Bloch-type space of temperature functions on a finite cylinder

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Abstract. We define the Bloch-type space $\mathcal{B}_T$ as the linear space of temperature functions on the cylinder $S_T = \mathbb{S}^1 \times (0, T)$ such that $\sup_{(x,t) \in \Omega_T} t |\frac{\partial u}{\partial t}(x, t)| < \infty$, $\Omega_T = [0, 2] \times (0, T)$; we prove that $(b^1(S_T))^* = \mathcal{B}_T$, where $b^1(S_T)$ is the Bergman space of temperature functions on S_T belonging to $L^1(\Omega_T, dxdt)$.

1. Introduction

For $\mathbb{D}$ the open unit disk in the complex plane $\mathbb{C}$, the classic Bergman space A^p is the subspace of analytic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ such that $f \in L^p(\mathbb{D})$, $0 < p < \infty$. If $1 \leq p < \infty$ then A^p is a Banach space, and for $p > 1$ it is well known that $(A^p)^* = A^q$ under the integral pairing (see [4, Theorem 1.16])

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z) \overline{g(z)} dA(z), \quad f \in A^p, \quad g \in A^q,$$

where $dA(z) = \pi^{-1}dxdy$, and q is the conjugate exponent of p : $p^{-1} + q^{-1} = 1$.

An analytic function f in $\mathbb{D}$ is in the Bloch space $\mathcal{B}$ if $\sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty$. When normed with

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)|,$$

the Bloch space $\mathcal{B}$ is a Banach space. In addition, we have $(A^1)^* = \mathcal{B}$ under the integral pairing (see [4, Theorem 1.21])

$$\langle f, g \rangle = \lim_{r \rightarrow 1^-} \int_{\mathbb{D}} f(rz) \overline{g(z)} dA(z), \quad f \in A^1, \quad g \in \mathcal{B}.$$

For $\Omega \subset \mathbb{R}_+^2$ an open set, let

$$H(\Omega) = \left\{ u \in C^2(\Omega) : \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \text{ on } \Omega \right\}.$$

We shall call the elements of $H(\Omega)$ temperature functions on Ω .

For $T > 0$ finite, we define the linear space of temperature functions on the cylinder $S_T = \mathbb{S}^1 \times (0, T)$ as

$$H(S_T) = \{ u \in H(\mathbb{R} \times (0, T)) : u(x, t) = u(x + 2, t) \}.$$

Roughly speaking, a function in $H(S_T)$ determines a temperature distribution on a circular ring of fine wire with radius $1/\pi$, up to a certain time T .

In [7] we introduced the weighted Bergman-type spaces $b_{\alpha}^p(S_T)$ consisting of temperature functions on the cylinder S_T that belong to $L_{\alpha}^p(\Omega_T) = L^p(\Omega_T, t^{\alpha} dx dt)$, where $\Omega_T = [0, 2] \times (0, T)$, $1 \leq p < \infty$, and $\alpha > -1$. Notice that if $u \in b_{\alpha}^p(S_T)$ then $u|_{\Omega_T} \in L_{\alpha}^p(\Omega_T)$, hence $b_{\alpha}^p(S_T) \subset L_{\alpha}^p(\Omega_T)$.

As in the analytic case, we proved that $(b_{\alpha}^p(S_T), \|\cdot\|_{L^p(\Omega_T)})$ is a Banach space, and $(b_{\alpha}^p(S_T))^* = b_{(\beta - \frac{p}{p-1})}^q(S_T)$ under the integral pairing (see [7, Theorem 1.2])

$$\langle u, v \rangle_{\beta} = \int_{\Omega_T} u(z) \overline{v(z)} t^{\beta} dx dt, \quad z = (x, t),$$

provided that $\alpha, \beta > -1$, $p > \max\left(1, \frac{1+\alpha}{1+\beta}\right)$. In particular, for $\alpha > -1$ we have that $b_{\alpha}^2(S_T)$ is a Hilbert space with respect the inner product inherited from $L_{\alpha}^2(\Omega_T)$,

$$(u, v)_{b_{\alpha}^2(S_T)} = \int_{\Omega_T} u(z) \overline{v(z)} t^{\alpha} dx dt,$$

and the corresponding orthogonal projection $P_{\alpha} : L_{\alpha}^2(\Omega_T) \rightarrow b_{\alpha}^2(S_T)$ is called the weighted Bergman projection on $b_{\alpha}^2(S_T)$.

