A new class of linear operators on ℓ^2 and Schur multipliers for them

Anca-Nicoleta Marcoci and Liviu-Gabriel Marcoci

(Communicated by Nicolae Popa)

2000 Mathematics Subject Classification. 47B35.

Keywords and phrases. Schur multiplier, Toeplitz matrices, matricial Wiener algebra.

Abstract. We introduce the space $B_w(\ell^2)$ of linear (unbounded) operators on ℓ^2 which map decreasing sequences from ℓ^2 into sequences from ℓ^2 and we find some classes of operators belonging either to $B_w(\ell^2)$ or to the space of all Schur multipliers on $B_w(\ell^2)$. For instance we show that the space $B(\ell^2)$ of all bounded operators on ℓ^2 is contained in the space of all Schur multipliers on $B_w(\ell^2)$.

1. Introduction

Let $A = (a_{ij})_{i,j \geq 1}$ be an infinite matrix. We define

$$B_w(l^2) = \{A \text{ infinite matrix} : Ax \in l^2 \text{ for every } x \in l^2 \text{ with } |x_k| \searrow 0\},$$

where

$$l^2 = \left\{ x = (x_k)_k : \sum_{k=0}^{\infty} |x_k|^2 < \infty \right\} \text{ is the classical Hilbert space of sequences.}$$

The above space of matrices has appeared in the study of the matriceal analogue of some well-known Banach spaces as $C(\mathbb{T})$, $M(\mathbb{T})$, $L^1(\mathbb{T})$.

A similarity between the functions defined on $\mathbb{T}$ and the infinite matrices was remarked for the first time in 1978 by J. Arazy [1]. Later on, in 1983, A. Shields has exploited further this similarity starting with a few constructs used in harmonic analysis together with their matriceal analogues [9].

Recently, in [3], the Fejer's theory developed for Fourier series was extended in the framework of matrices.

The analogy is as follows: we identify a function f with the Toeplitz matrix $A = (a_{ij})_{i,j \geq 1}$,

$$a_{ij} = a_{i-j} \text{ for all } i, j \in \mathbb{N}^*$$

where $(a_k)_{k \in \mathbb{Z}}$ is the sequence of Fourier coefficients of the function f .

The *Schur product* of two matrices is defined by

$$A * B = (a_{ij} \cdot b_{ij})_{i,j \geq 1},$$

where $A = (a_{ij})_{i,j \geq 1}$, $B = (b_{ij})_{i,j \geq 1}$. We denote by

$$M(l^2) = \{M : M * A \in B(l^2) \text{ for every } A \in B(l^2)\}$$

the *space of all Schur multipliers* equipped with the following norm

$$\|M\| = \sup_{\|A\|_{B(l^2)} \leq 1} \|M * A\|_{B(l^2)}.$$

Let us define $A_k = (a'_{ij})_{i,j \geq 1}$, $k \in \mathbb{Z}$ to be the matrix with the elements

$$a'_{ij} = \begin{cases} a_{ij} & \text{if } j - i = k \\ 0 & \text{otherwise} \end{cases}$$

A_k is called the *Fourier coefficient of k -order of the matrix A* . (See e.g. [3].) We have now a similarity between the expansion in Fourier series of a periodical function f on $\mathbb{T}$

$$f = \sum_k a_k e^{ikt}$$

and the decomposition of matrix in diagonal matrices

$$A = \sum_{k \in \mathbb{Z}} A_k$$

In this way we can say that $B(l^2)$ represents the matriceal analogue of $L^\infty(\mathbb{T})$ and $M(l^2)$ is the analogue of $M(\mathbb{T})$. Moreover in [3] was introduced the space of continuous matrices denoted by $C(l^2)$. A is a *continuous matrix*

if $\lim_{n \rightarrow \infty} \sigma_n(A) = A$, where the limit is taken in the norm of $B(l^2)$ and $\sigma_n(A)$

is the Cesaro sum associated to $S_n(A) := \sum_{k=-n}^n A_k$.

This space is a Banach space with the following norm

$$\|A\|_{C(l^2)} = \max(\sup_n \|\sigma_n(A)\|_{B(l^2)}, \|A\|_{B(l^2)}).$$

In the way described earlier this space represents the matriceal analogue of the space $C(\mathbb{T})$.

Also, N. Popa has defined

$$A(l^2) = \left\{ A \text{ infinite matrix} : \sup_{l \in \mathbb{Z}_+} \sum_{k \in \mathbb{Z}} |a_k^l| < \infty \right\},$$

where

$$A = \begin{pmatrix} a_0^1 & a_1^1 & \dots \\ a_{-1}^1 & a_0^2 & \dots \\ \dots & \dots & \dots \end{pmatrix}.$$

This was the first attempt to define the matriceal analogue of the Wiener algebra $A(\mathbb{T})$ and we may call it *matriceal Wiener algebra*.

