing up, from (3) and (4) we get the proof of Lemma 2.6. $\square$

Lemma 2.7. *Let $\alpha_j > \alpha_{\omega_j} + 1$, $1 \leq j \leq n$. Then $(1-|z|^2)^\alpha Df(z) \in L_{\omega^{-1}}^\infty$ if and only if $(1-|z|^2)^{\alpha-1} f(z) \in L_{\omega^{-1}}^\infty$.*

Proof. Let $g(z) = (1-|z|^2)^\alpha Df(z) \in L_{\omega^{-1}}^\infty$, then $Df \in A^1(\beta)$ for sufficiently large β_j , $1 \leq j \leq n$.

From the representation of M. M. Djrbashian (see [5]), we have

$$Df(z) = \frac{\beta+1}{\pi^n} \int_{U^n} \frac{(1-|\zeta|^2)^\beta}{(1-\bar{\zeta}z)^{\beta+2}} Df(\zeta) dm_{2n}(\zeta).$$

Integrating with respect to z_1 , we obtain

$$\begin{aligned} \frac{\partial^{n-1}(f(z)z)}{\partial z_2 \dots \partial z_n} &= \frac{1}{\pi^n} \int_{U^n} \frac{(1-|\zeta|^2)^\beta Df(\zeta) dm_{2n}(\zeta)}{(1-\bar{\zeta}_1 z_1)^{\beta_1+1} \bar{\zeta}_1 \prod_{j=2}^n (1-\bar{\zeta}_j z_j)^{\beta_j+2}} \\ &\quad - \frac{C(\beta)}{\pi^n} \int_{U^n} \frac{(1-|\zeta|^2)^\beta Df(\zeta)}{\bar{\zeta}_1 \prod_{j=2}^n (1-\bar{\zeta}_j z_j)^{\beta_j+2}} dm_{2n}(\zeta). \end{aligned}$$

It is easy to see (using for example the Taylor expansion) that the second integral is 0. Integrating the last equality with respect to $z_2, \dots, z_n$, we

obtain

$$(5) \quad f(z)z = \frac{1}{\pi^n} \int_{U^n} \frac{(1 - |\zeta|^2)^\beta}{(1 - \bar{\zeta}z)^{\beta+1}\zeta} Df(\zeta) dm_{2n}(\zeta).$$

Next, using the last representation, Lemma 2.6 and maximum modulus principle, we get

$$\begin{aligned} |f(z)| &\leq \|g\|_{L_{\omega^{-1}}^\infty} \int_{U^n} \frac{(1 - |w|^2)^{\beta-\alpha} \omega(1 - |w|^2)}{|1 - \bar{w}z|^{\beta+1}} dm_{2n}(w) \\ &\leq \|g\|_{L_{\omega^{-1}}^\infty} \frac{\omega(1 - |z|^2)}{(1 - |z|^2)^{\alpha-1}} \end{aligned}$$

and therefore $(1 - |z|^2)^{\alpha-1} f(z) \in L_{\omega^{-1}}^\infty$.

Conversely, if

$$g_1(z) = (1 - |z|^2)^{\alpha-1} f(z) \in L_{\omega^{-1}}^\infty,$$

then $f \in A^1(\beta)$ for $\beta_j > \beta_{\omega_j} + \alpha_j - 2$. Lemmas 2.3 and 2.6 imply that

$$\begin{aligned} |Df(z)| &\leq C(\beta, \pi) \int_{U^n} \frac{(1 - |\zeta|^2)^\beta}{|1 - \bar{\zeta}z|^{\beta+3}} |f(\zeta)| dm_{2n} \\ &\leq \|g_1\|_{L_{\omega^{-1}}^\infty} \int_{U^n} \frac{(1 - |\zeta|^2)^{\beta-\alpha+1} \omega(1 - |\zeta|^2)}{|1 - \bar{\zeta}z|^{\beta+3}} \\ &\leq \|g_1\|_{L_{\omega^{-1}}^\infty} \frac{\omega(1 - |z|^2)}{(1 - |z|^2)^\alpha}. \end{aligned}$$

Thus $(1 - |z|^2)^\alpha Df(z) \in L_{\omega^{-1}}^\infty$. □

3. Main properties of B_ω spaces

For $\omega_j \in S$, we say that ω satisfies the condition Ω if

$$t^{\alpha_{\omega_j}} \leq \omega_j(t) \leq t^{\gamma_{\omega_j}}, \quad \alpha_{\omega_j}, \gamma_{\omega_j} < 0, \quad \text{or} \quad \omega_j(t) = \text{const}, \quad 1 \leq j \leq n.$$

Theorem 3.1. *Suppose that ω satisfies the condition Ω . Then $H^\infty(U^n)$ is the subset of B_ω .*

