Initial value problems for first order impulsive integro-differential equations of Volterra type in Banach spaces

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Abstract. In this paper, we use a new method and combining the partial ordering method to study the existence of the solutions for the first order nonlinear impulsive integro-differential equations of Volterra type on finite interval in Banach spaces and for the first order nonlinear impulsive integro-differential equations of Volterra type on infinite interval with infinite number impulsive times in Banach spaces. By introducing an interim space and using progressive estimation method, some restrictive conditions on impulsive terms, used before, such as, prior estimation, noncompactness measure estimations are deleted.

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1. Introduction

Impulsive differential equations arise naturally from a wide variety of applications, such as spacecraft control, inspection processes in operations research, drug administration, and threshold theory in biology, see [1], [8] and [10]. Initial value problems for first order impulsive integro-differential equations of Volterra type in Banach Spaces has been studied in [5], [6] and [9]. However, to obtain the solutions, some restricted conditions on impulsive terms, such as, prior estimation, noncompactness measure estimation, are needed in [5], [6] and [9]. In this paper, we use a new method and combining the partial ordering method to study the existence of the solutions for the first order nonlinear impulsive integro-differential equations of Volterra type on infinite interval with infinite number impulsive times in Banach spaces. By introducing an interim space and using progressive estimation method, the restrictive conditions on impulsive terms, such as, prior estimation, noncompactness measure estimations are deleted. Our results and methods are different from the ones in [5], [6] and [9].

Let E be a Banach space, $P \subset E$ be a closed convex cone in E . The partial order $\leq$ in E is induced by P , that is $x \leq y$, if and only if $y - x \in P$. P is said to be normal if there exists a positive constant K , such that $\theta \leq x \leq y$ implies $\|x\| \leq K \|y\|$, where θ denotes the zero element of E , and the smallest K is called the normal constant of P . P is said to be regular if $x_1 \leq x_2 \leq \cdots \leq y$ implies $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$ for some $x \in E$. For details on cone theory, see [6].

In this paper, we consider the following initial value problems for the first order nonlinear impulsive integro-differential equations of Volterra type on infinite interval with infinite number impulsive times in Banach spaces.

$$(1) \quad \begin{cases} u' = f(t, u, Tu), t \in J, t \neq t_i, \\ \Delta u|_{t=t_i} = I_i(u(t_i^-)), (i = 1, 2, \dots), \\ u(0) = u_0, \end{cases}$$

where $J = [0, \infty)$, $u_0 \in E$, $f \in C[J \times E \times E, E]$,

$$(Tu)(t) = \int_0^t k(t, s)u(s)ds, t \in J,$$

$k \in C[D, R_+]$ and $D = \{(t, s) \in J \times J : t \geq s\}$. $I_i \in C[E, E]$ ($i = 1, 2, 3, \dots$), $0 < t_1 < \cdots < t_i < \cdots$, $t_i \rightarrow \infty$, $\Delta u|_{t=t_i}$ denotes the jump of $x(t)$ at $t = t_i$, that is,

$$\Delta u|_{t=t_i} = u(t_i^+) - u(t_i^-),$$

where $u(t_k^+)$ and $u(t_k^-)$ denote the right and left limits of $u(t)$ at $t = t_k$, respectively.

Let

$$J_0 = [0, t_1], \quad J_m = (t_m, t_{m+1}], m = 1, 2, 3, \dots, \\ \overline{J}_m = [t_m, t_{m+1}], m = 1, 2, 3, \dots.$$

Let

$$k_i = \max\{k(t, s) : (t, s) \in \overline{J}_i \times \overline{J}_i\}, i = 0, 1, 2, \dots.$$

Let $J' = J \setminus \{t_1, \dots, t_m, \dots\}$. Let $PC[J, E] = \{u : u \text{ is a map of from } J \rightarrow E \text{ such that } u(t) \text{ is continuous at } t \neq t_i, \text{ left continuous at } t = t_i \text{ and } u(t_i^+) \text{ exists, } i = 1, 2, 3, \dots\}$, $\Delta t_i = t_{i+1} - t_i$. Then $u \in PC[J, E] \cap C^1[J', E]$ is called a *solution* of the initial value problem (1) if it satisfies (1).