In this paper we define the Bloch-type space $\mathcal{B}_T$ as the linear space of temperature functions v on S_T such that $\sup \left\{ t \left| \frac{\partial v}{\partial t}(x, t) \right| : (x, t) \in \Omega_T \right\} < \infty$. When $\mathcal{B}_T$ is endowed with the norm

$$\|v\|_{\mathcal{B}_T} = \max_{x \in [0, 2]} \left| v\left(x, \frac{T}{2}\right) \right| + \sup \left\{ t \left| \frac{\partial v}{\partial t}(x, t) \right| : (x, t) \in \Omega_T \right\},$$

we obtain that $(\mathcal{B}_T, \|\cdot\|_{\mathcal{B}_T})$ is a Banach space. Furthermore, we show that $(b^1(S_T))^* = \mathcal{B}_T$ under the integral pairing

$$\langle u, v \rangle_0 = \int_{\Omega_T} u(z) \overline{(\mathcal{L}_0^1 v)(z)} dx dt, \quad u \in b^1(S_T), \quad v \in \mathcal{B}_T,$$

where $\mathcal{L}_0^1 : \mathcal{B}_T \rightarrow L^\infty(\Omega_T)$ is a bounded operator such that $P_0 \mathcal{L}_0^1 v = v$ for every $v \in \mathcal{B}_T$.

In fact, we have the following results,

Theorem 1. *For $\alpha > -1$, there is a bounded linear differential operator $\mathcal{L}_\alpha^1 : \mathcal{B}_T \rightarrow L^\infty(\Omega_T)$ such that $P_\alpha \mathcal{L}_\alpha^1 v = v$ for all $v \in \mathcal{B}_T$.*

Theorem 2. *If $\alpha > -1$ then $(b_\alpha^1(S_T))^* = \mathcal{B}_T$ under the integral pairing*

$$\langle u, v \rangle_\alpha = \int_{\Omega_T} u(z) \overline{(\mathcal{L}_\alpha^1 v)(z)} t^\alpha dx dt, \quad u \in b_\alpha^1(S_T), \quad v \in \mathcal{B}_T.$$

The proof of this theorem is based on the methods and techniques in [1], [3], [5], [6], which involve the construction of the so-called Bell operator $\mathcal{L}^1$ satisfying $P\mathcal{L}^1 v = v$ for every $v \in \mathcal{B}$, where P is the Bergman projection and $\mathcal{B}$ the Bloch space in the corresponding setting (harmonic functions on the unit ball in $\mathbb{R}^n$, holomorphic functions on strictly pseudoconvex domains with C^4 boundary in $\mathbb{C}^n$).

2. Notation and preliminary results

Throughout this paper we will write $z = (x, t)$, $w = (y, \tau)$, $dz = dx dt$, $dw = dy d\tau$, $\mathbb{S}^1 = \{e^{\pi i \theta} : \theta \in [0, 2]\}$, $\mathbb{Z}^* = \{n \in \mathbb{Z} : n \neq 0\}$, and α will be a real number greater than -1 . Also $K(x, t)$ will denote the Gauss-Weierstrass kernel. For $t > 0$, let

$$\begin{aligned} \theta(x, t) &= \sum_{n \in \mathbb{Z}} K(x + 2n, t) \\ &= \frac{1}{2} \sum_{n \in \mathbb{Z}} e^{-\pi^2 n^2 t + \pi n i x}, \quad [9, \text{Chapter V, Theorem 4.1}] \\ \varphi(x, t) &= -2 \frac{\partial \theta}{\partial x}(x, t), \end{aligned}$$

and for $t \leq 0$, let $K = \theta = \varphi = 0$. It follows that

$$\int_0^2 \theta(x, t) dx = \int_{-\infty}^{\infty} K(x, t) dx = 1, \quad \text{for all } t > 0.$$

Let $Q = (0, 1) \times (0, 1)$, $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ (the parabolic boundary of Q), where $\Gamma_1 = \{0\} \times [0, 1)$, $\Gamma_2 = \{1\} \times [0, 1)$, $\Gamma_3 = (0, 1) \times \{0\}$, and let λ be the one-dimensional Lebesgue measure on Γ .