Proof. Let $f \in H^\infty(U^n)$. We consider the torus

$$\tilde{T}^n = \tilde{T}_1 \dots \tilde{T}_n, \quad \tilde{T}_j = \left\{ \zeta_j, \zeta_j = z_j + \eta_j \frac{(1 - |z_j|^2)}{\omega_j(1 - |z_j|^2)} e^{i\theta_j} \right\} \quad j = 1, \dots, n.$$

We take η_j so that $\tilde{T}_j$ lies in the unit disc. Using the Cauchy formula for $\tilde{T}^n$, we get

$$Df(z) = \frac{\partial^n(f(z)z)}{\partial z_1 \dots \partial z_n} = \frac{1}{(2\pi i)^n} \int_{\tilde{T}^n} \frac{f(\zeta)\zeta d\zeta}{(\zeta - z)^2}.$$

It follows that

$$\begin{aligned} \left| \frac{\partial^n(f(z)z)}{\partial z_1 \dots \partial z_n} \right| &\leq \frac{1}{(2\pi)^n} \int_{\tilde{T}^n} \frac{|f(\zeta)\zeta| \prod_{j=1}^n \eta_j \frac{(1-|z_j|^2)}{\omega_j(1-|z_j|^2)} d\theta_1 \dots d\theta_n}{\prod_{j=1}^n \eta_j^2 \frac{(1-|z_j|^2)^2}{\omega_j^2(1-|z_j|^2)}} \\ &\leq C \int_{\tilde{T}^n} \frac{d\theta_1 \dots d\theta_n}{\prod_{j=1}^n \eta_j \frac{1-|z_j|^2}{\omega_j(1-|z_j|^2)}} \\ &= C_2 \frac{\omega(1-|z|^2)}{1-|z|^2} \end{aligned}$$

and hence $f \in B_\omega(U^n)$. $\square$

Remark 3.2. Note that $B_\omega \not\subset H^\infty$. As an example we can take the function $f(z) = \log(1-z_1) \dots \log(1-z_n)$.

Theorem 3.3. Let ω satisfy the condition Ω . Then the classes B_ω are invariant with respect to multiplication by monomials: $f \in B_\omega$ if and only if $z_j f \in B_\omega$ for any $j, j \in [1, n]$.

Proof. Let $f \in B_\omega$ and $j = 1$. Then

$$\frac{\partial^n(f(z)zz_1)}{\partial z_1 \dots \partial z_n} = \frac{\partial^n(f(z)z)}{\partial z_1 \dots \partial z_n} z_1 + \frac{\partial^{n-1}(f(z)z)}{\partial z_2 \dots \partial z_n}.$$

Using the fact that $h(z)z = \int_0^z (h(t)t)' dt$, $h \in H(U)$, we get

$$\left| \frac{\partial^{n-1}(f(z)z)}{\partial z_2 \dots \partial z_n} \right| \leq M_f \frac{\omega(1-|z|^2)}{(1-|z|^2)}.$$

It follows that $f(z)z_1 \in B_\omega$.

Conversely, let $f(z)z_1 \in B_\omega$. We will prove that $f \in B_\omega$. We have $f(z) = \frac{1}{z_1} f(z)z_1$ and

$$Df(z) = \frac{\partial^n(f(z)z)}{\partial z_1 \dots \partial z_n} = -\frac{1}{z_1^2} \frac{\partial^{n-1}(f(z)zz_1)}{\partial z_2 \dots \partial z_n} + \frac{1}{z_1} \frac{\partial^n(f(z)zz_1)}{\partial z_1 \dots \partial z_n}.$$

Therefore,

$$|Df(z)| \leq \frac{1}{|z_1|^2} \left| \frac{\partial^{n-1}(f(z)zz_1)}{\partial z_2 \dots \partial z_n} \right| + \frac{1}{|z_1|} \left| \frac{\partial^n(f(z)zz_1)}{\partial z_1 \dots \partial z_n} \right|.$$

If $|z_1| \geq \frac{1}{2}$, then

$$|Df(z)| \leq 4 \left| \frac{\partial^{n-1}(f(z)zz_1)}{\partial z_2 \dots \partial z_n} \right| + 2 \left| \frac{\partial^n(f(z)zz_1)}{\partial z_1 \dots \partial z_n} \right|$$

and

$$(6) \quad \frac{(1-|z|^2)}{\omega(1-|z|^2)} |Df(z)| < 6M_f < \infty.$$

If $|z_1| < 1/2$, then we use maximum of modulus principle for the holomorphic function $\frac{\partial^n f(z)}{\partial z_1 \dots \partial z_n} z_1^2$. Thus, (6) is true for all $|z_1| < 1$ and $f \in B_\omega$. $\square$