For fixed $u_i \in C[\overline{J}_i, E], i = 0, 1, 2, \dots, k$, we introduce the interim spaces as follows:

$$(2) \quad PC[J_{0,k}, u_0, \dots, u_k, E] \\ = \{u(t) \in PC[J_{0,k}, E] : u(t)|_{J_i} = u_i(t), i = 0, 1, \dots, k-1\},$$

where $J_{0,k} = \bigcup_{i=0}^k J_i, k \geq 1$. Let $J'_{0,k} = J_{0,k} \setminus \{t_1, \dots, t_k\}, k \geq 1$, the meaning of $PC[J_{0,k}, E]$ is same as $PC[J, E]$.

2. Main Results

In this section, we discuss the existences of the solutions for the initial value problem (1) and need the following comparison result.

Lemma 1. Let $p \in PC[J_{0,i}, E] \cap C^1[J'_{0,i}, E]$ (where $J_{0,0} = J_0$) be such that

$$(3) \quad \begin{cases} p' \leq -Mp - NT_i p, & t \in J_i, \\ p(t_i^+) \leq \theta, \\ p(t) = \theta, & \text{if } i \geq 1, t \in J_{0,i-1}, \end{cases}$$

where $M \geq 0, N \geq 0$ are constants, and

$$(4) \quad T_i p(t) = \int_{t_i}^t k(t, s) p(s) ds, t \in J_i.$$

If

$$(5) \quad \begin{cases} Nk_i \Delta t_i (e^{M \Delta t_i} - 1) \leq M, & \text{if } M > 0, \\ Nk_i (\Delta t_i)^2 \leq 1, & \text{if } M = 0. \end{cases}$$

Then, $p(t) \leq \theta, t \in J_i$.

The proof of Lemma 1 is similar to the proof of Lemma 4.1.1 in [6] or refer to [3].

We give the following conditions for the next results:

(H₁) There exist $v_0, w_0 \in PC[J, E] \cap C^1[J', E]$ such that $v_0(t) \leq w_0(t)$ ($t \in J$) and

$$\begin{aligned} v'_0 &\leq f(t, v_0, Tv_0), \quad t \in J, \quad t \neq t_i, \\ \Delta v_0|_{t=t_i} &\leq I_i(v_0(t_i)) \quad (i = 1, 2, 3, \dots), \quad v_0(0) \leq u_0, \end{aligned}$$

$$\begin{aligned} w'_0 &\geq f(t, w_0, Tw_0), \quad t \in J, \quad t \neq t_i, \\ \Delta w_0|_{t=t_i} &\geq I_i(w_0(t_i)) \quad (i = 1, 2, 3, \dots), \quad w_0(0) \geq u_0. \end{aligned}$$

(H₂) There exist constants $M \geq 0$ and $N \geq 0$ such that

$$f(t, u, v) - f(t, \bar{u}, \bar{v}) \geq -M(u - \bar{u}) - N(v - \bar{v}), \quad t \in J,$$

whenever $v_0(t) \leq \bar{u} \leq u \leq w_0(t)$, $(Tv_0)(t) \leq \bar{v} \leq v \leq (Tw_0)(t)$.

(H₃) For any $i = 1, 2, 3, \dots$

$$I_i(u) - I_i(\bar{u}) \geq -(u - \bar{u}),$$

whenever $v_0(t_i) \leq \bar{u} \leq u \leq w_0(t_i)$.

(H₄) For any $r > 0$, and any $i = 1, 2, 3, \dots$, there exist constants $c_{r,i} \geq 0$ and $c_{r,i}^* \geq 0$ such that

$$\alpha(f(\bar{J}_i, B, B^*)) \leq c_{r,i}\alpha(B) + c_{r,i}^*\alpha(B^*), \quad B, B^* \subset B_r,$$

where $B_r = \{u \in E : \|u\| \leq r\}$ and α denotes the Kuratowski measure of noncompactness in E .

Let $[v_0, w_0] = \{u \in PC[J, E] : u_0(t) \leq u(t) \leq w_0(t), \quad t \in J\}$.

