A convolution type characterization for L^p - multipliers for the Heisenberg group

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Abstract. It is well known that if m is an L^p - multiplier for the Fourier transform on $\mathbb{R}^n$ ($1 < p < \infty$), then there exists a pseudomeasure σ such that $T_m f = \sigma * f$. A similar result is proved for the group Fourier transform on the Heisenberg group H^n . Though this result is already known in generality for amenable groups, a simple proof is provided in this paper.

1. Introduction

One of the fundamental problems in Multiplier theory is the characterization of L^p multipliers ($1 < p < \infty$) for the Fourier transform. Different types of such characterizations have been considered under various situations in the literature. One of the important characterizations is to obtain sufficient conditions on $m \in L^\infty(\mathbb{R}^n)$ such that $T_m f = (mf)^\vee$ gives rise to a bounded operator T_m on $L^p(\mathbb{R}^n)$, which is popularly known as Hormander's multiplier theorem. According to another characterization,

if m is an L^p - multiplier for the Fourier transform, then T_m can be expressed as a convolution operator using a pseudo measure σ . In particular if $p = 1$, σ turns out to be a finite Borel measure. Yet another type of characterization is to identify the class of multipliers (same as translation invariant operators on a locally compact abelian group) for L^p as the dual space of a certain function space. These characterizations have been proved not only for $\mathbb{R}^n$ but also for a general locally compact abelian group. For details, we refer to Hormander [12], Stein [18], Figà-Talamanca [8] and Larsen [13]. The identification of the class of (right) translation invariant operators as the dual space for an amenable group has been studied by several authors. (See [3], [7], [10]). This type of result is shown for some non amenable groups by Cowling in [2] and for a general locally compact group in [5]. Hormander's multiplier theorem has been proved for Weyl transform by Mauceri [14] in 1980 and for the Heisenberg group Fourier transform by Michele and Mauceri [15] in 1979. In 1998, it was proved by Radha and Thangavelu [17] that Weyl multiplier for $L^p(\mathbb{C}^n)$ corresponds to a twisted convolution operator using a 'pseudomeasure'.

In this paper, we prove a similar result for an L^p - multiplier for the group Fourier transform on the Heisenberg group H^n . However this result is not new. In fact, the more general abstract result has already been established for amenable groups in [7] and [11]. We also refer to Cowling [4] for further details and Pier [16] where the result of Herz is presented. We provide a simple proof in this paper.

It is important to mention here that unlike the Fourier transform on $\mathbb{R}^n$, the multiplier associated with the Heisenberg group Fourier transform is vector valued.

1.1. Notations and Background. Let $\mathcal{B} = \mathcal{B}(L^2(\mathbb{R}^n))$ denote the class of bounded operators on $L^2(\mathbb{R}^n)$. Let $\mathcal{B}_1 = \mathcal{B}_1(L^2(\mathbb{R}^n))$ denote the ideal of trace class operators. $\mathcal{B}_1$ is a Banach space under the norm $\|A\|_1 = \text{tr}(|A|) = \text{tr}(A^*A)^{\frac{1}{2}}$. Let $\mathcal{B}_2 = \mathcal{B}_2(L^2(\mathbb{R}^n))$ denote the Hilbert space of Hilbert - Schmidt operators on $L^2(\mathbb{R}^n)$ with the inner product $\langle A, B \rangle = \text{tr}(B^*A)$.

Let H^n denote the Heisenberg group. It is a unimodular nilpotent Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$ and the group operation is defined by

$$(z, t) \cdot (w, s) = (z + w, t + s + \frac{1}{2} \text{Im}(z\bar{w})).$$

The Haar measure is given by $dzdt$.

By Stone-Von Neumann theorem (Folland [9]), the only infinite dimensional unitary irreducible representations (upto unitary equivalence)

are given by π_λ , $\lambda \in \mathbb{R}^*$, where π_λ is defined by

$$\pi_\lambda(z, t)\varphi(\xi) = e^{i\lambda t} e^{i\lambda(x\xi + \frac{1}{2}xy)} \varphi(\xi + y),$$

where $z = x + iy$ and $\varphi \in L^2(\mathbb{R}^n)$.