Proposition 3.4. *If ω satisfies the condition Ω , then B_ω is an algebra.*

Proof. The proof is standard. It is evident that if $f, g \in B_\omega$, then $\alpha f + \beta g \in B_\omega$ for all $\alpha, \beta \in \mathbb{C}$. Let us show that $fg \in B_\omega$. To this end, we use the Leibniz formula

$$\frac{\partial^k(f(z)g(z))}{\partial z_1 \dots \partial z_k} = \sum_{j=0}^k C_k^j \frac{\partial^j f(z)}{\partial z_{i_1} \dots \partial z_{i_j}} \cdot \frac{\partial^{k-j} g(z)}{\partial z_{i_{j+1}} \dots \partial z_{i_k}}.$$

and the fact that $f, g \in B_\omega$. It follows that $f \cdot g \in B_\omega$. $\square$

The following proposition shows that, in some cases, the spaces B_ω are ω weighted Lipschitz classes (see [7]).

Proposition 3.5. *Let $t^{\alpha_{\omega_j}} \leq \omega_j(t) \leq t^{\gamma_{\omega_j}}$, $0 < \alpha_{\omega_j}, \gamma_{\omega_j} < 1$, $1 \leq j \leq n$ then $B_\omega = \tilde{\Lambda}_a(\omega)$, where $\tilde{\Lambda}_a(\omega)$ denotes the weighted Lipschitz spaces for polydisk.*

Proof. Using (1), it is not difficult to see that there exist $C_1, C_2 \neq 0$ such that

$$C_1 \omega_j(1 - |z_j|) \leq \omega_j(1 - |z_j|^2) \leq C_2 \omega_j(1 - |z_j|), \quad j = 1, 2, \dots, n.$$

Let $f \in B_\omega$ and $(i_1, \dots, i_k) = (1, \dots, k)$. Then

$$|Df(z)| \leq M_f \frac{\omega(1 - |z|)}{(1 - |z|)}.$$

Integrating this inequality with respect to $z_{k+1}, \dots, z_n$, we derive

$$\left| \frac{\partial^k (f(z)z')}{\partial z_1 \dots \partial z_k} \right| \leq C_2 M_f \prod_{j=1}^k \frac{\omega_j(1 - |z_j|)}{(1 - |z_j|)} \quad z' = (z_1, \dots, z_k).$$

From the invariance of $\tilde{\Lambda}_a(\omega)$ with respect to the monomial multiplication, it follows that $f \in \tilde{\Lambda}_a(\omega)$.

Let now $f \in \tilde{\Lambda}_a(\omega)$. Using the characterisation and invariance with respect to the monomial multiplication of holomorphic Lipschitz classes again, we see that $f \in B_\omega$. $\square$

Hence, in view of the results of [7], we obtain

Corollary 3.6. Let $t^{\alpha_{\omega_j}} \leq \omega_j(t) \leq t^{\gamma_{\omega_j}}$, $0 < \alpha_{\omega_j}, \gamma_{\omega_j} < 1$, $1 \leq j \leq n$. Then:

- (i) B_ω , is an algebra and $B_\omega \subset H^\infty(U^n)$;
- (ii) B_ω is invariant with respect monomial multiplication.

4. Projection theorems and diagonal mappings

For $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j > 0$, $1 \leq j \leq n$, we introduce the operator

$$Q_\alpha f(z) = \alpha \int_{U^n} \frac{f(\zeta)}{(1 - \bar{\zeta}z)^{\alpha+1}} dm_{2n}(\zeta)$$

which is not a projection.

Theorem 4.1. Let $\alpha_j > \beta_{\omega_j}$, $1 \leq j \leq n$. Then the map Q_α is bounded from $L_{\tilde{\omega}}^\infty$ to B_ω , where $\tilde{\omega}(t) = t^{\alpha-1}\omega(t)$. Moreover, if $\alpha_j > \max\{\beta_{\omega_j} + 1, \alpha_{\omega_j} + 1\}$, $1 \leq j \leq n$ then Q_α is surjective.

Proof. We show that (2) is true for the function $F(z) = Q_\alpha f(z)$:

$$\begin{aligned} |DF(z)| &\leq \alpha \int_{U^n} \frac{|f(w)|\omega(1 - |w|^2)}{|1 - \bar{w}z|^{\alpha+2}\omega(1 - |w|^2)} dm_{2n}(w) \\ &\leq \alpha \|f\|_{L_{\tilde{\omega}-1}^\infty} \int_{U^n} \frac{(1 - |w|^2)^{\alpha-1}\omega(1 - |w|^2)}{|1 - \bar{w}z|^{\alpha+2}} dm_{2n}(w). \end{aligned}$$

Using Lemma 2.6, we get

$$\|F\|_{B_\omega} \leq C(\omega, \alpha) \|f\|_{L_{\tilde{\omega}-1}^\infty},$$

and we see that Q_α is a bounded operator from $L_{\tilde{\omega}-1}^\infty$ into B_ω .

