On an abstract nonlinear second order integrodifferential equation

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Abstract. The aim of the present paper is to study the global existence of solutions of nonlinear second order integrodifferential equation in Banach space. Our analysis is based on an application of the Leray-Schauder alternative and rely on a priori bounds of solutions..

1. Introduction

Let X be a Banach space with norm $\|\cdot\|$. Let $B = C([0, T], X)$ be the Banach space of all continuous functions from $[0, T]$ into X endowed with supremum norm

$$\|x\|_B = \sup \{\|x(t)\| : t \in [0, T]\}.$$

The purpose of this paper is to prove the global existence of solutions of the following initial value problem for second order integrodifferential equations of the form

$$(1.1) \quad (r(t)x'(t))' = f\left(t, x(t), \int_0^t k(t, s)g(s, x(s))ds\right), \quad t \in [0, T]$$

$$(1.2) \quad x(0) = x_0, \quad x'(0) = 0;$$

where $r(t)$ is real-valued positive and sufficiently smooth function defined on $[0, T]$, $f: [0, T] \times X \times X \rightarrow X$, $g: [0, T] \times X \rightarrow X$, $k: [0, T] \times [0, T] \rightarrow R$ are continuous functions and x_0 is a given element of X .

Let $x \in L^1(0, T; X)$. By a solution of the initial value problem (1.1) – (1.2), we mean the function $x \in C([0, T]; X)$ given by

$$(1.3) \quad x(t) = x_0 + \int_0^t \frac{1}{r(s)} \int_0^s f\left(\tau, x(\tau), \int_0^\tau k(\tau, \eta)g(\eta, x(\eta))d\eta\right) d\tau ds,$$

$t \in [0, T]$.

The main purpose of this paper is to study the global existence of solutions of initial value problem (1.1)-(1.2) by using an application of the topological transversality theorem known as Leray-Schauder alternative. The interesting and useful aspect of the method employed here is that it yields simultaneously the existence of a solution and the maximal interval of existence. Our work is motivated by interesting results obtained by W.F. Trench [4,5] and influenced by the work of the authors [1] and D.B. Pachpatte [3].

2. Statement of result

Our Theorem is based on the following theorem, which is a version of the topological transversality theorem given in J. Dugundji and A. Granas [2, p. 61]

Theorem G. *Let Y be a convex subset of a normed linear space E and assume $0 \in Y$. Let $F: Y \rightarrow Y$ be a completely continuous operator and let $U(F) = \{x \in Y: x = \lambda Fx \text{ for some } 0 < \lambda < 1\}$. Then either $U(F)$ is unbounded or F has a fixed point.*

For convenience, we list the following hypotheses used in our subsequent discussion.

(H_1) There exists a constant $L > 0$ such that

$$\|g(t, x(t))\| \leq LH(\|x(t)\|)$$

for every $t \in [0, T]$, and $x \in X$, where $H: R_+ \rightarrow (0, \infty)$ is continuous nondecreasing function.

(H_2) There exists a continuous function $p: [0, T] \rightarrow R_+$ such that

$$\|f(t, x, y)\| \leq p(t)(\|x\| + \|y\|),$$

for every $t \in [0, T]$, and $x, y \in X$.

- (H_3) For each $t \in [0, T]$, the function $g(t, \cdot): [0, T] \times X \rightarrow X$ is continuous and for each $x \in X$ the function $g(\cdot, x): [0, T] \times X \rightarrow X$ is strongly measurable.
- (H_4) For each $t \in [0, T]$, the function $f(t, \cdot, \cdot): [0, T] \times X \times X \rightarrow X$ is continuous and for each $x, y \in X$ the function $f(\cdot, x, y): [0, T] \times X \times X \rightarrow X$ is strongly measurable.
- (H_5) For every positive integer m there exists $\alpha_m \in L^1(0, T)$ such that $\|f(t, x, y)\| \leq \alpha_m(t)$ for x, y satisfying $\|x\| \leq m, \|y\| \leq m$ and for a.e. $t \in [0, T]$.

We next establish the following Theorem which deals with the global existence of solution of initial value problem (1.1)-(1.2).

Theorem. *Suppose that the hypotheses (H_1) – (H_5) hold. If T satisfies the following condition*

$$(2.1) \quad \int_0^T N(s)ds < \int_c^\infty \frac{ds}{s + M^*H(s)T},$$

where $N(t) = \frac{1}{r(t)} \int_0^t p(s)ds$, $c = \|x_0\|$, $M^* = L_1L$ and $L_1 > 0$ defined by (3.2), then the initial value problem (1.1)–(1.2) has a solution x defined on $[0, T]$.