Consider the heat kernel $\tilde{K}(x, t; \xi, \tau)$ on $Q \times \Gamma$ defined as follows

$$\tilde{K}(x, t; \xi, \tau) = \begin{cases} \varphi(x, t - \tau), & \xi = 0, 0 \leq \tau < 1, \\ \varphi(1 - x, t - \tau), & \xi = 1, 0 \leq \tau < 1, \\ \theta(x - \xi, t) - \theta(x + \xi, t), & \tau = 0, 0 < \xi < 1. \end{cases}$$

It is well known that if $u \in H(Q) \cap C(\overline{Q})$ then (see [2, Theorem 6.3.1])

$$(1) \quad u(x, t) = \int_{\Gamma} \tilde{K}(x, t; \xi, \tau) u(\xi, \tau) d\lambda(\xi, \tau), \quad \text{for all } (x, t) \in Q.$$

Conversely, if $v \in C(\Gamma)$ then

$$(2) \quad u(x, t) = \int_{\Gamma} \tilde{K}(x, t; \xi, \tau) v(\xi, \tau) d\lambda(\xi, \tau)$$

is a temperature function on Q .

Remark 3. Suppose f is a 2-periodic continuous function on $\mathbb{R}$. Then u is a temperature function on $\mathbb{R}_+^2$, 2-periodic in the variable x , and $u(x, t) \rightarrow f(x)$ as $t \rightarrow 0$ uniformly on $[0, 2]$ if and only if (see [9, Chapter V, Theorem 8])

$$u(x, t) = \int_0^2 \theta(x - y, t) f(y) dy.$$

Next we give a basic but useful result.

Proposition 4. If $u \in H(S_T)$ then $|u(x, t)| \leq \max_{x \in [0, 2]} |u(x, \frac{T}{2})|$ for all $\frac{T}{2} < t < T$.

Proof. Let $u \in H(S_T)$, by the remark above and the uniqueness of the solution of the heat equation on a finite cylinder (see [9, p. 100]), we can write

$$u(x, t) = \int_0^2 \theta\left(x - y, t - \frac{T}{2}\right) u\left(y, \frac{T}{2}\right) dy, \quad \text{for all } \frac{T}{2} < t < T,$$

therefore

$$\begin{aligned} |u(x, t)| &\leq \max_{x \in [0, 2]} \left| u\left(x, \frac{T}{2}\right) \right| \int_{-\infty}^{\infty} K\left(x - y, t - \frac{T}{2}\right) dy \\ &= \max_{x \in [0, 2]} \left| u\left(x, \frac{T}{2}\right) \right|, \quad \text{for all } t > \frac{T}{2}. \quad \square \end{aligned}$$

We define the linear space of continuous functions on the cylinder $\overline{S_T}$ as

$$C(\overline{S_T}) = \{u \in C(\mathbb{R} \times [0, T]) : u(x, t) = u(x + 2, t)\}.$$

For $1 \leq p < \infty$, $\alpha > -1$ the subspace $H(S_T) \cap C(\overline{S_T})$ is dense in $b_\alpha^p(S_T)$ ([7, Corollary 3.8]).

We define the incomplete gamma function γ as

$$\gamma(\beta, \sigma) = \int_0^\sigma t^{\beta-1} e^{-t} dt \quad \text{if } \beta, \sigma > 0.$$

For any $n \in \mathbb{Z}^*$, let

$$u_n(x, t) = \sqrt{\frac{2\alpha\pi^{2(1+\alpha)}n^{2(1+\alpha)}}{\gamma(1+\alpha, 2\pi^2n^2T)}} e^{-\pi^2n^2t + \pi nix},$$

and $u_0(x, t) = \sqrt{\frac{1+\alpha}{2T^{1+\alpha}}}$, then $\{u_n\}_{n \in \mathbb{Z}}$ is an orthonormal basis of $b_\alpha^2(S_T)$ ([7, Lemma 3.9]).