Next we show that Q_α is surjective: for any $f \in B_\omega$ there exists a function $h \in L_{\omega^{-1}}^\infty$ so that $f(z) = Q_\alpha h(z)$.

Let $f \in B_\omega$ and $\alpha_j > \max\{\beta_{\omega_j} + 1, \alpha_{\omega_j} + 1\}$, $1 \leq j \leq n$. The function $\frac{\alpha+1}{\alpha\pi^n}(1-|z|^2)^\alpha Df(z)$ belongs to $L_{\omega^{-1}}^\infty$. Using Lemma 2.7, we see that the function $h(z) = \frac{\alpha-1}{\alpha\pi^n}(1-|z|^2)^{\alpha-1}f(z)$ also belongs to $L_{\omega^{-1}}^\infty$. We set

$$Q_\alpha h_1(z) = \frac{\alpha-1}{\pi^n} \int_{U^n} \frac{(1-|w|^2)^{\alpha-1}}{(1-\bar{w}z)^{\alpha+1}} f(w) dm_{2n}(w).$$

On the other hand, the function $f(z)$ is holomorphic and belongs to $A^1(\alpha-1)$ if $\alpha_j > \beta_{\omega_j} + 1$, $1 \leq j \leq n$. It can be written as

$$f(z) = \frac{\alpha-1}{\pi^n} \int_{U^n} \frac{(1-|w|^2)^{\alpha-1}}{(1-\bar{w}z)^{\alpha+1}} f(w) dm_{2n}(w).$$

Thus, we get $f(z) = Q_\alpha h(z)$ $z \in U^n$. □

Consider now the projection

$$P_\alpha f(z) = \frac{(\alpha+1)}{\pi^n} \int_{U^n} \frac{(1-|w|^2)^\alpha}{(1-\bar{w}z)^{\alpha+2}} f(w) dm_{2n}(w) (\alpha_j > 0, 1 \leq j \leq n),$$

it is not difficult to prove the following result:

Proposition 4.2. *Let $\alpha_j > \beta_{\omega_j}$, $1 \leq j \leq n$. Then $P_\alpha : B_\omega \rightarrow L_{\omega^{-1}}^\infty$ and is a bounded.*

The question is whether there is an inverse operator $R_{\alpha,\beta}$ which maps B_ω to $L_{\omega^{-1}}^\infty$ and furthermore, if this is the case, whether $P_\alpha(R_{\alpha,\beta}(f))(z) = f(z)$ for all $f \in B_\omega$. The answer is positive.

Let us define the 'inverse' operator as

$$R_{\alpha,\beta} f(z) = (1-|z|^2)^\beta \int_{U^n} \frac{(1-|\zeta|^2)^{\alpha-1}}{(1-\bar{\zeta}z)^{\alpha+\beta+1}} f(\zeta) dm_{2n}(\zeta),$$

$$\alpha = (\alpha_1, \dots, \alpha_n) \quad \beta = (\beta_1, \dots, \beta_n), \quad \alpha_j > 0, \quad \beta_j \geq 1, \quad 1 \leq j \leq n.$$

We first show that $P_\alpha R_{\alpha,\beta} f(z) = f(z)$, $z \in U^n$. To this end, let us calculate $P_\alpha R_{\alpha,\beta} f(z)$ using the Fubini theorem:

$$\begin{aligned} P_\alpha R_{\alpha,\beta} f(z) &= \frac{\alpha}{\pi^n} \int_{U^n} \frac{(1-|\zeta|^2)^{\alpha+\beta-1}}{(1-\bar{\zeta}z)^{\alpha+1}} \\ &\quad \times \int_{U^n} \frac{(1-|w|^2)^{\alpha-1} f(w)}{(1-\bar{w}\zeta)^{\alpha+\beta+1}} dm_{2n}(w) dm_{2n}(\zeta) \\ &= \frac{\alpha+1}{\pi^n} \int_{U^n} (1-|w|^2)^{\alpha-1} f(w) \end{aligned}$$

$$\begin{aligned}
& \times \int_{U^n} \frac{(1-|\zeta|^2)^{\alpha+\beta-1} dm_{2n}(\zeta)}{(1-\bar{\zeta}z)^{\alpha+1}(1-\bar{w}\zeta)^{\alpha+\beta+1}} dm_{2n}(w) \\
& = \int_{U^n} \frac{(1-|w|^2)^{\alpha-1}}{(1-\bar{w}z)^{\alpha+1}} f(w) dm_{2n}(w) = f(z), \quad (\alpha_j > \beta_{\omega_j} + 1).
\end{aligned}$$

