On indirect variational formulations for operator equations¹

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Abstract. A scheme for the construction of indirect variational formulations for a wide class of equations is suggested.

1. Introduction

An important problem in applications of variational methods is a representation of a given system of equations in the form of the Euler - Lagrange equations. It means the construction of a functional F_N such that its extremals are solutions of the given system of equations. This is known as the classical inverse problem of the calculus of variations [1,2,4,7].

In spite of the remarkable number of papers on the subject different approaches for constructing of integral variational principles for equations with nonpotential operators should be developed. They will allow to obtain so-called indirect variational formulations of given problems.

The main aim of the present paper is a constructive determination of indirect variational formulations for any operator equation with the Gâteaux differentiable operator N .

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2. Bilinear forms and variational principles

Let N be an operator such that its domain of definition $D(N) \subseteq U \subseteq V$ and range of values $R(N) \subseteq V$, where U and V are linear normed spaces over $\mathbb{R}$, i. e.

$$N(u) = v, \quad u \in U, \quad v \in V.$$

If there exists a limit

$$(1) \quad \delta N(u, h) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \{N(u + \varepsilon h) - N(u)\}, \quad u \in D(N), (u + \varepsilon h) \in D(N),$$

then it is called the Gâteaux variation of the operator N at the point u or the first variation of the operator N at the point u .

$\delta N(u, h)$ is homogeneous with respect to h : $\delta N(u, \lambda h) = \lambda \delta N(u, h)$, but the operator $\delta N(u, \cdot) : U \rightarrow V$ is not always additive with respect to h .

If $\delta N(u, h)$ is a linear operator with respect to h , when u is a fixed element of $D(N)$, then we say that the operator N is Gâteaux differentiable at the point u . The expression $\delta N(u, h)$ is called the Gâteaux differential and denoted by $DN(u, h)$. In this case we will also write $DN(u, h) = N'_u h$ and say that N'_u is the Gâteaux derivative of operator N at the point u .

If N is a linear operator then $N'_u h = Nh$, i. e. the Gâteaux derivative of the linear operator coincides with it.

Further assume that for any given operator $N : D(N) \subset U \rightarrow V$ there exists its Gâteaux derivative at any point $u \in D(N)$. The domain of definition $D(N'_u)$ consists of elements $h \in U$ such that $(u + \varepsilon h) \in D(N)$ for all ε sufficiently small. In this case $h \in D(N'_u)$ is called an admissible element.

If the Gâteaux derivative of the operator N exists, then the following equality holds [3]

$$(2) \quad N(u + \varepsilon h) = N(u) + \varepsilon N'_u h + r(u, \varepsilon h), \quad u \in D(N),$$

where for any fixed element $h \in D(N'_u)$

$$\lim_{\varepsilon \rightarrow 0} \frac{r(u, \varepsilon h)}{\varepsilon} = 0_V.$$

Note that for any linear operator $\tilde{N}_u$ which may depend on u in a nonlinear way, the Gâteaux derivative is defined by

$$(3) \quad \tilde{N}'_u(g; h) = \lim_{\varepsilon \rightarrow 0} \frac{\tilde{N}_{u+\varepsilon h} g - \tilde{N}_u g}{\varepsilon}.$$

The second Gâteaux derivative N''_u of the operator N is given by

$$(4) \quad N''_u(h_1, h_2) = \frac{\partial^2}{\partial \varepsilon^1 \partial \varepsilon^2} N(u + \varepsilon^1 h_1 + \varepsilon^2 h_2)|_{\varepsilon^1 = \varepsilon^2 = 0}.$$

In the most general applications N''_u satisfies the symmetry condition

$$N''_u(h_1, h_2) = N''_u(h_2, h_1).$$

Now we need some notations and notions about bilinear forms and potential operators.

Definition 1. A mapping $\Phi(\cdot, \cdot) : V \times U \rightarrow \mathbb{R}$ is said to be a nonlocal bilinear form if it is linear with respect to every argument.