The weighted Bergman projection $P_\alpha : L_\alpha^2(\Omega_T) \rightarrow b_\alpha^2(S_T)$ can be written as the integral operator

$$P_\alpha u(z) = \int_{\Omega_T} N_\alpha(z, w) u(w) \tau^\alpha dw,$$

where $N_\alpha(z, w)$ is a symmetric, real-valued function called the Bergman reproducing kernel of $b_\alpha^2(S_T)$, it is given by

$$N_\alpha(z, w) = \sum_{n \in \mathbb{Z}} u_n(z) \overline{u_n(w)} = \sum_{n \in \mathbb{Z}} u_n(x, t) \overline{u_n(y, \tau)},$$

and satisfies $N_\alpha(\cdot, w) \in b_\alpha^2(S_T)$ for every $w \in S_T$ (see [7, Section 3]).

Since $\gamma(1+\alpha, 2\pi^2n^2T) \geq \gamma(1+\alpha, 2\pi^2T)$, and $e^{-x} \leq C_\lambda x^{-\lambda}$ for $x > 0$, $\lambda \geq 0$ we have that the series defining N_α converges absolutely and uniformly on $\Omega' \times \mathbb{R}_+^2$ provided $\Omega' \subset \mathbb{R}_+^2$ is compact, furthermore, the function N_α and its derivatives are bounded on $\Omega' \times \mathbb{R}_+^2$, therefore $N_\alpha \in C^\infty(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$ and $N_\alpha(\cdot, w) \in H(S_T)$ for all $w \in \mathbb{R}_+^2$. On the other hand, the function

$$\Theta_\alpha(z, w) = \theta_\alpha(x - y, t + \tau) = \frac{1}{2} \pi^{2(1+\alpha)} \sum_{n \in \mathbb{Z}} n^{2(1+\alpha)} e^{-\pi^2n^2(t+\tau) + \pi ni(x-y)},$$

satisfies ([8, Lema 2.14])

$$\int_{\Omega_T} |\Theta_\alpha(z, w)| \tau^\beta dw \leq C_{\alpha, \beta} t^{\beta-\alpha} \quad \text{if } \alpha > \beta > -1.$$

Finally, during the proof of Theorem (1.3) in [8] we have constructed an isomorphism $\mathcal{M}_\alpha : b_\beta^1(S_T) \rightarrow b_\beta^1(S_T)$, applying the theory of Fourier

multipliers on $L^1(S^1)$, such that

$$\begin{aligned}\mathcal{M}_\alpha \left(\frac{\partial^n N_\alpha}{\partial x^n}(z, w) \right) &= i^n \Theta_{\alpha + \frac{n}{2}}(z, w), \quad \text{if } \alpha + \frac{n}{2} > \beta > -1 \\ \mathcal{M}_\alpha \left(\frac{\partial^n N_\alpha}{\partial t^n}(z, w) \right) &= (-1)^n \Theta_{\alpha + n}(z, w), \quad \text{if } \alpha + n > \beta > -1\end{aligned}$$

where $\mathcal{M}_\alpha$ is acting on the variable w . In particular, we have

$$\begin{aligned}(3) \quad \int_{\Omega_T} \left| \frac{\partial N_\alpha}{\partial x}(z, w) \right| \tau^\alpha dw &\leq C_\alpha \int_{\Omega_T} \left| \Theta_{\alpha + \frac{1}{2}}(z, w) \right| \tau^\alpha dw \leq C_\alpha t^{-\frac{1}{2}}, \\ \int_{\Omega_T} \left| \frac{\partial N_\alpha}{\partial t}(z, w) \right| \tau^\alpha dw &\leq C_\alpha \int_{\Omega_T} |\Theta_{\alpha + 1}(z, w)| \tau^\alpha dw \leq C_\alpha t^{-1}.\end{aligned}$$

3. Basic properties of the Bloch-type space $\mathcal{B}_T$

We recall the definition of the normed linear space $\mathcal{B}_T$.