It is easy to see that the subspace of all Toeplitz matrices from $A(l^2)$ can be identified with $A(\mathbb{T})$ but since $A(l^2) \not\subseteq C(l^2)$ it is necessary to find a larger space than $B(l^2)$. In order to see that $A(l^2) \not\subseteq C(l^2)$ let us consider

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Then $\sup_{l \in \mathbb{Z}_+} \sum_{k \in \mathbb{Z}} |a_k^l| = 1$ and $A \in A(l^2)$ but $\|Ae_{\frac{n(n+1)}{2}}\|_2 = n$, where $e_k = (0, \dots, 0, 1, 0, \dots)$

One solution would be to choose a larger space than $B(l^2)$, and, for this reason we introduce $B_w(l^2)$. Clearly $B_w(l^2)$ is a Banach space with the norm

$$\|A\|_{B_w(l^2)} = \sup_{\substack{\|x\|_2 \leq 1 \\ |x_k| \searrow 0}} \|Ax\|_2.$$

The following result due to E. Sawyer [8] (See also [7]) will be used very often in the sequel. If $v = (v(n))_n$ is a weight on $\mathbb{N}^*$, we put $\tilde{v} = \sum_{n=0}^{\infty} v(n)\chi_{[n,n+1)}$ and $\tilde{V}(t) = \int_0^t \tilde{v}(s)ds$. Let us mention moreover that the relation $f \approx g$ means that there are two positive constants a and b such that $af \leq g \leq bf$.

Then we have

Theorem 1. *Let $w = (w(n))_n$, $v = (v(n))_n$ be weights on $\mathbb{N}^*$ and let*

$$S = \sup_{f \searrow} \frac{\sum_{n=0}^{\infty} f(n)v(n)}{\left(\sum_{n=0}^{\infty} f(n)^p w(n) \right)^{\frac{1}{p}}}.$$

Then

(i) *If $0 < p \leq 1$*

$$S = \sup_{n \geq 0} \frac{V(n)}{W^{\frac{1}{p}}(n)},$$

with W defined by $W(n) = \sum_{k=0}^n w(k)$ and similarly for V .

(ii) *If $1 < p < \infty$,*

$$S \approx \left(\int_0^{\infty} \left(\frac{\tilde{V}(t)}{\tilde{W}(t)} \right)^{p'-1} \tilde{v}(t) dt \right)^{\frac{1}{p'}} \approx \left(\int_0^{\infty} \left(\frac{\tilde{V}(t)}{\tilde{W}(t)} \right)^{p'} \tilde{w}(t) dt \right)^{\frac{1}{p'}} + \frac{\tilde{V}(\infty)}{\tilde{W}^{\frac{1}{p}}(\infty)}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

The paper is organised as follows: in Section 2, we show that the matriceal Wiener algebra $A(\ell^2)$ is a subset of $B_w(\ell^2)$ (Proposition 2) and we give some criteria for diagonal matrices to belong to $B_w(\ell^2)$. Moreover, we consider the Schur product of matrices and remark that $B_w(\ell^2)$ is not closed under this product. Using the Sawyer result we prove in Section 3 the main result of the paper, namely that linear and bounded operators on ℓ^2 are Schur multipliers on $B_w(\ell^2)$, a result which is not obvious, since $B_w(\ell^2)$ is not Schur algebra. Finally, we collected in Section 4 all results concerning the Toeplitz matrices. For instance there is no difference between Toeplitz matrices from $B(\ell^2)$ and those from $B_w(\ell^2)$.

2. Preliminary results

Proposition 2. *$A(l^2) \subseteq B_w(l^2)$ with $\|A\|_{B_w(l^2)} \leq \|A\|_{A(l^2)}$.*