Definition 2. A nonlocal bilinear form $\Phi(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ is called symmetric if

$$\Phi(v, g) = \Phi(g, v) \quad \text{for all } g, v \in V.$$

Definition 3. A bilinear mapping $\Phi(u; \cdot, \cdot) : V \times U \rightarrow \mathbb{R}$ is said to be a local bilinear form if it depends on u . In particular $\Phi(u; \cdot, \cdot) : V \times U \rightarrow \mathbb{R}$ is called nondegenerate if the following conditions hold

$$\begin{cases} \text{if } \Phi(u; v, h) = 0 \quad \text{for all } v \in V, u \in U, & \text{then } h = 0_U, \\ \text{if } \Phi(u; v, h) = 0 \quad \text{for all } h \in U, u \in U, & \text{then } v = 0_V. \end{cases}$$

An operator C_u is called symmetric with respect to the local bilinear form Φ , if $\Phi(u; v, C_u g) = \Phi(u; g, C_u v)$ for all $u \in D(N)$ and $g, v \in D(C_u)$.

A mapping $\Phi(\cdot; v, g) : U \rightarrow \mathbb{R}$ is a particular kind of operator. Its Gâteaux derivative Φ'_u is given by

$$(5) \quad \Phi'_u(h; v, g) = \frac{d}{d\varepsilon} \Phi(u + \varepsilon h; v, g)|_{\varepsilon=0} \equiv \langle h; v, g \rangle_u.$$

Definition 4. The operator $N : D(N) \subset U \rightarrow V$ is said to be potential on the set $D(N)$ with respect to the local bilinear form $\Phi(u; \cdot, \cdot) : V \times U \rightarrow \mathbb{R}$, if there exists a functional $F_N : D(F_N) = D(N) \rightarrow \mathbb{R}$ such that

$$\delta F_N[u, h] = \Phi(u; N(u), h) \quad \forall u \in D(N), \quad \forall h \in D(N'_u).$$

The functional F_N is called the potential of the operator N , and in turn the operator N is called the gradient of the functional F_N . In this case we will write $N = \text{grad}_\Phi F_N$.

An element $u \in D(N)$ such that $\delta F_N[u, h] = 0$ for all $h \in D(N'_u)$, is said to be a critical point of the functional F_N .

The following theorem is needed for the sequel.

Theorem 1. [4] Consider the operator $N : D(N) \subset U \rightarrow V$ and the local bilinear form $\Phi(u; \cdot, \cdot) : V \times U \rightarrow \mathbb{R}$ such that for any fixed elements $u \in D(N)$, $g, h \in D(N'_u)$ the function $\psi(\varepsilon) = \Phi(u; N(u + \varepsilon h), g)$ belongs to class $C^1[0, 1]$. For N to be potential on the convex open set $D(N)$ with respect to Φ it is necessary and sufficient to have

$$(6) \quad J_{N,h,g}(u) \equiv \Phi(u; N'_u h, g) + \langle h; N(u), g \rangle_u = \Phi(u; N'_u g, h) + \langle g; N(u), h \rangle_u$$

for all $u \in D(N)$, $g, h \in D(N'_u)$.

Under this condition the potential F_N is given by

$$(7) \quad F_N[u] = \int_0^1 \Phi(u_0 + \lambda(u - u_0); N(u_0 + \lambda(u - u_0)), (u - u_0)) d\lambda + F_N[u_0],$$

where u_0 is a fixed element of $D(N)$.

Remark 1. In the case of a nonlocal bilinear form the condition (6) takes the form

$$(8) \quad \Phi(N'_u h, g) = \Phi(N'_u g, h) \quad \text{for all } u \in D(N), g, h \in D(N'_u).$$

3. Indirect variational formulations

Consider the problem

$$(9) \quad N(u) = 0_V, \quad u \in D(N),$$

where the operator N is not potential with respect to the local bilinear form $\Phi(u; \cdot, \cdot) : V \times U \rightarrow \mathbb{R}$.

Now we shall construct an indirect variational formulation for the above equation (9).

Definition 5. An invertible linear operator $M_u : D(M_u) \subset R(N) \rightarrow V$ such that the operator $\tilde{N} = M_u N$ is potential on $D(N)$ with respect to the same bilinear form Φ is called a variational multiplier for N .