Definition 5. For $T > 0$, let

$$\mathcal{B}_T = \left\{ v \in H(S_T) : \sup \left\{ t \left| \frac{\partial v}{\partial t}(x, t) \right| : (x, t) \in \Omega_T \right\} < \infty \right\},$$

and

$$\|v\|_{\mathcal{B}_T} = \max_{x \in [0, 2]} \left| v\left(x, \frac{T}{2}\right) \right| + \sup \left\{ t \left| \frac{\partial v}{\partial t}(x, t) \right| : (x, t) \in \Omega_T \right\}.$$

The following result is about the growth of functions in $\mathcal{B}_T$.

Lemma 6. If $v \in \mathcal{B}_T$ then

$$(4) \quad |v(x, t)| \leq \|v\|_{\mathcal{B}_T} \left(1 + \ln \frac{T}{t} \right), \quad \text{for all } (x, t) \in \Omega_T.$$

Proof. For $t < \frac{T}{2}$ we have

$$\begin{aligned}\left| v\left(x, \frac{T}{2}\right) - v(x, t) \right| &= \left| \int_0^1 \frac{d}{d\sigma} v\left(x, \sigma \frac{T}{2} + (1 - \sigma)t\right) d\sigma \right| \\ &= \left| \int_0^1 \left(\frac{T}{2} - t \right) \frac{\partial v}{\partial t}\left(x, \sigma \frac{T}{2} + (1 - \sigma)t\right) d\sigma \right| \\ &\leq \sup \left\{ t \left| \frac{\partial v}{\partial t}(x, t) \right| : (x, t) \in \Omega_T \right\} \\ &\quad \times \int_0^1 \frac{\left(\frac{T}{2} - t \right)}{\sigma \frac{T}{2} + (1 - \sigma)t} d\sigma \\ &\leq \|v\|_{\mathcal{B}_T} \ln \frac{T}{2t}.\end{aligned}$$

When $t > \frac{T}{2}$ the inequality holds by Proposition 4. □

Remark 7. From the 2-periodicity in the variable x of the functions in $\mathcal{B}_T$, together with (4), we note that norm convergence in $\mathcal{B}_T$ implies the uniform convergence on compact subsets in $\mathbb{R} \times (0, T)$.

Next we show that the normed linear space $(\mathcal{B}_T, \|\cdot\|_{\mathcal{B}_T})$ is complete.

Proposition 8. *For every $T > 0$, $\mathcal{B}_T$ is a Banach space.*

Proof. Let (u_j) a Cauchy sequence in $\mathcal{B}_T$, by the remark above there is a function u such that $u = \lim u_j$. Likewise, the sequence $(\frac{\partial}{\partial t} u_j)$ converges uniformly on compact subsets in $\mathbb{R} \times (0, T)$, therefore $\frac{\partial u_j}{\partial t} \rightarrow \frac{\partial u}{\partial t}$ on $\mathbb{R} \times (0, T)$.

Pick $(x_0, t_0) \in \Omega_T$. Let $R = (a, b) \times (c, d)$, with $d - c = (b - a)^2$, such that $(x_0, t_0) \in R$ and $\overline{R} \subset \Omega_T$. Consider the mapping $\Psi : Q \rightarrow R$ given by $\Psi(\xi, \tau) = ((b - a)\xi + a, (d - c)\tau + c)$. Since $u_j \in H(R) \cap C(\overline{R})$, (1) implies that

$$u_j(x, t) = \int_{\Gamma_R} \tilde{K}(\Psi^{-1}(x, t); \Psi^{-1}(\xi, \tau)) u_j(\xi, \tau) d\lambda_R(\xi, \tau), \quad \text{for } (x, t) \in R,$$

where $\Gamma_R = \Psi(\Gamma)$ and λ_R is the one-dimensional Lebesgue measure normalized on each segment of Γ_R . By the dominated convergence theorem we have

$$u(x, t) = \int_{\Gamma_R} \tilde{K}(\Psi^{-1}(x, t); \Psi^{-1}(\xi, \tau)) u(\xi, \tau) d\lambda_R(\xi, \tau).$$