Theorem 2. *Let P be normal and the conditions $(H_1) \sim (H_4)$ be satisfied. Assume that the inequality (5) holds. Then the initial value problem (1) has at least a solution $u \in PC[J, E] \cap C^1[J', E]$ in $[v_0, w_0]$.*

Proof. Let $[v_0, w_0]_0 = \{u \in C[J_0, E] : u_0(t) \leq u(t) \leq w_0(t), \quad t \in J_0\}$. It follows from conditions $(H_1), (H_2), (H_4)$ and by using Theorem 4.1.1 in [6] or Theorem 3.1.1 in [5] that initial value problem

$$\begin{aligned} u' &= f(t, u, Tu), \quad t \in J_0, \\ u(0) &= u_0, \end{aligned}$$

has a minimal solution and a maximal solution $\bar{u}_0$, $u_0^* \in [v_0, w_0]_0$, respectively. By (2)

$$PC[J_{0,1}, \bar{u}_0, E] = \{u(t) \in PC[J_{0,1}, E] : u(t)|_{J_0} = \bar{u}_0(t)\}.$$

Let $[v_0, w_0]_1 = \{u \in PC[J_{0,1}, \bar{u}_0, E] : v_0(t) \leq u(t) \leq w_0(t), \quad t \in J_1\}$.

For any $\eta \in [v_0, w_0]_1$, we consider the initial value problem of linear integro-differential equation in E :

$$(6) \quad u' + Mu = -NTu + g(t), \quad t \in J_1, u(t_1^+) = \bar{u}_0(t_1^-) + I_1(\bar{u}_0(t_1^-)),$$

where

$$(7) \quad g(t) = f(t, \eta(t), (T\eta)(t)) + M\eta(t) + N(T\eta)(t).$$

It is easy to show that $u \in PC[J_{0,1}, \bar{u}_0, E]$ is a solution of the linear initial value problem (6) if and only if $u \in PC[J_{0,1}, \bar{u}_0, E]$ is a solution of the following integral equation:

$$(8) \quad u(t) = e^{-M(t-t_1)}\{u(t_1^+)\} + \int_{t_1}^t [g(s) - N(Tu)(s)]e^{M(s-t_1)}ds, \quad t \in J_1.$$

Since

$$\begin{aligned} \int_{t_1}^t N(Tu)(s)e^{M(s-t)}ds &= \int_{t_1}^t \int_0^{t_1} Ne^{M(s-t)}k(s,r)\bar{u}_0(r)drds \\ &\quad + \int_{t_1}^t N\left(\int_s^t e^{M(r-t)}k(r,s)dr\right)u(s)ds. \end{aligned}$$

(8) can be written as follows:

$$(9) \quad u(t) = F(t) - \int_{t_1}^t K_1(t,s)u(s)ds, \quad t \in J_1,$$

where

$$\begin{aligned} F(t) &= e^{-M(t-t_1)}\{u(t_1^+)\} + \int_{t_1}^t g(s)e^{M(s-t_1)}ds \\ &\quad + \int_{t_1}^t \int_0^{t_1} Ne^{M(s-t)}k(s,r)\bar{u}_0(r)drds, \\ K_1(t,s) &= Ne^{-Mt} \int_s^t e^{Mr}k(r,s)dr, \quad t_1 \leq s \leq t \leq t_2. \end{aligned}$$

Clearly, the integral equation (9) is a linear Volterra equation in E . Thus, by Theorem 1.4.2 in [5], it has a unique solution v in $C[\bar{J}_1, E]$. Letting

$$u(t) = \begin{cases} \bar{u}_0(t), & \text{if } t \in J_0, \\ v(t), & \text{if } t \in J_1. \end{cases}$$

Therefore, $u \in PC[J_{0,1}, \bar{u}_0, E]$ is the unique solution of the initial value problems (6). Letting $u = A\eta$, then the operator A maps $[v_0, w_0]_1$ into $PC[J_{0,1}, \bar{u}_0, E]$.