Theorem 2. Let $N : D(N) \subseteq U \rightarrow V$ be a twice continuously Gâteaux differentiable operator. Let C_u be any linear operator such that the following conditions take place:

- (a) it is symmetric with respect to the local bilinear form $\Phi_1(u; \cdot, \cdot) : V \times U \rightarrow \mathbb{R}$;
- (b) $D(C_u) \supseteq R(N'_u)$;

- (c) for any fixed elements $u \in D(N), g, h \in D(N'_u)$ the function $\psi(\varepsilon) = \Phi_1(u + \varepsilon h; N(u + \varepsilon h), C_{u+\varepsilon h} N'_{u+\varepsilon h} g)$ belongs to class $C^1[0, 1]$;
 (d) for all $u \in D(N)$ and $g, h \in D(N'_u)$,

$$(10) \quad C'_u(N'_u g; h) = C'_u(N'_u h; g)$$

- (e) for all $u \in D(N)$ and $g, h \in D(N'_u)$

$$(11) \quad \Phi'_{1u}(h; N(u), C_u N'_u g) = \Phi'_{1u}(g; N(u), C_u N'_u h).$$

Then the operator N is potential on $D(N)$ with respect to the local bilinear form

$$(12) \quad \Phi(u; v, g) = \Phi_1(u; v, C_u N'_u g)$$

and the potential F_N is given by

$$(13) \quad \begin{aligned} F_N[u] = & F_N[u_0] + \frac{1}{2} \Phi_1(u; N(u), C_u N(u)) - \frac{1}{2} \Phi_1(u_0; N(u_0), C_{u_0} N(u_0)) \\ & - \frac{1}{2} \int_0^1 [\Phi_1(\tilde{u}(\lambda); N(\tilde{u}(\lambda)), C'_{\tilde{u}(\lambda)}(N(\tilde{u}(\lambda)); \frac{\partial \tilde{u}(\lambda)}{\partial \lambda})) \\ & - \Phi'_{1u}(\frac{\partial \tilde{u}(\lambda)}{\partial \lambda}; N(\tilde{u}(\lambda)), C_{\tilde{u}(\lambda)} N(\tilde{u}(\lambda))) d\lambda, \end{aligned}$$

where $\tilde{u}(\lambda) = u_0 + \lambda(u - u_0)$; u_0 is a fixed element of $D(N)$.

Proof. Using (12) one gets

$$(14) \quad \begin{aligned} \Phi'_u(h; v, g) &= \frac{d}{d\varepsilon} \Phi_1(u + \varepsilon h; v, C_{u+\varepsilon h} N'_{u+\varepsilon h} g)|_{\varepsilon=0} \\ &= \Phi_1(u; v, C_u N''_u(g; h)) + \Phi_1(u; v, C'_u(N'_u g; h)) \\ &\quad + \Phi'_{1u}(h; v, C_u N'_u g). \end{aligned}$$

Thus for the left-hand side of (6) we have

$$(15) \quad \begin{aligned} J_{N,h,g}(u) = & \Phi_1(u; N'_u h, C_u N'_u g) + \Phi_1(u; N(u), C_u N''_u(g; h)) \\ & + \Phi_1(u; N(u), C'_u(N'_u g; h)) + \Phi'_{1u}(h; N(u), C_u N'_u g). \end{aligned}$$

On the other hand

$$(16) \quad \begin{aligned} J_{N,g,h}(u) = & \Phi_1(u; N'_u g, C_u N'_u h) + \Phi_1(u; N(u), C_u N''(h; g)) \\ & + \Phi_1(u; N(u), C'_u(N'_u h; g)) + \Phi'_{1u}(g; N(u), C_u N'_u h). \end{aligned}$$

Now, bearing in mind the symmetry of C_u and the conditions (10), (11) we conclude that

$$(17) \quad J_{N,h,g}(u) = J_{N,g,h}(u) \quad \forall u \in D(N), \quad \forall g, h \in D(N'_u).$$

According to Theorem 1 the given operator N is potential on $D(N)$ with respect to the local bilinear form (12) and the potential is given by

$$(18) \quad F_N[u] = F_N[u_0] + \int_{\lambda=0}^{\lambda=1} \Phi_1 \left(\tilde{u}(\lambda); N(\tilde{u}(\lambda)), C_{\tilde{u}(\lambda)} N'_{\tilde{u}(\lambda)} \frac{\partial \tilde{u}(\lambda)}{\partial \lambda} \right) d\lambda.$$