Proof. Let $A \in A(l^2)$ be an upper triangular matrix. Then for every $(x_l)_{l=0}^\infty \in l^2$ with $|x_l| \searrow 0$ and $\|x_l\| \leq 1$ we have

$$\begin{aligned} \sum_{k=1}^\infty \left| \sum_{l=k-1}^\infty a_{l-k+1}^k x_l \right|^2 &\leq \sum_{k=1}^\infty \left(\sum_{l=k-1}^\infty |a_{l-k+1}^k| |x_l| \right)^2 \leq \\ &\leq \sum_{k=1}^\infty \left(\sum_{l=k-1}^\infty |a_{l-k+1}^k| \right)^2 \sup_{l \geq k-1} |x_l|^2 \leq \sum_{k=1}^\infty \left(\sup_k \sum_{l=k-1}^\infty |a_{l-k+1}^k| \right)^2 \sup_{l \geq k-1} |x_l|^2 \leq \\ &\leq \|A\|_{A(l^2)}^2 \sum_{k=1}^\infty \sup_{l \geq k-1} |x_l|^2 = \|A\|_{A(l^2)}^2 \|x\|_2^2 \leq \|A\|_{A(l^2)}^2 \end{aligned}$$

which implies that $\|A\|_{B_w(l^2)} \leq \|A\|_{A(l^2)}$. $\square$

Proposition 3. Let $A = A_0$ given by $(a_k)_{k=1}^\infty$. Then $A \in B_w(l^2)$ if and only if

$$\sup_{n \in \mathbb{N}^*} \left(\frac{\sum_{k=1}^n |a_k|^2}{n} \right)^{\frac{1}{2}} < \infty.$$

Moreover

$$\|A\|_{B_w(l^2)} = \sup_{n \in \mathbb{N}^*} \left(\frac{\sum_{k=1}^n |a_k|^2}{n} \right)^{\frac{1}{2}}.$$

Proof. The sufficiency follows immediately from the factorization

$$l^2 = d(1, 2) \cdot g(1, 2),$$

where

$$d(1, 2) = \left\{ x : \sum_{n=1}^\infty \sup_{k \geq n} |x_k|^2 < \infty \right\}$$

and

$$g(1, 2) = \left\{ x : \sum_{k=1}^\infty |x_k|^2 = O(n) \right\},$$

see, e.g., [5]. For the necessity take $x^n = \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}, 0, \dots \right)$, $\|x^n\|_2 = 1$, $|x_k^n|_k \searrow 0$, $n \geq 1$

$$\|Ax^n\|_2 = \left(\frac{1}{n} \sum_{k=1}^n |a_k|^2 \right)^{\frac{1}{2}} \leq \|A\|_{B_w(l^2)}$$

$$\text{and } \sup_n \left(\frac{\sum_{k=1}^n |a_k|^2}{n} \right)^{\frac{1}{2}} < \infty.$$

□

If we translate this sequence $(a_k)_{k=1}^\infty$ above or below of the main diagonal we obtain the following similar result.

Corollary 4. (a) Let $k > 0$ and $A = A_k$ given by $(a_k)_{k=1}^\infty$. Then $A \in B_w(l^2)$ if and only if $\sup_{n \in \mathbb{N}^*} \left(\frac{1}{n+k} \sum_{l=1}^n |a_l|^2 \right)^{\frac{1}{2}} < \infty$ and

$$\|A\|_{B_w(l^2)} = \sup_{n \in \mathbb{N}^*} \left(\frac{1}{n+k} \sum_{l=1}^n |a_l|^2 \right)^{\frac{1}{2}},$$

(b) Let $k < 0$ and $A = A_k$ given by $(a_k)_{k=1}^\infty$. Then $A \in B_w(l^2)$ if and only if

$$\sup_{n \in \mathbb{N}^*} \left(\frac{1}{n} \sum_{l=1}^n |a_l|^2 \right)^{\frac{1}{2}} < \infty$$

$$\text{and } \|A\|_{B_w(l^2)} = \sup_{n \in \mathbb{N}^*} \left(\frac{1}{n} \sum_{l=1}^n |a_l|^2 \right)^{\frac{1}{2}}.$$

Remark 5. 1. Clearly $B(l^2) \subseteq B_w(l^2)$ and using Proposition 3 for the matrix $A = A_0$ given by $(a_k)_{k=1}^\infty$,

$$a_k = \begin{cases} \sqrt{2^n} & \text{if } k = 2^n \\ 0 & \text{if } k \neq 2^n \end{cases} \quad n \geq 1$$

we can easily show that the inclusion is proper.