Since the function u is continuous on Γ_R then u is a temperature function on R . Since $(x_0, t_0) \in \Omega_T$ was arbitrary we conclude that $u \in H(S_T)$, and $\|u - u_j\|_{\mathcal{B}_T} \rightarrow 0$ as $j \rightarrow \infty$. $\square$

Proposition 9. *Let $1 \leq p < \infty$ and $\alpha > -1$. If $v \in \mathcal{B}_T$ then $v \in b_\alpha^p(S_T)$. Therefore $P_\alpha v = v$ for all $v \in \mathcal{B}_T$.*

Proof. Since $t^\alpha dx dt$ is a finite measure on Ω_T , by the previous lemma it is sufficient to show that $\int_0^T \left| \ln \frac{T}{t} \right|^p t^\alpha dt < \infty$. Let $\varepsilon > 0$ such that $\varepsilon < \alpha + 1$, then $|\ln u|^p / u^{\alpha+1-\varepsilon} \rightarrow 0$ as $u \rightarrow \infty$, so

$$\int_0^T \left| \ln \frac{T}{t} \right|^p t^\alpha dt = T^{\alpha+1} \int_1^\infty \frac{|\ln u|^p}{u^{\alpha+2}} du \leq C_{\alpha, T} \int_1^\infty \frac{1}{u^{1+\varepsilon}} du < \infty. \quad \square$$

Proposition 10. *For $\alpha > -1$ the weighted Bergman projection P_α maps $L^\infty(\Omega_T)$ continuously into $\mathcal{B}_T$. Additionally, $\|t^{1/2} \frac{\partial}{\partial x} (P_\alpha \varphi)\|_{L^\infty(\Omega_T)} < \infty$ for every $\varphi \in L^\infty(\Omega_T)$.*

Proof. Let $\varphi \in L^\infty(\Omega_T)$, differentiating under the integral sign yields

$$\frac{\partial (P_\alpha \varphi)}{\partial t}(z) = \int_{\Omega_T} \frac{\partial N_\alpha}{\partial t}(z, w) \varphi(w) \tau^\alpha dw, \quad z \in \Omega_T$$

by (3) we get

$$\left| \frac{\partial (P_\alpha \varphi)}{\partial t}(z) \right| \leq C_\alpha \|\varphi\|_{L^\infty(\Omega_T)} t^{-1}, \quad z \in \Omega_T.$$

The proof of the other conclusion is similar. $\square$

Finally, we prove the main results.

Proof of Theorem 1. Let $\psi_\alpha \in C^\infty(\mathbb{R})$ a nonnegative function satisfying that $\psi_\alpha \equiv 1/(\alpha+1)$ for $t \leq \frac{T}{2}$ and $\psi_\alpha \equiv 0$ for $t > T - \delta$, with $\delta > 0$ small enough. Thus, for $v \in \mathcal{B}_T$ we define

$$\begin{aligned} (\mathcal{L}_\alpha^1 v)(x, t) &= v(x, t) - \left[\alpha \psi_\alpha(t) v(x, t) + \left(\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} \right) (\psi_\alpha(t) t v(x, t)) \right] \\ &= \{1 - (1 + \alpha) \psi_\alpha(t) - t \psi'_\alpha(t)\} v(x, t) - 2t \psi_\alpha(t) \frac{\partial v}{\partial t}(x, t). \end{aligned}$$

Clearly $\mathcal{L}_\alpha^1$ is a linear differential operator on $\mathcal{B}_T$. By Proposition 4, it follows that

$$|(\mathcal{L}_\alpha^1 v)(x, t)| \leq C_{\psi_\alpha} \|v\|_{\mathcal{B}_T} \quad \text{for all } (x, t) \in \Omega_T.$$

Letting $g(x, t) = \alpha \psi_\alpha(t) v(x, t) + \left(\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} \right) (\psi_\alpha(t) t v(x, t))$, we claim that $g \in L_\alpha^2(\Omega_T)$, and $(g, u)_{L_\alpha^2(\Omega_T)} = 0$ for all $u \in H(S_T) \cap C(\overline{S_T})$. So, $P_\alpha \mathcal{L}_\alpha^1 v = P_\alpha v - P_\alpha g = P_\alpha v = v$ since $v \in b_\alpha^2(S_T)$.