The group Fourier transform of $f \in L^1(H^n)$ is defined as

$$\hat{f}(\lambda) = \int_{H^n} f(z, t) \pi_\lambda(z, t) dz dt, \quad \lambda \in \mathbb{R}^*.$$

Notice that $\hat{f}(\lambda) \in \mathcal{B}$ and $\|\hat{f}(\lambda)\|_{\mathcal{B}} \leq \|f\|_{L^1(H^n)}$. As in the case of $\mathbb{R}^n$, the group Fourier transform $\hat{f}$ satisfies the basic properties, namely if $f \in L^1 \cap L^2(H^n)$, $\hat{f}(\lambda)$ is a Hilbert-Schmidt operator. Further if we define $d\mu(\lambda) = (2\pi)^{-n-1} |\lambda|^n d\lambda$, and $L^2(\mathbb{R}^*, \mathcal{B}, d\mu)$ denote the collection of square integrable $\mathcal{B}$ -valued functions on $\mathbb{R}^*$ under the measure $d\mu$, then the group Fourier transform is an isometric isomorphism between $L^2(H^n)$ and $L^2(\mathbb{R}^*, \mathcal{B}, d\mu)$. The inversion formula is given by

$$f(z, t) = \int \text{tr}(\pi_\lambda(z, t)^* \hat{f}(\lambda)) d\mu(\lambda).$$

If $f, g \in L^1(H^n)$,

$$(f * g)(z, t) = \int f((z, t)(w, s)^{-1}) g(w, s) dw ds$$

denotes the convolution, then $(f * g)^\wedge(\lambda) = \hat{f}(\lambda) \hat{g}(\lambda)$. Under this convolution operation $L^1(H^n)$ becomes a non-commutative algebra. For further results of the group Fourier transform we refer to Thangavelu [19].

2. The Main Result

Let $m \in L^\infty(\mathbb{R}^*, \mathcal{B}, d\mu)$. ie m is a $\mathcal{B}$ -valued L^∞ -function defined on $\mathbb{R}^*$ with the measure $d\mu$. Now for $f \in L^2(\mathbb{R}^n)$, define $T_m f$ by $(T_m f)^\wedge = m \hat{f}$ or more explicitly

$$(T_m f)^\wedge(\lambda) = m(\lambda) \hat{f}(\lambda) \quad \lambda \in \mathbb{R}^*.$$

This definition is meaningful because

$$\begin{aligned}
\|T_m f\|_{L^2(H^n)}^2 &= \|(T_m f)^\wedge\|_{L^2(\mathbb{R}^*, \mathcal{B}_2, d\mu)}^2 \\
&= \|m \hat{f}\|_{L^2(\mathbb{R}^*, \mathcal{B}_2, d\mu)}^2 \\
&= \int \|m \hat{f}(\lambda)\|_{\mathcal{B}_2}^2 d\mu \\
&\leq \int \|m(\lambda)\|_{\mathcal{B}}^2 \|\hat{f}(\lambda)\|_{\mathcal{B}_2}^2 d\mu \\
&\leq \|m\|_{L^\infty(\mathbb{R}^*, \mathcal{B}, d\mu)}^2 \|\hat{f}\|_{L^2(\mathbb{R}^*, \mathcal{B}_2, d\mu)}^2 \\
&= \|m\|^2 \|f\|_{L^2(H^n)}^2.
\end{aligned}$$

This helps us to define a multiplier for $L^p(H^n)$.

Definition 2.1. Let $m \in L^\infty(\mathbb{R}^*, \mathcal{B}, d\mu)$. Define for $f \in L^2 \cap L^p(H^n)$, $(T_m f)^\wedge = m \hat{f}$. If $T_m f \in L^p(H^n)$ and T_m is a bounded operator, then we say that m is a multiplier for $L^p(H^n)$. Let $\mathcal{A}(H^n)$ denote the class of complex valued functions f on H^n such that f is the inverse Fourier transform $\check{g}$ of a function $g \in L^1(\mathbb{R}^*, \mathcal{B}, d\mu)$. Since

$$\check{g}(z, t) = \int_{\mathbb{R}^*} \text{tr}(\pi(z, t)^* g(\lambda)) d\mu(\lambda),$$

we have

$$\begin{aligned}
|\check{g}(z, t)| &\leq \int_{\mathbb{R}^*} \|\pi_\lambda(z, t)\|_{\mathcal{B}} \|f(\lambda)\|_{\mathcal{B}_1} d\mu(\lambda) \\
&= \int_{\mathbb{R}^*} \|g(\lambda)\|_{\mathcal{B}_1} d\mu \\
&= \|g\|_{L^1(\mathbb{R}^*, \mathcal{B}_1, d\mu)}.
\end{aligned}$$

Thus, the definition of $\mathcal{A}(H^n)$ is meaningful. Define for $f \in \mathcal{A}(H^n)$,

$$\|f\|_{\mathcal{A}} = \|\check{g}\|_{\mathcal{A}} = \|f\|_{L^1(\mathbb{R}^*, \mathcal{B}_1, d\mu)}.$$

Then it is easy to verify that $\mathcal{A}(H^n)$ is a Banach space under $\|\cdot\|_{\mathcal{A}}$. Let $\mathcal{P}(H^n)$ denote the class of continuous linear functionals on $\mathcal{A}(H^n)$. *viz.* $\mathcal{P}(H^n) = [\mathcal{A}(H^n)]^*$, where $*$ denotes the dual. Since the construction of $\mathcal{P}(H^n)$ is similar to that of the class of pseudomeasures on $\mathbb{R}^n$ in the abstract sense, we call the members of $\mathcal{P}(H^n)$ as pseudomeasures on H^n .