3. Proof of result

In order to prove the Theorem we apply Theorem G. First we establish the priori bounds for a solution to the initial value problem $(1.1)_\lambda - (1.2)$, $\lambda \in (0, 1)$ where

$$(1.1)_\lambda \quad (r(t)x'(t))' = \lambda f\left(t, x(t), \int_0^t k(t, s)g(s, x(s))ds\right), \quad t \in [0, T].$$

Let $x(t)$ be a solution of $(1.1)_\lambda - (1.2)$. Then it satisfies the equivalent integral equation

$$(3.1) \quad x(t) = x_0 + \lambda \int_0^t \frac{1}{r(s)} \int_0^s f\left(\tau, x(\tau), \int_0^\tau k(\tau, \eta)g(\eta, x(\eta))d\eta\right) d\tau ds,$$

where $t \in [0, T]$. Since k is continuous on a compact set $[0, T] \times [0, T]$, there exists $L_1 > 0$ such that

$$(3.2) \quad |k(t, s)| \leq L_1 \quad \text{for } 0 \leq s \leq t \leq T.$$

Using (3.1),(3.2), hypotheses $(H_1), (H_2)$ and the fact that $\lambda \in (0, 1)$, we obtain

$$\begin{aligned}
 (3.3) \quad \|x(t)\| &\leq \|x_0\| + \int_0^t \frac{1}{r(s)} \int_0^s p(\tau) [\|x(\tau)\| \\
 &\quad + \|\int_0^\tau k(\tau, \eta) g(\eta, x(\eta)) d\eta\|] d\tau ds \\
 &\leq \|x_0\| + \int_0^t \frac{1}{r(s)} \int_0^s p(\tau) [\|x(\tau)\| \\
 &\quad + \int_0^\tau L_1 L H(\|x(\eta)\|) d\eta] d\tau ds
 \end{aligned}$$

Define a function $u(t)$ by the right hand side of (3.3). By using the fact that H is continuous non-decreasing function, we get $\|x(t)\| \leq u(t)$, $u(0) = \|x_0\| = c$ and

$$(3.4) \quad u(t) \leq \|x_0\| + \int_0^t \frac{1}{r(s)} \int_0^s p(\tau) [u(\tau) + \int_0^\tau L_1 L H(u(\eta)) d\eta] d\tau ds$$

Clearly $u(t)$ is an increasing function. Denote by $v(t)$ the right hand side of (3.4), then $u(t) \leq v(t)$, $v(0) = \|x_0\| = c$ and

$$(3.5) \quad v(t) = \|x_0\| + \int_0^t \frac{1}{r(s)} \int_0^s p(\tau) [u(\tau) + \int_0^\tau L_1 L H(u(\eta)) d\eta] d\tau ds$$

It is clear that $v(t)$ is also an increasing function. Differentiating, we have

$$v'(t) \leq \{v(t) + M^* H(v(t))T\} \frac{1}{r(t)} \int_0^t p(s) ds.$$

Hence

$$(3.6) \quad \frac{v'(t)}{v(t) + M^* H(v(t))T} \leq N(t).$$

Integrating (3.6) from 0 to t and by making use of the change of variables $t \rightarrow s = v(t)$ and the condition (2.1), we get

$$(3.7) \quad \int_c^{v(t)} \frac{ds}{s + M^* H(s)T} \leq \int_0^t N(s) ds \leq \int_0^T N(s) ds < \int_c^\infty \frac{ds}{s + M^* H(s)T}.$$

From (3.7) we conclude by the mean value theorem that there is a constant Q independent of $\lambda \in (0, 1)$ such that $v(t) \leq Q$ for $t \in [0, T]$. Then we

have successively $u(t) \leq Q$, $\|x(t)\| \leq Q$ for $t \in [0, T]$ and consequently

$$\|x\|_B = \sup\{\|x(t)\| : t \in [0, T]\} \leq Q.$$

Now, we rewrite the problem (1.1)–(1.2) as follows: If $y \in B$ and $x(t)$ defined by $x(t) = x_0 + y(t)$, $t \in [0, T]$, then it is easy to observe that $y(t)$ satisfies an integral equation

$$y(t) = \int_0^t \frac{1}{r(s)} \int_0^s f\left(\tau, y(\tau) + x_0, \int_0^\tau k(\tau, \eta)g(\eta, y(\eta) + x_0)d\eta\right) d\tau ds,$$

if and only if $x(t)$ satisfies IVP (1.1)–(1.2) or their equivalent integral equation

$$x(t) = x_0 + \int_0^t \frac{1}{r(s)} \int_0^s f\left(\tau, x(\tau), \int_0^\tau k(\tau, \eta)g(\eta, x(\eta))d\eta\right) d\tau ds.$$