Theorem 4.3. Let $\alpha_j > \max\{\beta_{\omega_j} + 1, \alpha_{\omega_j} + 1\}$ and $\beta_j > \alpha_{\omega_j}$, $1 \leq j \leq n$. Then

(a) The operator $R_{\alpha,\beta}$ is bounded from B_ω to $L_{\omega^{-1}}^\infty$, and there exist constants $C_1(\omega)$, $C_2(\omega)$ such that

$$(7) \quad C_1(\omega) \|f\|_{B_\omega} \leq \|R_{\alpha,\beta} f\|_{L_{\omega^{-1}}^\infty} \leq C_2(\omega) \|f\|_{B_\omega}.$$

(b) $f \in B_\omega$ if and only if $R_{\alpha,\beta} f \in L_{\omega^{-1}}^\infty$.

Proof. (a) Let $f \in B_\omega$. Theorem 4.1 implies that there exists a function $g \in L_{\omega^{-1}}^\infty$ such that $Q_\alpha g(z) = f(z)$, $z \in U^n$. Using the Fubini theorem, we get

$$\begin{aligned}
R_{\alpha,\beta} f(z) &= \alpha(1-|z|^2)^\beta \int_{U^n} g(w) \int_{U^n} \frac{(1-|\zeta|^2)^{\alpha-1} dm_{2n}(\zeta)}{(1-\bar{\zeta}z)^{\alpha+\beta+1}(1-\bar{w}\zeta)^{\alpha+1}} dm_{2n}(w) \\
&= \pi^n (1-|z|^2)^\beta \int_{U^n} \frac{g(w)}{(1-\bar{w}z)^{\alpha+\beta+1}} dm_{2n}(w).
\end{aligned}$$

Thus,

$$\begin{aligned}
|R_{\alpha,\beta} f(z)| &\leq \|g\|_{L_{\omega^{-1}}^\infty} (1-|z|^2)^\beta \int_{U^n} \frac{(1-|w|^2)^{\alpha-1} \omega(1-|w|^2)}{|1-\bar{w}z|^{\alpha+\beta+1}} dm_{2n}(w) \\
&\leq C(\omega) \|g\|_{L_{\omega^{-1}}^\infty} \omega(1-|z|^2),
\end{aligned}$$

and hence

$$\|R_{\alpha,\beta} f\|_{L_{\omega^{-1}}^\infty} \leq C(\omega) \|g\|_{L_{\omega^{-1}}^\infty}, \quad g \in L_{\omega^{-1}}^\infty.$$

Using Theorem 4.1 and the open mapping theorem, we get

$$(8) \quad \|R_{\alpha,\beta} f\|_{L_{\omega^{-1}}^\infty} \leq C_2(\omega) \|f\|_{B_\omega}$$

i.e. $R_{\alpha,\beta}$ is bounded from B_ω to $L_{\omega^{-1}}^\infty$. To prove the other inequality in (7), we use the fact, that $P_\alpha R_{\alpha,\beta} f(z) = f(z)$ for all $f \in B_\omega$. Then by Proposition 4.2, there exists $C_0 > 0$ such that

$$(9) \quad \|f\|_{B_\omega} = \|P_\alpha R_{\alpha,\beta} f\|_{B_\omega} \leq C_0(\omega) \|R_{\alpha,\beta} f\|_{L_{\omega^{-1}}^\infty}$$

Taking $C_1 = C_0^{-1}$, we obtain the left-hand side inequality in (7).

(b) The proof follows from (8) and (9). $\square$

Let us formulate the problem of diagonal mapping in the general case: let $f \in X \subset H(U^n)$, then the function $g(z) = f^*(z) = f(z, \dots, z)$ is analytic on the unit disk.

Problem. Find all functions g of the class $H(U^n)$ for which there is $f \in X$, such that

$$g(z) = f^*(z) = f(z, \dots, z).$$

Assume that $n = 2$, $X = B_\omega$. Then we have the following theorem

Theorem 4.4. Let $\omega_1, \omega_2 \in S$ and $f \in B^2(\omega)$. Then $f^* \in B_{\omega^*}^1$, where $\omega^*(t) = \omega_1(t)\omega_2(t)$. Conversely, for any $g \in B_{\omega^*}^1$ there exists a function $f \in B_\omega$ such that $g(z) = f^*(z)$, $z \in U$.

Proof. Let $f \in B^2(\omega)$, then

$$M_f = \sup_{z \in U} \left\{ \frac{(1 - |z|^2)^2}{\omega_1(1 - |z|^2)\omega_2(1 - |z|^2)} |D^2 g(z)| \right\} < +\infty$$

and hence $g \in B_{\omega^*}^1$.