Now, we prove that

$$(i) \quad v_0 \leq Av_0 \text{ and } Aw_0 \leq w_0, t \in J_{0,1},$$

$$(ii) \quad A \text{ is increasing in } [v_0, w_0]_1.$$

If $t \in J_0$, for any $\eta \in [v_0, w_0]_1$, $A\eta(t) = \bar{u}_0(t)$, thus, (i) and (ii) hold. If $t \in J_1$, let $v_1 = Av_0$ and $p = v_0 - v_1$. Then, from (6), we have

$$\begin{aligned} v_1' + Mv_1 &= -NTv_1 + f(t, v_0, Tv_0) + Mv_0 + NTv_0, \quad t \in J_1, \\ v_1(t_1^+) &= \bar{u}_0(t_1^-) + I_1(\bar{u}_0(t_1^-)), \end{aligned}$$

and so, from the condition $(H_1), (H_3)$, it follows that

$$\begin{aligned} p' &\leq -Mp - NTP, \quad t \in J_1, \\ p(t_1^+) &= v_0(t_1^+) - v_1(t_1^+) \\ &\leq v_0(t_1^-) + I_1(v_0(t_1^-)) - \bar{u}_0(t_1^-) - I_1(\bar{u}_0(t_1^-)) \\ &\leq \bar{u}_0(t_1^-) + I_1(\bar{u}_0(t_1^-)) - \bar{u}_0(t_1^-) - I_1(\bar{u}_0(t_1^-)) \\ &= \theta. \end{aligned}$$

By the definition of $PC[J_{0,1}, \bar{u}_0, E]$, we have

$$\begin{aligned} p' &\leq -Mp - NTP \\ &= -Mp - N \int_0^t k(t, s)p(s)ds \\ &= -Mp - N \int_0^{t_1} k(t, s)p(s)ds - N \int_{t_1}^t k(t, s)p(s)ds \\ &= -Mp - N \int_{t_1}^t k(t, s)p(s)ds \\ &= -Mp - NT_1p, t \in J_1, \\ p(t) &= \theta, t \in J_0. \end{aligned}$$

Thus, by Lemma 1, it follows that $p(t) \leq \theta$ for all $t \in J_1$, that is, $v_0 \leq Av_0$, for $t \in J_1$. Similarly, we can prove that $Aw_0 \leq w_0$, $t \in J_1$. Let

$\eta_1, \eta_2 \in [v_0, w_0]_1$ with $\eta_1 \leq \eta_2$ and $p = u_1 - u_2$, where $u_1 = A\eta_1$ and $u_2 = A\eta_2$. From (6), it follows that

$$\begin{aligned} u'_i + Mu_i &= -NTu_i + f(t, \eta_i, T\eta_i) + M\eta_i + NT\eta_i, \quad t \in J_1, \\ u_i(t_1^+) &= \bar{u}_0(t_1^-) + I_1(\bar{u}_0(t_1^-)), i = 1, 2, \end{aligned}$$

and so, by using the condition (H_2) , we can show that

$$\begin{aligned} p' &\leq -Mp - NTP, \quad t \in J_1, \\ p(t_1^+) &= u_1(t_1^+) - u_2(t_1^+) \\ &= u_1(t_1^-) + I_1(u_1(t_1^-)) - u_2(t_1^-) - I_1(u_2(t_1^-)) \\ &= \bar{u}_0(t_1^-) + I_1(\bar{u}_0(t_1^-)) - \bar{u}_0(t_1^-) - I_1(\bar{u}_0(t_1^-)) \\ &= \theta. \end{aligned}$$

By the definition of $PC[J_{0,1}, \bar{u}_0, E]$, we have

$$\begin{aligned} p' &\leq -Mp - NTP \\ &= -Mp - N \int_0^t k(t, s)p(s)ds \\ &= -Mp - N \int_0^{t_1} k(t, s)p(s)ds - N \int_{t_1}^t k(t, s)p(s)ds \\ &= -Mp - N \int_{t_1}^t k(t, s)p(s)ds \\ &= -Mp - NT_1p, t \in J_1, \\ p(t) &= \theta, t \in J_0. \end{aligned}$$

Therefore, by Lemma 1, it follows that $p(t) \leq \theta$ for all $t \in J_1$, that is, $A\eta_1 \leq A\eta_2$, for all $t \in J_1$ and so (ii) is proved.