Define $F: B_0 \rightarrow B_0$, $B_0 = \{y \in B : y(0) = 0\}$ by

$$(3.8) \quad (Fy)(t) = \int_0^t \frac{1}{r(s)} \int_0^s f\left(\tau, y(\tau) + x_0, \int_0^\tau k(\tau, \eta)g(\eta, y(\eta) + x_0)d\eta\right) d\tau ds,$$

where $t \in [0, T]$.

Now, we prove that $F: B_0 \rightarrow B_0$ is continuous. Let $\{y_n\}$ be a sequence of elements of B_0 converging to y in B_0 . Then by using (H_3) and (H_4) , we get

$$\begin{aligned} & f\left(s, y_n(s) + x_0, \int_0^s k(s, \tau)g(\tau, y_n(\tau) + x_0)d\tau\right) \\ & \rightarrow f\left(s, y(s) + x_0, \int_0^s k(s, \tau)g(\tau, y(\tau) + x_0)d\tau\right), \end{aligned}$$

for each $t \in [0, T]$. From equation (3.8), we get

$$(3.9) \quad (Fy_n)(t) = \int_0^t \frac{1}{r(s)} \int_0^s f\left(\tau, y_n(\tau) + x_0, \int_0^\tau k(\tau, \eta)g(\eta, y_n(\eta) + x_0)d\eta\right) d\tau ds.$$

Now, $\|Fy_n - Fy\|_B = \sup_{0 \leq t \leq T} \|(Fy_n)(t) - (Fy)(t)\|$. By using hypotheses (H_3) and (H_4) and the dominated convergence theorem, we have

$$\begin{aligned}
& \| (Fy_n)(t) - (Fy)(t) \| \\
& \leq \int_0^t \frac{1}{r(s)} \int_0^s \left\| f(\tau, y_n(\tau) + x_0, \int_0^\tau k(\tau, \eta)g(\eta, y_n(\eta) + x_0)d\eta) \right. \\
& \quad \left. - f(\tau, y(\tau) + x_0, \int_0^\tau k(\tau, \eta)g(\eta, y(\eta) + x_0)d\eta) \right\| d\tau ds \\
& \rightarrow 0 \text{ as } n \rightarrow \infty
\end{aligned}$$

and it follows that $\|Fy_n - Fy\|_B \rightarrow 0$ as $n \rightarrow \infty$ i.e. $Fy_n \rightarrow Fy$ in B_0 as $y_n \rightarrow y$ in B_0 . Thus F is continuous.

Now, we prove that F maps a bounded set of B_0 into a precompact set in B_0 . Let $B_m = \{y \in B_0 : \|y\|_B \leq m\}$ for $m \geq 1$. We first show that F maps B_m into an equicontinuous family of functions with values in X . Let $y \in B_m$, $t_1, t_2 \in [0, T]$ and $0 \leq t_1 \leq t_2 \leq T$. Using (3.2), hypotheses $(H_1), (H_2)$ and condition (2.1), we have

(3.10)

$$\begin{aligned}
& \| (Fy)(t_1) - (Fy)(t_2) \| \\
& \leq \int_{t_1}^{t_2} \frac{1}{r(s)} \int_0^s \left\| f(\tau, y(\tau) + x_0, \int_0^\tau k(\tau, \eta)g(\eta, y(\eta) + x_0)d\eta) \right\| d\tau ds \\
& \leq \int_{t_1}^{t_2} \frac{1}{r(s)} \int_0^s p(\tau) \left[m + c + \int_0^\tau L_1 LH(m + c)d\eta \right] d\tau ds \\
& \leq [m + c + M^*H(m + c)T] \int_{t_1}^{t_2} N(s)ds.
\end{aligned}$$

The right hand side of (3.10) tends to zero as $t_2 - t_1 \rightarrow 0$. Thus F_{B_m} is an equicontinuous family of functions with values in X .