To prove (13) we find

$$(19) \quad \begin{aligned} \frac{d}{d\lambda} \Phi_1(\tilde{u}(\lambda); N(\tilde{u}(\lambda)), C_{\tilde{u}(\lambda)} N(\tilde{u}(\lambda))) &= \Phi_1 \left(\tilde{u}(\lambda); N(\tilde{u}(\lambda)), C_{\tilde{u}(\lambda)} N'_{\tilde{u}(\lambda)} \frac{\partial \tilde{u}(\lambda)}{\partial \lambda} \right) \\ &\quad + \Phi_1 \left(\tilde{u}(\lambda); N(\tilde{u}(\lambda)), C'_{\tilde{u}(\lambda)}(N(\tilde{u}(\lambda)); \frac{\partial \tilde{u}(\lambda)}{\partial \lambda}) \right) \\ &\quad + \Phi_1 \left(\tilde{u}(\lambda); N'_{\tilde{u}(\lambda)} \frac{\partial \tilde{u}(\lambda)}{\partial \lambda}, C_{\tilde{u}(\lambda)} N(\tilde{u}(\lambda)) \right) \\ &\quad + \Phi'_{1u} \left(\frac{\partial \tilde{u}(\lambda)}{\partial \lambda}; N(\tilde{u}(\lambda)), C_{\tilde{u}(\lambda)} N(\tilde{u}(\lambda)) \right). \end{aligned}$$

Taking into consideration the symmetry of $C_{\tilde{u}(\lambda)}$ we have

$$(20) \quad \Phi_1 \left(\tilde{u}(\lambda); N'_{\tilde{u}(\lambda)} \frac{\partial \tilde{u}(\lambda)}{\partial \lambda}, C_{\tilde{u}(\lambda)} N(\tilde{u}(\lambda)) \right) = \Phi_1 \left(\tilde{u}(\lambda); N(\tilde{u}(\lambda)), C_{\tilde{u}(\lambda)} N'_{\tilde{u}(\lambda)} \frac{\partial \tilde{u}(\lambda)}{\partial \lambda} \right).$$

Hence, from (19) one obtains

$$(21) \quad \begin{aligned} \Phi_1 \left(\tilde{u}(\lambda); N(\tilde{u}(\lambda)), C_{\tilde{u}(\lambda)} N'_{\tilde{u}(\lambda)} \frac{\partial \tilde{u}(\lambda)}{\partial \lambda} \right) &= \frac{1}{2} \frac{d}{d\lambda} \Phi_1(\tilde{u}(\lambda); N(\tilde{u}(\lambda)), C_{\tilde{u}(\lambda)} N(\tilde{u}(\lambda))) \\ &\quad - \Phi_1 \left(\tilde{u}(\lambda); N(\tilde{u}(\lambda)), C'_{\tilde{u}(\lambda)}(N(\tilde{u}(\lambda)); \frac{\partial \tilde{u}(\lambda)}{\partial \lambda}) \right) \\ &\quad - \Phi'_{1u} \left(\frac{\partial \tilde{u}(\lambda)}{\partial \lambda}; N(\tilde{u}(\lambda)), C_{\tilde{u}(\lambda)} N(\tilde{u}(\lambda)) \right). \end{aligned}$$

The use of (18), (21) shows that the sought potential F_N can be represented in the form (13). $\square$

Remark 2. When $C_u = C$ is a constant operator then the functional (13) takes the form

$$F_N[u] = F_N[u_0] + \frac{1}{2}\Phi_1(u; N(u), CN(u)) - \frac{1}{2}\Phi_1(u_0; N(u_0), CN(u_0)) \\ - \frac{1}{2} \int_{\lambda=0}^{\lambda=1} \Phi'_{1u}(\frac{\partial \tilde{u}(\lambda)}{\partial \lambda}; N(\tilde{u}(\lambda)), CN(\tilde{u}(\lambda))) d\lambda.$$

Remark 3. When $C_u = C$ is a constant operator and bilinear form Φ_1 is nonlocal then

$$F_N[u] = F_N[u_0] + \frac{1}{2}\Phi_1(N(u), CN(u)) - \frac{1}{2}\Phi_1(N(u_0), CN(u_0)).$$

Hence, in this case we can take

$$F_N[u] = \frac{1}{2}\Phi_1(N(u), CN(u)).$$

This formula was obtained by Tonti [7] as a solution of the inverse problem of the calculus of variations in an extended sense.