Let

$$v_n^1 = Av_{n-1}^1, \quad w_n^1 = Aw_{n-1}^1 \quad (n = 1, 2, 3, \dots),$$

where $v_0^1 = v_0, w_0^1 = w_0$. By the conclusions (i) and (ii), we have

$$(10) \quad v_0(t) \leq v_1^1(t) \leq \dots \leq v_n^1(t) \leq \dots \leq w_n^1(t) \leq \dots \leq w_1^1(t) \leq w_0(t), \quad t \in J_{0,1}.$$

Then, by the normality of P , the set $V = \{v_n^1 : n = 1, 2, \dots\}$ is a bounded set in $C[\bar{J}_1, E]$, here, for $v_n^1 \in C[\bar{J}_1, E]$, we understand that $v_n^1(t_1) = v_n^1(t_1^+)$. Since the condition (H_4) implies that the set $f(\bar{J}_1, B_r, B_r)$ is bounded for any $r > 0$, So, there exists a constant $c_0 > 0$ such that

$$\begin{aligned} (11) \quad &\|f(t, v_{n-1}(t), (Tv_{n-1})(t)) + Mv_{n-1}(t) - N(T(v_n - v_{n-1}))(t)\| \leq c_0, \\ &t \in \bar{J}_1 \quad (n = 1, 2, 3, \dots). \end{aligned}$$

By the definition of v_n^1 and (8), we have

$$(12) \quad v_n^1(t) = e^{-M(t-t_1)}(u(t_1^+) + \int_{t_1}^t [f(s, v_{n-1}^1(s), (Tv_{n-1}^1)(s)) + Mv_{n-1}^1(s) - N(T(v_n^1 - v_{n-1}^1))(s)]e^{M(s-t_1)}ds), \quad t \in J_1 \quad (n = 1, 2, 3, \dots).$$

Then, from (11) and (12), all the functions in V is equicontinuous on $\overline{J}_1$ and so by Theorem 1.2.2 in [6], the function $m(t) = \alpha(V(t))$ is continuous on $\overline{J}_1$, where $V(t) = \{v_n^1(t) : n = 1, 2, \dots\}$ for a fixed $t \in \overline{J}_1$. Applying Corollary 1.2.1 in [6] to (12), we have

$$(13) \quad \begin{aligned} m(t) &\leq 2 \int_{t_1}^t \alpha(\{e^{-M(t-s)}[f(s, v_{n-1}^1(s), (Tv_{n-1}^1)(s)) \\ &\quad + Mv_{n-1}^1(s) - N(T(v_n^1 - v_{n-1}^1))(s)] : n = 1, 2, \dots\})ds \\ &\leq 2 \int_{t_1}^t [\alpha(f(s, V(s), (TV)(s)) + M\alpha(V(s)) + 2N\alpha(TV)(s))]ds, \end{aligned}$$

where $(TV)(t) = \{(Tv_n^1)(t) : n = 1, 2, \dots\}$. On the other hand, by Corollary 1.2.1 in [6], we have

$$(14) \quad \begin{aligned} \alpha((TV)(t)) &\leq 2 \int_0^t \alpha(\{k(t, s)v_n^1(s) : n = 0, 1, 2, \dots\})ds \\ &\leq k_1 2 \int_{t_1}^t m(s)ds, \quad t \in J_1. \end{aligned}$$

Thus, from the condition (H_4) , there exists constant $c \geq 0$ and $c^* \geq 0$ such that

$$(15) \quad \alpha(f(t, V(t), (TV)(t))) \leq c\alpha(V(t)) + c^*\alpha((TV)(t)), \quad t \in \overline{J}_1.$$

Therefore, from (13)~(15), it follows that

$$(16) \quad \begin{aligned} m(t) &\leq 2(c + M) \int_{t_1}^t m(s)ds + 4k_1(c^* + 2N) \int_{t_1}^t ds \int_{t_1}^s m(s')ds' \\ &= 2(c + M) \int_{t_1}^t m(s)ds + 4k_1(c^* + 2N) \int_{t_1}^t (t - s)m(s)ds \\ &\leq \tau \int_{t_1}^t m(s)ds, \quad t \in \overline{J}_1, \end{aligned}$$

where $\tau = 2(c + M) + 4k_1\Delta t_1(c^* + 2N) = \text{const} \geq 0$. Let

$$y(t) = \int_{t_1}^t m(s)ds, \quad z(t) = y(t)e^{-\tau t}, \quad t \in \overline{J}_1.$$