2. While $B(l^2)$ is closed under Schur multiplication $B_w(l^2)$ is not. For example it is easy to see that $A * A \notin B_w(l^2)$ where A is the matrix defined previously.
3. The space $B_w(l^2)$ cannot be compared with $M(l^2)$ meaning that

$$(1) \quad B_w(l^2) \not\subseteq M(l^2)$$

and

$$(2) \quad M(l^2) \not\subseteq B_w(l^2).$$

For (1) there is $A = A_0$ given by $(a_k)_{k=1}^\infty$ such that $(a_n)_n \notin l^\infty$, $A \in B_w(l^2)$ and $A \notin M(l^2)$.

$$\text{For (2) we take } A = \begin{pmatrix} 1 & 1 & \dots & 1 & \dots \\ 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \in M(l^2) \text{ but}$$

$$A \notin B_w(l^2).$$

4. Let $A = A_0$ given by $(a_k)_{k=1}^\infty$. Then

$$A \in M(B_w(l^2), B_w(l^2)) = \{M : M * B \in B_w(l^2) \text{ for every } B \in B_w(l^2)\}$$

if and only if $(a_n)_n \in l^\infty$.

$A \in M(B_w(l^2), B_w(l^2))$ then

$$A * B \in B_w(l^2) \text{ for } B = B_0 \in B_w(l^2) \text{ given by } (b_k)_{k=1}^\infty.$$

From Proposition 3,

$$(b_k)_{k=1}^\infty \in g(1, 2), \|Ax\|_{l^2}^2 = \sum_{k=1}^\infty |a_k b_k x_k|^2 < \infty$$

for every $b = (b_k)_k \in g(1, 2)$, and $x \in d(1, 2)$. Using the factorization $l^\infty \cdot g(1, 2) = g(1, 2)$ and $l^2 = d(1, 2) \cdot g(1, 2)$, provided in [5] we get that $(a_n)_n \in l^\infty$. Now let $(a_n)_n \in l^\infty$. It is easy to see that

$$\|B_0\|_{B_w(l^2)} \leq \|B\|_{B_w(l^2)}$$

3. Main result

Lemma 6. $\sup_{|x_n| \searrow 0} \frac{\left| \sum_{n=1}^\infty a_n x_n \right|}{\left(\sum_{n=1}^\infty |x_n|^2 \right)^{\frac{1}{2}}} = \sup_{|x_n| \searrow 0} \frac{\sum_{n=1}^\infty |a_n| |x_n|}{\left(\sum_{n=1}^\infty |x_n|^2 \right)^{\frac{1}{2}}} \approx \left(\sum_{n=1}^\infty \left(\frac{1}{n} \sum_{k=1}^n |a_k| \right)^2 \right)^{\frac{1}{2}},$

where $(a_n)_n$ and $(x_n)_n$ are sequences of complex numbers.

Proof. We denote $S = \sup_{|x_n| \searrow 0} \frac{\sum_{n=1}^\infty |a_n| |x_n|}{\left(\sum_{n=1}^\infty |x_n|^2 \right)^{\frac{1}{2}}}$. From Theorem 1 we have

$$S \approx \left(\int_0^\infty \left(\frac{\tilde{V}(t)}{\tilde{W}(t)} \right)^2 \tilde{w}(t) dt \right)^{\frac{1}{2}} + \frac{\tilde{V}(\infty)}{\tilde{W}^{\frac{1}{p}}(\infty)},$$

where $v(n) = |a_n|$, $w(n) = 1$, $f(n) = |x_n|$ for every n nonnegative integer.

In this case

$$\tilde{v} = \sum_{n=0}^{\infty} v(n) \chi_{[n, n+1)} = \sum_{n=0}^{\infty} |a_n| \chi_{[n, n+1)}(s), \text{ where } a_0 = 0,$$

therefore, for $t \in (j, j+1)$, we have

$$\begin{aligned} \tilde{V}(t) &= \int_0^t \tilde{v}(s) ds = \int_0^j \tilde{v}(s) ds + \int_j^t \tilde{v}(s) ds = \\ &= \sum_{m=0}^{j-1} \int_m^{m+1} \tilde{v}(s) ds + \int_j^t \tilde{v}(s) ds = \sum_{m=0}^{j-1} |a_m| + |a_j|(t-j), \\ \tilde{V}(\infty) &= \int_0^{\infty} \tilde{v}(s) ds = \int_0^{\infty} \sum_{n=0}^{\infty} |a_n| \chi_{[n, n+1)}(s) ds = \sum_{n=1}^{\infty} |a_n| \text{ and} \\ \widetilde{W}(\infty) &= \int_0^{\infty} \tilde{w}(s) ds = \infty, \end{aligned}$$

since $\tilde{w}(s) = \sum_{n=0}^{\infty} \chi_{[n, n+1)}(s)$.