We next show that F_{B_m} is uniformly bounded. From the equation (3.8) and using (3.2), hypotheses $(H_1), (H_2)$, the condition (2.1) and the fact that $\|y\|_B \leq m$, we have

$$\begin{aligned}
& \| (Fy)(t) \| \\
& \leq \int_0^t \frac{1}{r(s)} \int_0^s \left\| f(\tau, y(\tau) + x_0, \int_0^\tau k(\tau, \eta)g(\eta, y(\eta) + x_0)d\eta) \right\| d\tau ds \\
& \leq \int_0^t \frac{1}{r(s)} \int_0^s p(\tau) \left[m + c + \int_0^\tau L_1 LH(m + c)d\eta \right] d\tau ds \\
& \leq [m + c + M^*H(m + c)T] \int_0^t N(s)ds
\end{aligned}$$

$$\leq [m + c + M^*H(m + c)T] \int_0^T N(s)ds.$$

This yields that the set $\{(Fy)(t) : \|y\|_B \leq m, 0 \leq t \leq T\}$ is bounded in X and hence, $\{F_{B_m}\}$ is uniformly bounded.

We have already shown that F_{B_m} is an equicontinuous and uniformly bounded collection. To prove the set F_{B_m} is precompact in B , it is sufficient, by Arzela-Ascoli's argument, to show that the set $\{(Fy)(t) : y \in B_m\}$ is precompact in X for each $t \in [0, T]$. Since $(Fy)(0) = 0$ for $y \in B_m$, it suffices to show this for $0 < t \leq T$. Let $0 < t \leq T$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $y \in B_m$, we define

$$(3.11) \quad (F_\epsilon y)(t) = \int_0^{t-\epsilon} \frac{1}{r(s)} \int_0^s f(\tau, y(\tau) + x_0, \int_0^\tau k(\tau, \eta)g(\eta, y(\eta) + x_0)d\eta) d\tau ds.$$

The set $y_\epsilon(t) = \{(F_\epsilon y)(t) : y \in B_m\}$ is precompact in X for every ϵ , $0 < \epsilon < t$. Moreover for every $y \in B_m$, we get

$$(3.12) \quad \begin{aligned} & (Fy)(t) - (F_\epsilon y)(t) \\ &= \int_{t-\epsilon}^t \frac{1}{r(s)} \int_0^s f(\tau, y(\tau) + x_0, \int_0^\tau k(\tau, \eta)g(\eta, y(\eta) + x_0)d\eta) d\tau ds \end{aligned}$$

By making use of (3.2), hypotheses $(H_1), (H_2)$, the condition (2.1) and the fact that $\|y(s)\| \leq m$, we have

$$\begin{aligned} & \|(Fy)(t) - (F_\epsilon y)(t)\| \\ & \leq \int_{t-\epsilon}^t \frac{1}{r(s)} \int_0^s p(\tau)[\|y(\tau) + x_0\| + \|\int_0^\tau k(\tau, \eta)g(\eta, y(\eta) + x_0)d\eta\|] d\tau ds \\ & \leq \int_{t-\epsilon}^t \frac{1}{r(s)} \int_0^s p(\tau)[\|y(\tau)\| + \|x_0\| + \int_0^\tau L_1 L H(\|y(\eta)\| + \|x_0\|) d\eta] d\tau ds \\ & \leq \int_{t-\epsilon}^t \frac{1}{r(s)} \int_0^s p(\tau)[m + c + \int_0^\tau L_1 L H(m + c) d\eta] d\tau ds \\ & \leq [m + c + M^*H(m + c)T] \int_{t-\epsilon}^t N(s)ds \end{aligned}$$

This implies that there exist precompact sets arbitrarily close to the set $\{(Fy)(t): y \in B_m\}$. Hence the set $\{(Fy)(t): y \in B_m\}$ is precompact in X . Thus we have shown that F is completely continuous operator.

Moreover, the set

$$U(F) = \{y \in B_0: y = \lambda Fy, \lambda \in (0, 1)\},$$

is bounded in B , since for every y in $U(F)$, the function $x(t) = y(t) + x_0$ is a solution of (1.1) $_{\lambda}$ – (1.2) for which we have proved $\|x\|_B \leq Q$ and hence $\|y\|_B \leq Q + c$. Now, by virtue of Theorem G, the operator F has a fixed point in B_0 . Therefore, the initial value problem (1.1) – (1.2) has a solution. Thus the proof of the Theorem is complete. $\square$

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