Now show that for any $g \in B_{\omega^*}^1$ there exists $f \in B_\omega$ so that $g(z) = f^*(z)$. Indeed, taking $\gamma > (\beta_{\omega_1} + \beta_{\omega_2} + 2)/2$ it is easy to show that $Dg(z) \in A^1(\gamma n - 2)$ and, hence,

$$Dg(z) = C(\gamma, \pi) \int_U \frac{(1 - |\zeta|^2)^{2\gamma-2}}{(1 - \bar{\zeta}z)^{2\gamma}} Dg(\zeta) dm_2(\zeta).$$

Let us write

$$Df(z) = C(\gamma, \pi) \int_U \frac{(1 - |\zeta|^2)^{2\gamma-2}}{(1 - \bar{\zeta}z_1)^\gamma (1 - \bar{\zeta}z_2)^\gamma} Dg(\zeta) dm_2(\zeta).$$

Then $f(z) = D^-(Df(z))$ and $Df^*(z) = Dg(z)$. Therefore from Lemma 2.3 we get $f^*(z) = g(z)$, $z \in U$.

Let us show that $f \in B_\omega$. In fact,

$$\begin{aligned} |Df(z)| &\leq C(\gamma, \pi) \leq \int_U \frac{(1 - |\zeta|^2)^{2\gamma-2} |Dg(\zeta)|}{|1 - \bar{\zeta}z_1|^\gamma |1 - \bar{\zeta}z_2|^\gamma} dm_2(\zeta) \\ &\leq \|g\|_{B_{\omega^*}} \int_U \frac{(1 - |\zeta|^2)^{2\gamma-3} |\omega^*(1 - |\zeta|^2)|}{|1 - \bar{\zeta}z_1|^\gamma |1 - \bar{\zeta}z_2|^\gamma} dm_2(\zeta). \end{aligned}$$

Using Holder inequality and Lemma 2.6, we get

$$|Df(z)| \leq C^* \left(\int_U \frac{(1 - |\zeta|^2)^{2\gamma-3} |\omega_1^2(1 - |\zeta|^2)|}{|1 - \bar{\zeta}z_1|^{2\gamma}} dm_2(\zeta) \right)^{1/2}$$

$$\begin{aligned} & \times \left(\int_U \frac{(1-|\zeta|^2)^{2\gamma-3} |\omega_2^2(1-|\zeta|^2)}{|1-\bar{\zeta}z_2|^{2\gamma}} dm_2(\zeta) \right)^{1/2} \\ & \leq \frac{\omega_1(1-|z_1|^2)\omega_2(1-|z_2|^2)}{(1-|z_1|)(1-|z_2|)} \end{aligned}$$

Hence $f \in B_\omega^2$. □

5. Linear continuous functionals on $A^p(\omega)$

We need to establish the following theorem before proving the duality result.

Theorem 5.1. *Let $\beta_j > 0$, $1 \leq j \leq n$. Then $f \in B_\omega$ if and only if*

$$(10) \quad M_f^\beta = \sup_{z \in U^n} \left\{ \frac{(1-|z|^2)^{\beta+1}}{\omega(1-|z|^2)} D^{\beta+1} f(z) \right\} < \infty.$$

Proof. Let $f \in B_\omega$. Then, by Theorem 4.1, there exists a function $g \in L_{\omega^{-1}}^\infty$ such that

$$f(z) = \alpha \int_{U^n} \frac{g(w)}{(1-\bar{w}z)^{\alpha+1}} dm_{2n}(w).$$

For $\beta_j + 1 > \alpha_{\omega_j}$, we get

$$\begin{aligned} |D^{\beta+1} f(z)| & \leq \alpha \int_{U^n} \frac{|g(w)| \omega(1-|w|^2)}{|1-\bar{w}z|^{\alpha+\beta+2} \omega(1-|w|^2)} dm_{2n}(w) \\ & \leq \alpha \|g\|_{L_{\omega^{-1}}^\infty} \int_{U^n} \frac{(1-|w|^2)^\alpha \omega(1-|w|^2)}{|1-\bar{w}z|^{\alpha+\beta+3}} dm_{2n}(w) \\ & \leq C(\alpha, \beta, \omega) \|g\|_{L_{\omega^{-1}}^\infty} \frac{\omega(1-|z|^2)}{(1-|z|^2)^{\beta+1}}. \end{aligned}$$

It follows that

$$\sup_{z \in U^n} \left\{ \frac{(1-|z|^2)^{\beta+1}}{\omega(1-|z|^2)} D^{\beta+1} f(z) \right\} < \infty.$$

Conversely, if (10) takes place, then taking m_j , $1 \leq j \leq n$ sufficiently large, using the inequality

$$|Df(z)| \leq C(m, \pi, \beta) \int_{U^n} \frac{(1-|\zeta|^2)^m}{|1-\bar{\zeta}z|^{m+2-\beta}} |D^{\beta+1} f(\zeta)| dm_{2n}(\zeta),$$

and applying Lemma 2.6, we easily show that

$$\sup_{z \in U^n} \left\{ \frac{(1 - |z|^2)}{\omega(1 - |z|^2)} |Df(z)| \right\} < \infty. \quad \square$$

The following theorem describes the continuous linear functionals on $A^p(\omega)$.