Clearly, letting $\tilde{v}_M = \sum_{n=0}^M |a_n| \chi_{[n, n+1)}$, $\tilde{V}_M = \int_0^{\infty} \tilde{v}_M(s) ds = \int_0^M \sum_{n=0}^M |a_n| \chi_{[n, n+1)}(s) ds = \sum_{n=1}^M |a_n| < \infty$, we get

$$\begin{aligned} S &= \sup_M \left(\sup_{|x_n| \searrow} \frac{\sum_{n=1}^M |x_n| |a_n|}{\left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{\frac{1}{2}}} \right) \approx \sup_M \left[\left(\int_0^{\infty} \left(\frac{\tilde{V}_M(t)}{t} \right)^2 dt \right)^{\frac{1}{2}} + \frac{\tilde{V}_M(\infty)}{\widetilde{W}^{\frac{1}{2}}(\infty)} \right] \approx \\ &= \left(\int_0^{\infty} \left(\frac{\tilde{V}(t)}{t} \right)^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

But

$$\int_0^{\infty} \left(\frac{\tilde{V}(t)}{t} \right)^2 dt = \sum_{j=1}^{\infty} \int_j^{j+1} \left(\frac{\sum_{m=0}^{j-1} |a_m| + |a_j|(t-j)}{t} \right)^2 dt$$

$$\leq \sum_{j=1}^{\infty} \left(\int_j^{j+1} \frac{1}{t^2} dt \right) \left(\sum_{m=1}^j |a_m| \right)^2 \leq \sum_{j=1}^{\infty} \left(\frac{1}{j} \sum_{m=1}^j |a_m| \right)^2.$$

On the other hand

$$\int_0^{\infty} \left(\frac{\tilde{V}(t)}{t} \right)^2 dt \geq \sum_{j=1}^{\infty} \int_j^{j+1} \left(\frac{\sum_{m=1}^{j-1} |a_m|}{t} \right)^2 dt + \sum_{j=1}^{\infty} \int_j^{j+1} |a_j|^2 \frac{(t-j)^2}{t^2} dt \geq$$

$$\sum_{j=1}^{\infty} \left(\frac{1}{j} \sum_{m=1}^j |a_m| \right)^2$$

$$\text{which implies that } S \approx \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n |a_k| \right)^2 \right)^{\frac{1}{2}}.$$

□

Theorem 7. $B(l^2) \subseteq M(B_w(l^2), B_w(l^2))$.

Proof. Let $A \in B(l^2)$ and $B \in B_w(l^2)$. Then

$$\begin{aligned} \sum_j \left| \sum_k a_{jk} b_{jk} x_k \right|^2 &\leq \sum_j \left(\sum_k |a_{jk}| |b_{jk}| |x_k| \right)^2 \leq \sum_j \left(\sum_k |a_{jk}|^2 \right) \left(\sum_k |b_{jk}|^2 |x_k|^2 \right) \\ &\leq \sup_j \left(\sum_k |a_{jk}|^2 \right) \sum_j \left(\sum_k |b_{jk}|^2 |x_k|^2 \right) \end{aligned}$$

and $\sum_j \sum_k |b_{jk}|^2 |x_k|^2 = \sum_k c_k |x_k|^2$, where $c_k = \sum_j |b_{jk}|^2$. Using now Sawyer's formula for $p = 1$ we have

$$\sup_{|x_k| \downarrow} \frac{\sum_k c_k |x_k|^2}{\sum_k |x_k|^2} = \sup_N \frac{\sum_{k=1}^N c_k}{N} = \sup_N \frac{\sum_{k=1}^N \sum_j |b_{jk}|^2}{N}$$

which implies

$$\|A * B\|_{B_w(l^2)} \leq \|A\|_{2,\infty} \sup_N \left(\frac{\sum_{k=1}^N \sum_j |b_{jk}|^2}{N} \right)^{\frac{1}{2}}.$$

Using Proposition 3 the matrix

$$C = \begin{pmatrix} c_0^{\frac{1}{2}} & 0 & 0 & 0 & \dots \\ 0 & c_1^{\frac{1}{2}} & 0 & 0 & \dots \\ 0 & 0 & c_2^{\frac{1}{2}} & 0 & \dots \\ 0 & 0 & 0 & c_3^{\frac{1}{2}} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \in B_w(l^2) \text{ if and only if } \sup_{\substack{|x_k| \searrow \\ \|x\|_2 \leq 1}} \frac{\sum_k \left(\sum_j |b_{jk}|^2 \right) |x_k|^2}{\left(\sum_k |x_k|^2 \right)^{\frac{1}{2}}} < \infty.$$

$$\text{But } \sup_{\substack{|x_k| \searrow \\ \|x\|_2 \leq 1}} \frac{\sum_k \left(\sum_j |b_{jk}|^2 \right) |x_k|^2}{\left(\sum_k |x_k|^2 \right)^{\frac{1}{2}}} \leq \sup_{\substack{|x_k| \searrow \\ \|x\|_2 \leq 1}} \frac{\sum_j \left(\sum_k |b_{jk}| |x_k| \right)^2}{\left(\sum_k |x_k|^2 \right)^{\frac{1}{2}}} = \sup_{\substack{|x_k| \searrow \\ \|x\|_2 \leq 1}} \frac{\sum_j \left| \sum_{k=0}^{\infty} b_{jk} x_k \right|^2}{\left(\sum_k |x_k|^2 \right)^{\frac{1}{2}}} = \|B\|_{B_w(l^2)} < \infty \text{ which completes the proof. } \square$$