We shall show that there is an isometric isomorphism of $\mathcal{P}(H^n)$ onto $[L^1(\mathbb{R}^*, \mathcal{B}_1, d\mu)]^*$. It is important to notice that since the L^1 space under consideration consists of $\mathcal{B}_1$ -valued functions (vector valued), the dual

space is not known. In other words the well known relation $(L^p)^* = L^{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$, $1 \leq p < \infty$ for the complex-valued functions is not true in the case of vector valued functions. However, there is an isometric isomorphism of $L^{p'}(\mathbb{R}^n, A^*)$ into a subspace of $[L^p(\mathbb{R}^n, A)]^*$ defined as follows: For $f \in L^p(\mathbb{R}^n, A)$, $g \in L^{p'}(\mathbb{R}^n, A^*)$ ($1 \leq p < \infty$), define

$$L_g(f) = \int_{\mathbb{R}^n} g(t) (f(t)) dt,$$

then L_g is a bounded linear functional on $L^p(\mathbb{R}^n, A)$ and the mapping

$$g \longmapsto L_g$$

for $g \in L^{p'}(\mathbb{R}^n, A^*)$ is an isometric isomorphism carrying $L^{p'}(G, A^*)$ onto a subspace of $L^p(G, A)^*$. (Here A denotes a Banach space). For a more detailed study of vector valued functions and vector measures, we refer to Diestel and Uhl [6].

We now proceed towards establishing identification between $\mathcal{P}(H^n)$ and $[L^1(\mathbb{R}^*, \mathcal{B}_1, d\mu)]^*$. Let $\sigma \in \mathcal{P}(H^n)$. For $f \in L^1(\mathbb{R}^*, \mathcal{B}_1, d\mu)$, define $F_\sigma(f) = \sigma(\check{f})$. Since

$$(1) \quad |F_\sigma(f)| = |\sigma(\check{f})| \leq \|\sigma\|_{\mathcal{P}} \|\check{f}\|_{\mathcal{A}} = \|\sigma\|_{\mathcal{P}} \|f\|_{\mathcal{B}_1},$$

F_σ is a continuous linear functional on $L^1(\mathbb{R}^*, \mathcal{B}_1, d\mu)$. Conversely, if $\varphi \in [L^1(\mathbb{R}^*, \mathcal{B}_1, d\mu)]^*$, the function G_φ defined by $G_\varphi(g) = \varphi(f)$, where $g = \check{f}$, $f \in L^1(\mathbb{R}^*, \mathcal{B}_1, d\mu)$ satisfies

$$(2) \quad |G_\varphi(g)| \leq \|\varphi\| \|g\|_{\mathcal{A}},$$

which shows that G_φ is a continuous linear functional on $\mathcal{A}(H^n)$. Thus for $\sigma \in \mathcal{P}(H^n)$, there is a unique $\varphi \in [L^1(\mathbb{R}^*, \mathcal{B}_1, d\mu)]^*$, which we denote by $\hat{\sigma}$. Further from inequalities (1) and (2) it can be easily shown that $\|\sigma\| = \|\hat{\sigma}\|$. Thus $\sigma \longmapsto \hat{\sigma}$ is a isometric isomorphism from $\mathcal{P}(H^n)$ onto $[L^1(\mathbb{R}^*, \mathcal{B}_1, d\mu)]^*$. Now we are in a position to prove our main result of this section. Before stating the theorem, let us look at the following definitions.

Definition 2.2. For $g \in [L^\infty(\mathbb{R}^*, \mathcal{B}, dX)]$, $h \in L^1(\mathbb{R}^*, \mathcal{B}_1, d\mu)$, define for $\lambda \in \mathbb{R}^*$,

$$g \otimes h(\lambda) = g(\lambda)h(\lambda),$$

where dX denotes the Lebesgue measure. This definition is meaningful. For, first notice that as $g(\lambda) \in \mathcal{B}$, $h(\lambda) \in \mathcal{B}_1$, $g(\lambda)h(\lambda) \in \mathcal{B}_1$. Therefore

$$(g \otimes h)(\lambda) \in \mathcal{B}_1.$$

Consider

$$\begin{aligned}
 \int \|(g \otimes h)(\lambda)\|_{\mathcal{B}_1} d\mu &\leq \int \|g(\lambda)\|_{\mathcal{B}} \|h(\lambda)\|_{\mathcal{B}_1} d\mu \\
 &\leq \|g\|_{L^\infty(\mathbb{R}^*, \mathcal{B}, dX)} \int_{\mathbb{R}^*} \|h(\lambda)\|_{\mathcal{B}_1} d\mu \\
 &= \|g\|_{L^\infty(\mathbb{R}^*, \mathcal{B}, dX)} \|h\|_{L^1(\mathbb{R}^*, \mathcal{B}_1, d\mu)}.
 \end{aligned}$$

Thus $g \otimes h \in L^1(\mathbb{R}^*, \mathcal{B}_1, d\mu)$.