To prove the claim, we rewrite $g = ((1 + \alpha) \psi_\alpha + t \psi'_\alpha) v + 2t \psi_\alpha \frac{\partial v}{\partial t}$ then $g \in L_\alpha^2(\Omega_T)$. By the other hand, if $u \in H(S_T) \cap C(\overline{S_T})$ then integrating by parts we get for $\varepsilon > 0$ small enough,

$$\begin{aligned} & \int_\varepsilon^T \int_0^2 \bar{u} \left(\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} \right) (\psi_\alpha t v) t^\alpha dz \\ &= \int_\varepsilon^T \left[\bar{u} \frac{\partial v}{\partial x} - v \frac{\partial \bar{u}}{\partial x} \right] \Big|_{x=0}^{x=2} \psi_\alpha t^{\alpha+1} dt \\ & \quad + \int_0^2 \psi_\alpha t^{\alpha+1} \bar{u} v \Big|_{t=\varepsilon}^{t=T} dx - \int_\varepsilon^T \int_0^2 \alpha \psi_\alpha \bar{u} v t^\alpha dz \\ &= - \int_0^2 \frac{\varepsilon^{\alpha+1}}{\alpha+1} \bar{u}(x, \varepsilon) v(x, \varepsilon) dx - \int_\varepsilon^T \int_0^2 \alpha \psi_\alpha \bar{u} v t^\alpha dz. \end{aligned}$$

Lemma 6 and the dominated convergence theorem imply $(g, u)_{L_\alpha^2(\Omega_T)} = 0$. $\square$

Corollary 11. *The mapping $P_\alpha : L^\infty(\Omega_T) \rightarrow \mathcal{B}_T$ is onto.*

From Proposition 10 we have the following

Corollary 12. *If $v \in \mathcal{B}_T$ then $\sup \{t^{1/2} \left| \frac{\partial v}{\partial x}(x, t) \right| : (x, t) \in \Omega_T\} < \infty$.*

Proposition 13. For $\alpha > -1$, the bilinear form $\langle \cdot, \cdot \rangle_\alpha : b_\alpha^1(S_T) \times \mathcal{B}_T \rightarrow \mathbb{C}$ given by

$$\langle u, v \rangle_\alpha = \int_{\Omega_T} u(z) \overline{(\mathcal{L}_\alpha^1 v)(z)} t^\alpha dz$$

is continuous, and $\langle u, v \rangle_\alpha = (u, v)_{b_\alpha^2(S_T)}$ for $u \in b_\alpha^2(S_T)$, $v \in \mathcal{B}_T$.

Proof. For $u \in b_\alpha^1(S_T)$, $v \in \mathcal{B}_T$, we have

$$|\langle u, v \rangle_\alpha| \leq \|u\|_{b_\alpha^1(S_T)} \|\mathcal{L}_\alpha^1 v\|_{L^\infty(\Omega_T)} \leq C \|u\|_{b_\alpha^1(S_T)} \|v\|_{\mathcal{B}_T}.$$

On the other hand, if $u \in b_\alpha^2(S_T)$, $v \in \mathcal{B}_T$

$$\begin{aligned} \langle u, v \rangle_\alpha &= (u, \mathcal{L}_\alpha^1 v)_{b_\alpha^2(S_T)} \\ &= (P_\alpha u, \mathcal{L}_\alpha^1 v)_{b_\alpha^2(S_T)} \\ &= (u, P_\alpha \mathcal{L}_\alpha^1 v)_{b_\alpha^2(S_T)} \\ &= (u, v)_{b_\alpha^2(S_T)}. \end{aligned} \quad \square$$