4. Toeplitz matrices

We give now a necessary condition in order that an upper triangular Toeplitz matrix belong to $B(l^2)$:

Proposition 8. Let $A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & \dots \\ 0 & a_0 & a_1 & a_2 & a_3 & \dots \\ 0 & 0 & a_0 & a_1 & a_2 & \dots \\ 0 & 0 & 0 & a_0 & a_1 & \dots \\ 0 & 0 & 0 & 0 & a_0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$. If $A \in B(l^2)$

then

$$\tilde{A} \in B_w(l^2), \text{ where } \tilde{A} \text{ is the diagonal matrix } \begin{pmatrix} \tilde{a}_0 & 0 & 0 & \dots & \dots \\ 0 & \tilde{a}_1 & 0 & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \dots \\ 0 & 0 & \ddots & \tilde{a}_m & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

given by $(\tilde{a}_m)_{m=0}^{\infty}$ and $\tilde{a}_m = \sum_{j=0}^m a_j$.

Proof. Let $A \in B(l^2)$. Then

$$\sum_{k=0}^{\infty} \left| \sum_{j=0}^{\infty} a_j x_{j+k} \right|^2 < \infty$$

for every $(x_j)_j \in l^2$.

But $\|e_1 + e_2 + \dots + e_N\|_2^2 = N$, where $e_k = (0, 0, \dots, 0, 1, 0, \dots)$. Thus

$$A \left(\frac{1}{\sqrt{N}}(e_1 + e_2 + \dots + e_N) \right) = \frac{1}{\sqrt{N}}(\tilde{a}_N, \tilde{a}_{N-1}, \dots, \tilde{a}_1, \tilde{a}_0, 0, \dots)$$

which implies that $\sup_N \frac{\sum_{m=0}^N |\tilde{a}_m|^2}{N} < \infty$ and, by Proposition 3 it follows that $\tilde{A} \in B_w(l^2)$. $\square$

We remark here that if we translate this result to the case of functions we shall get a necessary condition for a function to belong to H^∞ i.e. if $f \in H^\infty$ then $\sup_N \frac{\sum_{k=0}^N \left| \sum_{m=0}^k \hat{f}(m) \right|^2}{N} < \infty$, where $\hat{f}(k)$ is the Fourier coefficient of k -order.

Theorem 9. $B_w(l^2) \cap \mathcal{T} = B(l^2) \cap \mathcal{T}$, where $\mathcal{T}$ is the set of all Toeplitz matrices.

Proof. Let A be a Toeplitz matrix. Clearly, if $A \in B(l^2)$ it follows that $A \in B_w(l^2)$. It is well known that a Toeplitz matrix $A = (a_{ij})$, where $a_{ij} = a_{i-j}$ for all $i, j \in \mathbb{N}^*$, maps l^2 into l^2 precisely when there exists a measurable function essentially bounded on $[0, 2\pi]$ with Fourier coefficients $\hat{f}(n) = a_n$ ($n = 0, \pm 1, \pm 2, \dots$) and $\|A\|_{B(l^2)} = \|f\|_\infty$ see e.g. [11].

Let $f(t) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi i k t}$, $(x_n)_{n=0}^\infty \in l^2$ with $|x_n| \searrow 0$ and $h(t) = \sum_{k=0}^{\infty} x_k e^{2\pi i k t}$. Then

$$\begin{aligned} \|Ax\|_2^2 &= \left(\sum_{k=0}^{\infty} \left| \sum_{j=-k}^{\infty} a_j x_{k+j} \right|^2 \right)^{\frac{1}{2}} = \left(\sum_{k=0}^{\infty} \left| \int_0^1 f(t) e^{2\pi i k t} h(-t) dt \right|^2 \right)^{\frac{1}{2}} \\ &\simeq \sup_{\|g\|_2 \leq 1} \left| \sum_{k=0}^{\infty} g_k \int_0^1 f(t) e^{2\pi i k t} h(-t) dt \right| = \sup_{\|g\|_2 \leq 1} \left| \int_0^1 f(t) g(t) h(-t) dt \right|, \end{aligned}$$

where $g(t) = \sum_{k=0}^{\infty} g_k e^{2\pi i k t}$. Hence

$$\|A\|_{B_w(l^2)} = \sup \left\{ \left| \int_0^1 f(t) g(t) h(-t) dt \right| : \|g\|_2 \leq 1, \|h\|_2 \leq 1, h \in L^2([0, 1]) \right\}.$$