Then, from (16), we have

$$z'(t) = [y'(t) - \tau y(t)]e^{-\tau t} = [m(t) - \tau \int_{t_1}^t m(s)ds]e^{-\tau t} \leq 0, \quad t \in \overline{J}_1$$

and so $z(t) \leq z(t_1) = y(t_1)e^{-\tau t_1} = 0$ for all $t \in \overline{J}_1$. Thus $z(t) \equiv 0$ for all $t \in \overline{J}_1$, which implies that $m(t) \equiv 0$ for all $t \in \overline{J}_1$ and, by Theorem 1.2.5 in [6], V is a relatively compact set in $C[\overline{J}_1, E]$. Thus, $\{v_n^1\}$ has a convergent subsequence in $C[\overline{J}_1, E]$, suppose that $\{v_{n_i}^1\}$ converges to $\bar{u}^1$ in $C[\overline{J}_1, E]$. Since $\{v_n^1\}$ is increasing, for any positive integer n , we have $v_{n_i}^1 \geq v_n^1$, whenever $n_i \geq n$. Taking $n_i \rightarrow \infty$, we get that $\bar{u}^1 \geq v_n^1$. Since P is normal, from inequality

$$\theta \leq \bar{u}^1 - v_n^1 \leq \bar{u}^1 - v_{n_i}^1, \forall n \geq n_i,$$

we have

$$\|\bar{u}^1 - v_n^1\| \leq K \|\bar{u}^1 - v_{n_i}^1\| \quad \forall n \geq n_i.$$

Since $\lim_{n \rightarrow \infty} \|\bar{u}^1 - v_{n_i}^1\| = 0$, we obtain $\lim_{n \rightarrow \infty} \|\bar{u}^1 - v_n^1\| = 0$. Thus, $\{v_n^1\}$ converges to $\bar{u}^1$ in $C[\overline{J}_1, E]$, that is, the sequence $\{v_n^1\}$ converges uniformly to $\bar{u}^1 \in C[\overline{J}_1, E]$ on $\overline{J}_1$. Since $v_n^1 \in [v_0, w_0]_1$, we have $v_n^1|_{J_0} = \bar{u}_0$, thus $\{v_n^1\}$ converges uniformly to $\bar{u}_0$ on J_0 . Let

$$\bar{v}_1(t) = \begin{cases} \bar{u}_0(t), & t \in J_0. \\ \bar{u}^1(t), & t \in J_1. \end{cases}$$

Then, $\{v_n^1\}$ converges uniformly to $\bar{v}_1$ on $J_{0,1}$.

Now, we have

$$(17) \quad \begin{aligned} & f(t, v_{n-1}^1, (Tv_{n-1}^1)(t)) + Mv_{n-1}^1(t) - N(T(v_n^1 - v_{n-1}^1))(t) \\ & \rightarrow f(t, \bar{v}_1(t), (T\bar{v}_1)(t)) + M\bar{v}_1(t) \quad (n \rightarrow \infty), \quad t \in \overline{J}_1. \end{aligned}$$

Further, from (11), it follows that

$$(18) \quad \begin{aligned} & \|f(t, v_{n-1}^1(t), (Tv_{n-1}^1)(t)) + Mv_{n-1}^1(t) - N(T(v_n^1 - v_{n-1}^1))(t) \\ & - f(t, \bar{v}_1(t), (T\bar{v}_1)(t)) - M\bar{v}_1(t)\| \\ & \leq 2c_0, \quad t \in \overline{J}_1 (n = 1, 2, 3, \dots) \end{aligned}$$

and so, from (17) and (18), by taking limits in both sides of (12), we get

$$(19) \quad \begin{aligned} \bar{v}_1(t) = e^{-M(t-t_1)} \{ & u(t_1^+) + \int_{t_1}^t [f(s, \bar{v}_1(s), (T\bar{v}_1)(s)) \\ & + M\bar{v}_1(s)] e^{M(s-t_1)} ds \}, \quad t \in \bar{J}_1. \end{aligned}$$

Hence we can easily show that $\bar{v}_1 \in PC[J_{0,1}, E] \cap C^1[J'_{0,1}, E]$ and $\bar{v}_1$ is a solution of the initial value problem

$$\begin{cases} u'(t) = f(t, u, Tu), & t \in J_{0,1}, t \neq t_1, \\ u(t_1^+) = \bar{u}_0(t_1^-) + I_1(\bar{u}_0(t_1^-)), \\ u(0) = u_0. \end{cases}$$

That is, $\bar{v}_1(t)$ is a solution of the initial value problem (1) on $J_{0,1}$.