Proof of Theorem 2. Consider the linear mapping $\Phi_\alpha : \mathcal{B}_T \rightarrow (b_\alpha^1(S_T))^*$ given by $\Phi_\alpha v = \langle \cdot, v \rangle_\alpha$. By the previous proposition Φ_α is continuous, and injective: If $\Phi_\alpha v = 0$ then $\langle v, v \rangle_\alpha = (v, v)_{b_\alpha^2(S_T)} = 0$, therefore $v \equiv 0$.

Let $F \in (b_\alpha^1(S_T))^*$, by the Hahn-Banach theorem the operator F can be extended continuously to $L^1(\Omega_T, t^\alpha dz)$, and the Riesz representation theorem implies the existence of $\varphi \in L^\infty(\Omega_T)$ such that

$$Fu = \int_{\Omega_T} u(z) \overline{\varphi(z)} t^\alpha dz, \text{ for all } u \in b_\alpha^1(S_T).$$

In particular, for $u \in b_\alpha^2(S_T)$ we have

$$\begin{aligned} Fu &= (u, \varphi)_{L_\alpha^2(\Omega_T)} = (P_\alpha u, \varphi)_{L_\alpha^2(\Omega_T)} \\ &= (u, P_\alpha \varphi)_{b_\alpha^2(S_T)} = \langle u, P_\alpha \varphi \rangle_\alpha, \end{aligned}$$

where the last equality follows from the previous proposition. Since $b_\alpha^2(S_T)$ is dense in $b_\alpha^1(S_T)$, then $F = \Phi_\alpha(P_\alpha \varphi)$. By the open mapping theorem we have that Φ_α is an isomorphism. $\square$

We finish with a simple example of Bloch-type temperature functions,

Example 14. Suppose $(a_n)_n$ is a bounded sequence of complex numbers and suppose $(\lambda_n)_n$ is a sequence of positive integers with $\lambda_n/\lambda_{n-1} \geq \sqrt{\lambda}$, where $1 < \lambda < \infty$. Let

$$v(x, t) = \sum_{n=1}^{\infty} a_n e^{-\pi^2 \lambda_n^2 t + \pi \lambda_n i x}.$$

Since $e^{-x} \leq C_\sigma x^{-\sigma}$ for $x > 0$, $\sigma \geq 0$ then $v \in H(S_T)$, and

$$\left| \frac{\partial v}{\partial t}(x, t) \right| \leq M \sum_{n=1}^{\infty} \lambda_n^2 (e^{-\pi^2 t})^{\lambda_n^2},$$

where $M > 0$ is a constant.

Setting $z = e^{-\pi^2 t}$, and $K = \lambda/(\lambda-1)$ we have that $0 < z < 1$, $1 < K < \infty$. Then,

$$\lambda_1^2 |z|^{\lambda_1^2} \leq |z| + \cdots + |z|^{\lambda_1^2} \leq K(|z| + \cdots + |z|^{\lambda_1^2}).$$

Since $\lambda_n^2 \leq K(\lambda_n^2 - \lambda_{n-1}^2)$ for every $n \geq 2$, we have

$$\lambda_n^2 |z|^{\lambda_n^2} \leq K(|z|^{\lambda_{n-1}^2+1} + \cdots + |z|^{\lambda_n^2}).$$

Let $\delta > 0$ small enough and $C_\delta > 0$ a constant such that $t \leq C_\delta(1 - e^{-\pi^2 t})$ for $0 < t < \delta$, therefore

$$\left| \frac{\partial v}{\partial t}(x, t) \right| \leq MK \sum_{n=1}^{\infty} |z|^n \leq \frac{MK}{1 - e^{-\pi^2 t}} \leq \frac{C_\delta MK}{t}.$$

Since $\frac{\partial v}{\partial t} \in C(\mathbb{R} \times [\delta, \infty))$ we obtain that $v \in \mathcal{B}_T$.

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