If we take $g(t) = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} e^{2\pi i j(t-t_0)}$ and $h(-t) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{2\pi i k(t-t_0)}$ for $n = 2p + 1$, $p \in \mathbb{N}^*$ and K_n -the Fejer's kernel we obtain

$$\|A\|_{B_w(l^2)} \geq \left| \int_0^1 f(t) \frac{1}{2p+1} \left| \sum_{j=-p}^p e^{2\pi i j(t-t_0)} \right|^2 dt \right| = \left| \int_0^1 f(t) K_{2p+1}(t-t_0) dt \right|$$

which implies that $\|A\|_{B_w(l^2)} \geq \|f\|_\infty = \|A\|_{B(l^2)}$. $\square$

Theorem 10. Let $A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \dots \\ 0 & a_0 & a_1 & a_2 & \dots \\ 0 & 0 & a_0 & a_1 & \dots \\ 0 & 0 & 0 & a_0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$ be an upper triangular

Toeplitz matrix. Then $A \in B(l^2)$ if and only if the sublinear operator T_A is bounded from l^2 into l^2 where

$$T_A(b)(j) = \frac{1}{j} \sum_{m=0}^j |(a * b)(m)|, \quad T_A(b)(0) = |a_0 b_0|$$

and $(a * b)(m) = \sum_{k+l=m} a_k b_l$, $a = (a_k)_{k \geq 0}$, $b = (b_k)_{k \geq 0} \in \ell^2$.

Proof. From Theorem 9 $A \in B(\ell^2)$ if and only if $A \in B_w(l^2)$ if and only if $\left(\sum_{j=0}^{\infty} a_j x_{j+k} \right)_{k \geq 0} \in l^2$ for any $x \in l^2$ with $|x_j| \searrow 0$. This is equivalent with

$$\sup_{\|b\|_{\ell^2} \leq 1} \left| \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} a_j x_{j+k} \right) b_k \right| < \infty \quad \forall x \in \ell^2, |x_j| \searrow 0.$$

$\sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} a_j x_{j+k} \right) b_k = \sum_{l=0}^{\infty} x_l c_l$, where $c_l = \sum_{k=0}^l a_{l-k} b_k$. From lemma 6

$$\sup_{|x_l| \searrow 0} \frac{\left| \sum_{l=0}^{\infty} x_l c_l \right|}{\left(\sum_l |x_l|^2 \right)^{\frac{1}{2}}} = \sup_{|x_l| \searrow 0} \frac{\sum_{l=0}^{\infty} |x_l| |c_l|}{\left(\sum_l |x_l|^2 \right)^{\frac{1}{2}}} \approx$$

$$\approx |a_0 b_0|^2 + \sum_{j=1}^{\infty} \left(\frac{1}{j} \sum_{m=0}^j \left| \sum_{k=0}^m a_{m-k} b_k \right| \right)^2 = |a_0 b_0|^2 + \sum_{j=1}^{\infty} \left(\frac{1}{j} \sum_{m=0}^j |(a * b)(m)| \right)^2$$

Thus the proof is complete. $\square$

Now we show that $M(B_w(l^2), B_w(l^2)) \cap \mathcal{T} \subseteq M(\ell^2) \cap \mathcal{T}$.

Theorem 11. *If M is a Toeplitz matrix with $M = (m_{jk}), m_{jk} = c_{j-k}$ ($j, k = 0, 1, 2, 3, \dots$) and $M \in M(B_w(l^2), B_w(l^2))$ there exists μ a bounded, complex, Borel measure on $\mathbb{T}$ with $\hat{\mu}(n) = c_n$ for $n = 0, \pm 1, \pm 2, \dots$. Moreover*

$$\|\mu\| \leq \|M\|_{M(B_w(l^2), B_w(l^2))}.$$

Proof. We follow the standard method [2] for solving the "problem of moments". Let $p(t) = \sum_{n=-\infty}^{\infty} p_n e^{int}$ be an arbitrary trigonometric polynomial and denote by P the Toeplitz matrix generated by p . Then

$$\left| \sum c_n p_n \right| = \lim_{N \rightarrow \infty} \left| \sum c_n p_n \left(1 - \frac{|n|}{N} \right) \right| = \lim_{N \rightarrow \infty} \left| \left\langle (M * P) x^{(N)}, x^{(N)} \right\rangle \right|$$

where

$$x^{(N)}(j) = \begin{cases} \frac{1}{\sqrt{N+1}} & \text{for } 0 \leq j \leq N \\ 0 & \text{for } j > N \end{cases}$$