Definition 2.3. For $\nu \in [L^1(\mathbb{R}^*, \mathcal{B}_1, d\mu)]^*$, $g \in L^\infty(\mathbb{R}^*, \mathcal{B}, dX)$, define $\nu \diamond g$ by

$$\begin{aligned}
 (\nu \diamond g)(h) &= \langle h, \nu \diamond g \rangle \\
 &= \nu(g \otimes h) \quad \forall h \in L^1(\mathbb{R}^*, \mathcal{B}_1, d\mu)
 \end{aligned}$$

The definition of $\nu \diamond g$ is meaningful. Further, by straight forward computations, it is easy to verify that $\nu \diamond g \in [L^1(\mathbb{R}^*, \mathcal{B}_1, d\mu)]^*$.

Remark 2.4. If $\nu \in L^\infty(\mathbb{R}^*, \mathcal{B}, d\mu)$, then $\nu \diamond g$ coincides with ordinary multiplication of ν and g . In fact, $\forall h \in L^1(\mathbb{R}^*, \mathcal{B}_1, d\mu)$,

$$\begin{aligned}
 \langle h, \nu \diamond g \rangle &= \nu(g \otimes h) \\
 &= \int_{\mathbb{R}^*} \langle g \otimes h(\lambda), \nu(\lambda) \rangle d\mu(\lambda) \\
 &= \int_{\mathbb{R}^*} \langle g(\lambda)h(\lambda), \nu(\lambda) \rangle d\mu(\lambda) \\
 (3) \quad &= \int_{\mathbb{R}^*} \text{tr}(\nu(\lambda)g(\lambda)h(\lambda)) d\mu(\lambda) \\
 &= \int_{\mathbb{R}^*} \langle h(\lambda), \nu(\lambda)g(\lambda) \rangle d\mu(\lambda) \\
 &= \int_{\mathbb{R}^*} \langle h(\lambda), (\nu g)(\lambda) \rangle d\mu(\lambda) \\
 &= \langle h, \nu g \rangle.
 \end{aligned}$$

The equation (3) follows from the following fact: If $B \in \mathcal{B}$. Define $F_B : \mathcal{B}_1 \rightarrow \mathbb{C}$ by

$$F_B(T) = \text{tr}(BT).$$

Then $B \mapsto F_B$ is an isomorphism of $\mathcal{B}$ onto $\mathcal{B}_1^*$ (see Conway [1]).

Theorem 2.5. Let m be an L^p -multiplier for the group Fourier transform on the Heisenberg group H^n . Then there exists a pseudomeasure σ such that

$$T_m f = \sigma * f \quad \forall f \in L^1 \cap L^2 \cap L^p(H^n).$$

Proof. For $f \in L^1 \cap L^2 \cap L^p(H^n)$, define T_m by $(T_m f)^\wedge = m\hat{f}$. Since $m \in L^\infty(\mathbb{R}^*, \mathcal{B}, d\mu) \subset [L^1(\mathbb{R}^*, \mathcal{B}_1, d\mu)]^*$, there exists a pseudomeasure $\sigma \in \mathcal{P}(H^n)$ such that $\hat{\sigma} = m$. As $f \in L^1(H^n)$, $\hat{f} \in L^\infty(\mathbb{R}^*, \mathcal{B}, dX)$.

Now consider $\hat{\sigma} \diamond \hat{f}$, which belongs to $[L^1(\mathbb{R}^*, \mathcal{B}_1, d\mu)]^*$. We denote $\sigma * f$ as the inverse image of $\hat{\sigma} \diamond \hat{f}$ under the map $\sigma \mapsto \hat{\sigma}$ from $\mathcal{P}(H^n)$ onto $[L^1(\mathbb{R}^*, \mathcal{B}_1, d\mu)]^*$. As $m = \hat{\sigma} \in L^\infty(\mathbb{R}^*, \mathcal{B}, d\mu)$, $\hat{\sigma} \diamond \hat{f} = \hat{\sigma}\hat{f}$, using Remark 2.4. Also, $(T_m f)^\wedge = m\hat{f} = \hat{\sigma}\hat{f} = \hat{\sigma} \diamond \hat{f}$. Further it can be easily verified that the Fourier transform of the function $T_m f$ coincides with the Fourier transform of a pseudomeasure, when $T_m f$ can be regarded as a pseudomeasure. Thus $T_m f = \sigma * f$, proving our assertion. $\square$

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