By using Theorem 1, it is easy to prove the following theorems.

Theorem 3. *A Gâteaux differentiable invertible operator M_u is a variational multiplier for $N \Leftrightarrow$ the operator N is potential on $D(N)$ with respect to the bilinear form*

$$\Phi_1(u; v, g) \equiv \Phi(u; M_u v, g).$$

Theorem 4. *If*

- (a) $\Phi(\cdot; \cdot) : V \times U \rightarrow \mathbb{R}$ is a fixed nonlocal bilinear form;
- (b) $N'_u = A \cdot K_u$, where A is a linear invertible operator such that it does not depend on u ; K_u is a symmetric operator,

then $M = A^{-1}$ is a variational multiplier for N .

Let us denote

$$(22) \quad J(M_u, h, g) = \Phi(u; M_u N'_u h, g) + \Phi(u; M'_u(N(u); h), g) + \Phi'_u(h; M_u N(u), g).$$

Theorem 5. M_u is a variational multiplier for $N \Leftrightarrow$ for all $u \in D(N)$ and $h \in D(N'_u)$

$$(23) \quad J(M_u, h, g) = J(M_u, g, h).$$

Proof. The criteria of potentiality (6) for $N_1 = M_u N(u)$ on the set

$D(N_1) = D(N)$ with respect to the bilinear form Φ can be written as

$$(24) \quad \Phi(u; N'_{1u}h, g) + \Phi'_u(h; N_1(u), g) = \Phi(u; N'_{1u}g, h) + \Phi'_u(g; N_1(u), h)$$

for all $u \in D(N_1)$ and $g, h \in D(N'_{1u})$. By using the equality $N'_{1u} = M_u N'_u h + M'_u(N(u); h)$ and (22), (24), we obtain (23). $\square$

4. Examples

1. Consider the following partial differential equation:

$$(25) \quad N(u) \equiv a_1 u_{tt} + b_1 u_{xx} + a_2 u_t^2 + b_2 u_x^2 = 0, \\ (x, t) \in Q_T = (0, l) \times (0, T),$$

where $u = u(x, t)$ is an unknown function; a_i, b_i ($i = 1, 2$) are constants.

We set

$$(26) \quad D(N) = \{u \in U = C^2(\overline{Q_T}) : u|_{t=0} = \varphi_1(x), u|_{t=T} = \varphi_2(x) \ (x \in (0, l)), \\ u|_{x=0} = \psi_1(t), u|_{x=l} = \psi_2(t) \ (t \in (0, T))\},$$

where $\varphi_i \in C[0, l]$ and $\psi_i \in C[0, T]$ ($i = 1, 2$).

We denote $V = C(\overline{Q_T})$ and determine the local bilinear form by setting

$$(27) \quad \langle v, h \rangle_u = \int_0^T \int_0^l e^u \cdot v \cdot h \, dx dt.$$

Let us prove that the operator N in (25) is potential on the set $D(N)$ (26) with respect to the bilinear form (27) under the conditions

$$(28) \quad a_1 = 2a_2, \quad b_1 = 2b_2.$$

To this end using (25) and (26), we get

$$(29) \quad \langle N'_u h, g \rangle_u = \int_0^T \int_0^l e^u (a_1 h_{tt} + b_1 h_{xx} + 2a_2 u_t h_t + 2b_2 u_x h_x) g \, dx dt,$$

$$(30) \quad \langle h; N(u), g \rangle_u = \int_0^T \int_0^l e^u h (a_1 u_{tt} + b_1 u_{xx} + a_2 u_t^2 + b_2 u_x^2) g \, dx dt.$$

From (30) we obtain the following equality:

$$\langle h; N(u), g \rangle_u = \langle g; N(u), h \rangle_u \quad \text{for all } u \in D(N), g, h \in D(N'_u).$$