Theorem 5.2. *Let Φ be a continuous linear functional on $A^p(\omega)$, $0 < p \leq 1$ and let $g(z) = \Phi((1 - zw)^{-1})$, $(z, w \in U^n)$. Then:*

(i) (a) $g \in B_{\tilde{\omega}}$, $\tilde{\omega}(t) = \omega^{1/p}(t)t^{2/p-1}$

(b) *The functional Φ is representable in the forms*

$$(11) \quad \Phi(f) = \lim_{\rho \rightarrow 1-0} \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f(\rho\zeta) g(\rho\bar{\zeta}) dm_n(\zeta),$$

$$(12) \quad \Phi(f) = \lim_{\rho \rightarrow 1-0} \frac{1}{(2\pi)^n} \int_{U^n} f(\zeta) D^{m+1} g(\rho^2\bar{\zeta}) (1 - |\zeta|^2)^m dm_{2n}(\zeta),$$

$$m = (m_1, \dots, m_n), \quad m_j > \frac{\alpha_{\omega_j} + 2}{p}, \quad 1 \leq j \leq n.$$

Moreover, there are positive constants $C_1(\tilde{\omega})$ and $C_2(\omega)$ such that

$$(13) \quad C_1(\tilde{\omega}) \|\Phi\| \leq \|g\|_{B_{\tilde{\omega}}} \leq C_2(\omega) \|\Phi\|.$$

(ii) *Conversely, for any function $g \in B_{\tilde{\omega}}$ relations (11) and (12) define a continuous linear functional on $A^p(\omega)$, which satisfies (13).*

Proof. It is easy to show that the linear functional Φ is continuous on $A^p(\omega)$ if and only if

$$\|\Phi\| = \sup_{\|f\|_{A^p(\omega)} \leq 1} |\Phi(f)| < +\infty.$$

(i) Let Φ be a continuous linear functional on the space $A^p(\omega)$ and let $e_z = (1 - \zeta z)^{-1}$ (remember $0 < \beta_{\omega_j} < 1$, $0 \leq j \leq n$). Then

$$\begin{aligned} |D^{\beta+1} g(z)| &= |\Phi(D^{\beta+1} e_z)| \leq \|\Phi\| \|D^{\beta+1} e_z\|_{A^p(\omega)} \\ &= \|\Phi\| \left(\int_{U^n} \frac{\omega(1 - |\zeta|^2)}{|1 - \zeta z|^{p(\beta+2)}} dm_{2n}(\zeta) \right)^{1/p} \\ &\leq C_2(\tilde{\omega}) \|\Phi\| \frac{\omega^{1/p}(1 - |z|^2)}{(1 - |z|^2)^{\beta+2-2/p}}. \end{aligned}$$

We get

$$|D^{\beta+1}g(z)| \frac{(1-|z|^2)^{\beta+2-2/p}}{\omega^{1/p}(1-|z|^2)} \leq C_2(\tilde{\omega})\|\Phi\|.$$

Hence, by Theorem 5.1, we have $g \in B_{\tilde{\omega}}$ and $\|g\|_{B_{\tilde{\omega}}} \leq C_2(\tilde{\omega})\|\Phi\|$.

We now show that Φ has the form (11). We have

$$g(z) = \Phi(e_z) = \sum_{|k|=0}^{\infty} \Phi(\zeta^k)z^k = \sum_{|k|=0}^{\infty} a_k z^k, \quad a_k = \Phi(\zeta^k).$$

It is well known, that

$$\lim_{\rho \rightarrow 1-0} \|f - f_{\rho}\|_{A^p(\omega)} \rightarrow 0, \quad f \in A^p(\omega), \quad f_{\rho}(z) = f(\rho z).$$

Thus

$$\Phi(f) = \lim_{\rho \rightarrow 1-0} \Phi(f_{\rho}) = \lim_{\rho \rightarrow 1-0} \sum_{|k|=0}^{\infty} a_k b_k \rho^k, \quad (f(z) = \sum_{|k|=0}^{\infty} b_k z^k).$$

We can also write

$$\Phi(f) = \lim_{\rho \rightarrow 1-0} \sum_{|k|=0}^{\infty} a_k b_k \rho^{2k}.$$

Calculations show that

$$\int_{T^n} f(\rho\zeta)g(\rho\bar{\zeta})dm_n(\zeta) = \sum_{|k|=0}^{\infty} a_k b_k \rho^{2k}$$

which proves (11).