Assume that we have get a solution $\bar{v}_{m-1}(t)$ of (1) on $J_{0,m-1}$ in $[v_0, w_0]_{m-1}$, such that $\bar{v}_{m-1}(t)$ satisfies

$$\bar{v}_{m-1}(t) = \begin{cases} \bar{u}_0(t), & t \in J_0, \\ \bar{u}^1(t), & t \in J_1, \\ \dots \\ \bar{u}^{m-1}(t), & t \in J_{m-1}, \end{cases}$$

and

$$\begin{cases} u'(t) = f(t, u, Tu), t \in J_{0,m-1}, & t \neq t_k, \\ u(t_k^+) = \bar{u}^{k-1}(t_k^-) + I_k(\bar{u}^{k-1}(t_k^-)), \\ u(0) = u_0, & k = 1, 2, \dots, m-1. \end{cases}$$

where $\bar{u}^k(t)$, is a uniformly limitation of the increasing sequence $\{v_n^k\} \subset [v_0, w_0]_k$ on $J_k, k = 1, 2, \dots, m-1$. Let

$$\begin{aligned} & [v_0, w_0]_m \\ & = \{u \in PC[J_{0,m}, \bar{u}_0, \bar{u}^1(t), \dots, \bar{u}^{m-1}(t), E] : v_0(t) \leq u(t) \leq w_0(t), t \in J_m\}. \end{aligned}$$

It is similar to the above proof of (10), we can get an increasing sequence

$$v_0(t) \leq v_1^m(t) \leq \dots \leq v_n^m(t) \leq \dots \leq w_n^m(t) \leq \dots \leq w_1^m(t) \leq w_0(t), t \in J_{0,m}.$$

in $[v_0, w_0]_m$ and $\{v_n^m(t)\}$ uniformly converges to $\bar{u}^m(t)$ on J_m . If we let

$$\bar{v}_m(t) = \begin{cases} \bar{u}_0(t), & t \in J_0, \\ \bar{u}^1(t), & t \in J_1, \\ \dots \\ \bar{u}^{m-1}(t), & t \in J_{m-1}, \\ \bar{u}^m(t), & t \in J_m. \end{cases}$$

Then, by the similar proof of (19), we have

$$\begin{aligned} \bar{v}_m(t) &= e^{-M(t-t_m)} \{ \bar{u}^{m-1}(t_m^-) + I(\bar{u}^{m-1}(t_m^-)) \\ &\quad + \int_{t_m}^t [f(s, \bar{v}_m(s), (T\bar{v}_m)(s)) + M\bar{v}_m(s)] e^{M(s-t_m)} ds \}, \quad t \in J_m. \end{aligned}$$

That is, $\bar{v}_m(t)$ is a solution of the following

$$\begin{cases} u'(t) = f(t, u, Tu), & t \in J_m, \\ u(t_m^+) = \bar{u}^{m-1}(t_m^-) + I_m(\bar{u}^{m-1}(t_m^-)), \end{cases}$$

Then $\bar{v}_m(t)$ is a solution of

$$\begin{cases} u'(t) = f(t, u, Tu), & t \in J_{0,m}, \quad t \neq t_k, \\ u(t_k^+) = \bar{u}^{k-1}(t_k^-) + I_k(\bar{u}^{k-1}(t_k^-)), \\ u(0) = u_0, & k = 1, 2, \dots, m, \end{cases}$$

thus, $\bar{v}_m(t)$ satisfies

$$\begin{cases} \bar{v}_m'(t) = f(t, \bar{v}_m, T\bar{v}_m), & t \in J_{0,m}, t \neq t_k, \\ \Delta \bar{v}_m(t_k^+) = I_k(\bar{v}_m(t_k^-)), \\ \bar{v}_m(0) = u_0, & k = 1, 2, \dots, m. \end{cases}$$