Hence, in the given case the condition (6) can be written in the form

$$\int_0^T \int_0^l e^u \cdot N'_u h \cdot g \, dx dt = \int_0^T \int_0^l e^u \cdot N'_u g \cdot h \, dx dt$$

for all $u \in D(N)$ and $g, h \in D(N'_u)$. By integrating (29) by parts and taking into consideration the conditions

$$g|_{t=0} = h|_{t=0} = g|_{t=T} = h|_{t=T} = 0,$$

$$g|_{x=0} = h|_{x=0} = g|_{x=l} = h|_{x=l} = 0,$$

we get

$$\begin{aligned} \langle N'_u h, g \rangle_u = & \int_0^T \int_0^l e^u [(a_1 - 2a_2)gu_{tt} + (a_1 - 2a_2)u_t^2 g + 2(a_1 - a_2)u_t g_t + a_1 g_{tt} \\ & + (b_1 - 2b_2)u_{xx}g + 2(b_1 - b_2)u_x g_x + (b_1 - 2b_2)gu_x^2 + b_1 g_{xx}] h \, dx dt. \end{aligned}$$

If we take (28) into account, it follows that

$$\langle N'_u h, g \rangle_u = \int_0^T \int_0^l e^u (a_1 g_{tt} + b_1 g_{xx} + 2a_2 u_t g_t + 2b_2 u_x g_x) h \, dx dt = \langle N'_u g, h \rangle_u$$

for all $u \in D(N)$ and $g, h \in D(N'_u)$.

Thus, under the conditions (28) the given operator N is potential on the set $D(N)$ (26) with respect to the local bilinear form (27). The corresponding functional is

$$F_N[u] = -\frac{1}{2} \int_0^T \int_0^l e^u (a_1 u_t^2 + b_1 u_x^2) \, dx dt.$$

Taking into account Theorem 3, we obtain that $M_u = e^u$ is a variational multiplier for N (25).

It follows from the equation $\delta F_N[u, h] = 0$, $u \in D(N)$ for all $h \in D(N'_u)$ that $e^u N(u) = 0$.

2. Consider the following system of partial differential equations:

$$(31) \quad \begin{cases} N^1(u) \equiv au_{ttx}^1 + bu_t^1 + u^1u_x^1 + u_x^2 - \frac{1}{2}u_{xx}^1 & = 0, \\ N^2(u) \equiv cu_{ttx}^2 + du_t^2 + (u^1u^2)_x + \frac{1}{2}u_{xx}^2 & = 0, \end{cases}$$

$$(x, t) \in Q = (-\infty, +\infty) \times (t_0, t_1),$$

where a, b, c, d are constants, $u(x, t) = (u^1(x, t), u^2(x, t))^T$ is an unknown vector function.

We denote by $D(N)$ the domain of definition of the operator $N = (N^1, N^2)^T$ in (31):

$$(32) \quad D(N) = \{u \in U = (U_1, U_2)^T, u^i \in U_i = C_{x,t}^{2,2}((-\infty, +\infty) \times [t_0, t_1]) :$$

$$u^i|_{t=t_0} = u_0^i(x), u^i|_{t=t_1} = u_1^i(x), u_t^i|_{t=t_0} = u_2^i(x), u_t^i|_{t=t_1} = u_3^i(x), \\ \lim_{|x| \rightarrow +\infty} u^i = 0, \lim_{|x| \rightarrow +\infty} u_x^i = 0 \ (i = 1, 2)\},$$

where $u_0^i(x), u_1^i(x), u_2^i(x), u_3^i(x)$ ($i = 1, 2$) are given functions.

Let us consider the nonlocal bilinear form

$$(33) \quad \Phi(v, g) = \int_{t_0}^{t_1} \int_{-\infty}^{+\infty} (v^1(x, t)g^1(x, t) + v^2(x, t)g^2(x, t))dxdt.$$

If $a \neq 0, c \neq 0$ then the equations (31) can not be represented in the form of the Euler-Lagrange equations on the set $D(N)$ (32) with respect to (33) because the operator [6]

$$P_2 = \begin{pmatrix} aD_x & 0 \\ 0 & cD_x \end{pmatrix}$$

is not symmetric on the set

$$D(N'_u) = \{u \in U = (U_1, U_2)^T, u^i \in U_i = C_{x,t}^{2,2}((-\infty, +\infty) \times [t_0, t_1]) :$$

$$u^i|_{t=t_0} = 0, u^i|_{t=t_1} = 0, u_t^i|_{t=t_0} = 0, u_t^i|_{t=t_1} = 0, \\ \lim_{|x| \rightarrow +\infty} u^i = 0, \lim_{|x| \rightarrow +\infty} u_x^i = 0 \ (i = 1, 2)\}.$$