(ii) Let us show that for any function $g \in B_{\tilde{\omega}}$ relations (11) and (12) define a continuous linear functional on $A^p(\omega)$, which satisfies (13). Let $f \in A^p(\omega)$. Then $f \in A^1(m)$ for some sufficiently large $m = (m_1, \dots, m_n)$ and from the Djrbashian representation we get

$$\begin{aligned} \int_{T^n} f(\rho\zeta)g(\rho\bar{\zeta})dm_n(\zeta) &= \frac{m+1}{\pi^n} \int_{T^n} g(\rho\bar{\zeta}) \\ &\times \int_{U^n} \frac{(1-|w|^2)^m f(w)}{(1-\rho\bar{w}\zeta)^{m+2}} dm_{2n}(w) dm_n(\zeta) \\ &= \frac{m+1}{\pi^n} \int_{U^n} f(w)(1-|w|^2)^m \\ &\times \int_{T^n} \frac{g(\rho\bar{\zeta})}{(1-\bar{w}\rho\zeta)^{m+2}} dm_n(\zeta) dm_{2n}(w). \end{aligned}$$

Nothing that

$$\frac{1}{(2\pi)^n} \int_{T^n} \frac{g(\rho\bar{\zeta})}{(1 - \bar{w}\rho\zeta)^{m+2}} dm_n(\zeta) = D^{m+1}g(\rho^2\bar{w}),$$

we obtain the formula

$$(14) \quad \int_{T^n} f(\rho\zeta)g(\rho\bar{\zeta})dm_n(\zeta) = \int_{U^n} f(\zeta)D^{m+1}g(\rho^2\bar{\zeta})(1 - |\zeta|^2)^m dm_{2n}(\zeta)$$

To show that the limits in (11) exist, we estimate the right hand-side of (14):

$$I = \int_{U^n} f(\zeta)D^{m+1}g(\rho^2\bar{\zeta})(1 - |\zeta|^2)^m dm_{2n}(\zeta).$$

We have

$$I \leq C\|g\|_{B_{\bar{w}}} \int_{U^n} \frac{\omega^{1/p}(1 - |\zeta|^2)}{(1 - |\zeta|^2)^{2-2/p}} |f(\zeta)| dm_{2n}(\zeta).$$

Let $k = (k_1, \dots, k_n)$, $k_j \geq 0$, $l_j \in Z$ and $-2^{k_j} \leq l_j \leq 2^{k_j+1} - 1$, $1 \leq j \leq n$.

We put

$$\Delta_{k_j, l_j} = \{z_j : 1 - 1/2^{k_j} \leq |z_j| < 1 - 1/2^{k_j+1}, \pi l_j/2^{k_j} \leq \arg z_j < \pi(l_j+1)/2^{k_j}\}$$

$$\Delta_{k, l} = \Delta_{k_1, l_1} \times \dots \times \Delta_{k_n, l_n}$$

Since $0 < p \leq 1$, we have

$$I^p \leq C_1 \|g\|_{B_{\bar{w}}}^p \sum_{|k|=0}^{\infty} \sum_{j=-2^k}^{2^k-1} \max_{\zeta \in \Delta_{k, j}} |f(\zeta)|^p \frac{\omega(1 - |\zeta_{k, j}|)}{(1 - |\zeta_{k, j}|)^{p-2}} |\Delta_{k, j}|^p$$

$$k = (k_1, \dots, k_n), \quad j = (j_1, \dots, j_n), \quad 2^k = (2^{k_1}, \dots, 2^{k_n}).$$

Hence

$$(1 - \eta)(1 - |\zeta_{k_l, j_l}|) \leq 1 - |\zeta_l| \leq (1 + \eta)(1 - |\zeta_{k_l, j_l}|), \quad \zeta_l \in \Delta_{k, j}$$

$$\frac{(1 - |\zeta_{k_l, j_l}|)^2}{4} \leq |\Delta_{k_l, j_l}| \leq 4(1 - |\zeta_{k_l, j_l}|)^2, \quad 1 \leq j \leq n$$

and therefore, by Lemma 4 in [2], we have

$$\begin{aligned} I^p &\leq C_1(\omega) \|g\|_{B_{\bar{w}}}^p \sum_{|k|=0}^{\infty} \sum_{j=-2^k}^{2^k-1} \max_{\zeta \in \Delta_{k, j}} |f(\zeta)|^p \omega(|\Delta_{k, j}|^{1/2}) \\ &\leq C_1(\omega) \|g\|_{B_{\bar{w}}}^p \|f\|_{A^p(\omega)}^p. \end{aligned}$$

Using the Cauchy formula, it is easy to show that $\Phi(e_z) = g(z)$. Thus, the inequality $C_1(\omega)\|\Phi\| \leq \|g\|_{\tilde{B}_\omega}$ is also true. $\square$

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