We consider now the linear functional $p \xrightarrow{\Lambda} \sum_n c_n p_n$ on the subspace of $C(\mathbb{T})$ generated by polynomials

$$\begin{aligned} |\Lambda(p)| &= \left| \sum_n c_n p_n \right| = \lim_{N \rightarrow \infty} \left| \left\langle (M * P) x^{(N)}, x^{(N)} \right\rangle \right| \leq \left\| (M * P) x^{(N)} \right\|_{l^2} \leq \\ &\leq \|(M * P)\|_{B_w(l^2)} \leq \|M\|_{M(B_w(l^2), B_w(l^2))} \cdot \|P\|_{B_w(l^2)} = (\text{by Theorem 9}) = \\ &\quad \|M\|_{M(B_w(l^2), B_w(l^2))} \cdot \|p\|_{\infty} \end{aligned}$$

which implies that

$$\|\Lambda\| \leq \|M\|_{M(B_w(l^2), B_w(l^2))}.$$

It is clear now that this map is well-defined and continuous and the existence of a measure satisfying all the requirements of the theorem follows easily from the Hahn-Banach and Riesz representation theorems. $\square$

Using an extension of the F. and M. Riesz theorem from [10] we have:

Corollary 12. *If $M \in M(B_w(l^2), B_w(l^2))$ is a Toeplitz matrix $M = (m_{jk}), m_{jk} = c_{j-k}$ ($k, j = 0, 1, 2, \dots$), and for $n < 0$, $c_n = 0$ with $n \notin E$, where E is a set of type $\Lambda(1)$ (see [10]) then $c_n \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. This follows immediately from Riemann-Lebesgue lemma. $\square$

Acknowledgements. Both authors thank to professor Nicolae Popa for comments, useful informations and for his help to study this space. We are also grateful to Institut de Matemàtica Barcelona, Spain for providing a stimulating atmosphere during our visit and professor Maria J. Carro for hospitality and helpful ideas to approach some proofs. Financial support from the Romanian Ministry of Education and Reserch (CEEX Program, Project CEEX 05-D11-23/2005) is gratefully acknowledged.

References

- [1] J. Arazy, *Some remarks on interpolation theorems and the boundedness of the triangular projection in unitary matrix spaces*, Integral Equations Operator Theory, 1 (1978), 453–495.
- [2] S. Banach, *Théorie des Opérations Linéaires*, second edition, Chelsea, New York, 1963.
- [3] S. Barza, L.E. Persson and N. Popa, *A matriceal analogue of Fejer's theory*, Math. Nachr., 260 (2003), 14–20.
- [4] G. Bennett, *Schur multipliers*, Duke Math.J., 44 (1977), 603–639.
- [5] G. Bennett, *Factorizing the Classical Inequalities*, Memoirs of Amer. Math. Soc., 576, 1996.
- [6] M.J. Carro and J. Soria, *Weighted Lorentz spaces and the Hardy operator*, J. Funct. Anal. 112 (1993), 480–494.
- [7] M.J. Carro, J.A. Raposo, J. Soria, *Recent Developments in the Theory of Lorentz Spaces and Weighted Inequalities*, Memoirs of Amer. Math. Soc., to appear.
- [8] E. Sawyer, *Boundedness of classical operators on classical Lorentz spaces*, Studia Math. 96 (1990), 145–158.
- [9] A.L. Shields, *An analogue of a Hardy-Littlewood-Fejer inequality for upper triangular trace class operators*, Math.Z., 182 (1983), 473–484.
- [10] W. Rudin, *Trigonometric series with gaps*, J. Math. Mech., 9(2) (1960), 203–227.
- [11] K. Zhu, *Operator Theory on Banach Function Spaces*, Marcel Dekker, New York, 1990.

Department of Mathematics and Informatics
Technical University of Civil Engineering Bucharest
124 Lacul Tei Boulevard
Bucharest 020396
Romania
(*E-mail* : *anca_marcoci@yahoo.com*)
(*E-mail* : *liviu_marcoci@yahoo.com*)

(*Received* : *June 2006*)