By using Theorem 4, we shall find a variational multiplier for the given system of equations. For that we obtain

$$N'_u h = \begin{pmatrix} ah_{ttx}^1 + bh_t^1 + h^1u_x^1 + u^1h_x^1 - \frac{1}{2}h_{xx}^1 + h_x^2 \\ h_x^1u^2 + h^1u_x^2 + ch_{ttx}^2 + dh_t^2 + u_x^1h^2 + u^1h_x^2 + \frac{1}{2}h_{xx}^2 \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} aD_{ttx} + bD_t + u_x^1 + u^1 D_x - \frac{1}{2}D_{xx} & D_x \\ u^2 D_x + u_x^2 & cD_{ttx} + dD_t + u_x^1 + u^1 D_x + \frac{1}{2}D_{xx} \end{pmatrix} \begin{pmatrix} h^1 \\ h^2 \end{pmatrix} \\
&= \begin{pmatrix} D_x (aD_{tt} + bD_x^{-1}D_t + u^1 - \frac{1}{2}D_x) & D_x \\ D_x u^2 & D_x (cD_{tt} + dD_x^{-1}D_t + u^1 + \frac{1}{2}D_x) \end{pmatrix} \begin{pmatrix} h^1 \\ h^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix} \begin{pmatrix} u^2 & cD_{tt} + dD_x^{-1}D_t + u^1 + \frac{1}{2}D_x \\ aD_{tt} + bD_x^{-1}D_t + u^1 - \frac{1}{2}D_x & 1 \end{pmatrix} \begin{pmatrix} h^1 \\ h^2 \end{pmatrix}.
\end{aligned}$$

In this case

$$A = \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix},$$

$$K_u = \begin{pmatrix} u^2 & cD_{tt} + dD_x^{-1}D_t + u^1 + \frac{1}{2}D_x \\ aD_{tt} + bD_x^{-1}D_t + u^1 - \frac{1}{2}D_x & 1 \end{pmatrix}.$$

The operator A is invertible and

$$(34) \quad A^{-1} = \begin{pmatrix} 0 & D_x^{-1} \\ D_x^{-1} & 0 \end{pmatrix},$$

where $D_x^{-1}v(x, t) = \int_{-\infty}^x v(y, t)dy$.

The operator K_u is symmetric with respect to the bilinear form (33) under the conditions $a = c$, $b = d$. Indeed,

$$\begin{aligned}
\Phi(K_u h, g) &= \int_{t_0}^{t_1} \int_{-\infty}^{+\infty} \left(\left(u^2 h^1 + ch_{tt}^2 + dD_x^{-1}h_t^2 + u^1 h^2 + \frac{1}{2}h_x^2 \right) g^1 \right. \\
&\quad \left. + \left(ah_{tt}^1 + bD_x^{-1}h_t^1 + u^1 h^1 - \frac{1}{2}h_x^1 + h^2 \right) g^2 \right) dx dt \\
&= \int_{t_0}^{t_1} \int_{-\infty}^{+\infty} \left(u^2 h^1 g^1 + ch^2 g_{tt}^1 + dD_x^{-1}g_t^1 h^2 + u^1 h^2 g^1 - \frac{1}{2}h^2 g_x^1 + ah^1 g_{tt}^2 \right. \\
&\quad \left. + bD_x^{-1}g_t^2 h^1 + u^1 h^1 g^2 + \frac{1}{2}h^1 g_x^2 + h^2 g^2 \right) dx dt \\
&= \Phi(h, K_u^* g),
\end{aligned}$$

i.e.

$$K_u^* = \begin{pmatrix} u^2 & aD_{tt} + bD_x^{-1}D_t + u^1 + \frac{1}{2}D_x \\ cD_{tt} + dD_x^{-1}D_t + u^1 - \frac{1}{2}D_x & 1 \end{pmatrix}.$$

Therefore, if $a = c$, $b = d$ then $K_u = K_u^*$. By using Theorem 4, we get that the operator A^{-1} in (34) is a variational multiplier for (31).

Remark 4. Construction of an indirect variational formulation for the boundary value problem for the general Navier - Stokes equations and the equation of continuity was done in [5].

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