By induction, we can get two sequences $\{\bar{v}_m(t)\} \subset [v_0, w_0]_m$ and $\{\bar{u}^m(t)\}$ such that

$$\bar{v}_m(t) = \begin{cases} \bar{u}_0(t), & t \in J_0, \\ \bar{u}^1(t), & t \in J_1, \\ \dots \\ \bar{u}^{m-1}(t), & t \in J_{m-1}, \\ \bar{u}^m(t), & t \in J_m. \end{cases}$$

and $\bar{v}_m(t)$ is a solution of

$$(20) \quad \begin{cases} u'(t) = f(t, u, Tu), & t \in J_{0,m}, t \neq t_k, \\ u(t_k^+) = \bar{u}^{k-1}(t_k^-) + I_k(\bar{u}^{k-1}(t_k^-)), \\ u(0) = u_0, & k = 1, 2, \dots, m, \end{cases}$$

that is, $\bar{v}_m(t)$ satisfies

$$(21) \quad \begin{cases} \bar{v}_m'(t) = f(t, \bar{v}_m, T\bar{v}_m), & t \in J_{0,m}, t \neq t_k, \\ \Delta \bar{v}_m(t_k^+) = I_k(\bar{v}_m(t_k^-)), \\ \bar{v}_m(0) = u_0, & k = 1, 2, \dots, m, \end{cases}$$

Let

$$(22) \quad \bar{v}(t) = \bar{u}^m(t), t \in J_m, m = 0, 1, 2, \dots$$

From (20), (21) and (22) we can know that $\bar{v}(t)$ satisfies

$$\begin{cases} \bar{v}'(t) = f(t, \bar{v}, T\bar{v}), & t \in J, t \neq t_k, \\ \Delta \bar{v}(t_k^+) = I_k(\bar{v}(t_k^-)), \\ \bar{v}(0) = u_0, & k = 1, 2, \dots, \end{cases}$$

that is, $\bar{v}(t)$ is a solution of (1) and $\bar{v} \in PC[J, E] \cap C^1[J', E] \cap [v_0, w_0]$. The proof is completed. $\square$

Remark 1. From the proof of Theorem 2 we can see that if $\bar{u}_0$ is different from u_0^* on J_0 , then the initial value problem (1) has at least two solutions in $[v_0, w_0]$.

Remark 2. Condition (H_4) is an important, most in use and general condition in nonlinear problems in abstract spaces. Concerning the checking and using of this kind of conditions, one can refer to [2],[4],[6] and [7].

Theorem 3. Let P be regular and the conditions $(H_1) \sim (H_3)$ be satisfied. Assume that the inequality (5) holds. Then the initial value problem (1) has at least a solution $u \in PC[J, E] \cap C^1[J', E]$ in $[v_0, w_0]$.

Proof. The proof is similar to that of Theorem 2. The only difference of the proof is that the conclusion $m(t) = \alpha(V(t)) = 0$ for all $t \in \bar{J}_1$ follows directly from the regularity of P and the fact that $\{v_n^1(t)\}$ is an increasing sequence. This complete the proof. $\square$

From the proof of Theorem 2 we can get the following corollary.

Corollary 4. *If $J = [0, a]$ and $0 < t_1 < \dots < t_m < a$. Let P be normal and the conditions $(H_1) \sim (H_4)$ be satisfied for $i = 1, 2, \dots, m$. Assume that the inequality (5) holds. Then the initial value problem (1) has at least a solution $u \in PC[J, E] \cap C^1[J', E]$ in $[v_0, w_0]$.*

3. Example

We consider the following initial value problem of infinite system for the scalar first order impulsive integro-differential equation:

$$(23) \quad \begin{cases} u'_n = \frac{1}{3n}[(t - u_n)^3 + u_{n+1}^4] - \frac{1}{2} \int_0^t e^{-ts} u_n(s) ds \\ \quad + \frac{1}{\sqrt{n}} \left(\int_0^t e^{-ts} u_{2n}(s) ds \right)^2, & 0 \leq t \leq 1, \quad t \neq \frac{1}{2}, \\ \Delta u_n|_{t=\frac{1}{2}} = -u_n \left(\frac{1}{2} \right), & u_n(0) = 0 \quad (n = 1, 2, 3, \dots). \end{cases}$$

Evidently, $u_n(t) \equiv 0$ ($n = 1, 2, 3, \dots$) is not a solution of the initial value problem (23).

Conclusion. The initial value problem (23) has at least a continuously differentiable solutions defined on $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ with

$$0 \leq u_n(t) \leq \begin{cases} \frac{t}{n} + \frac{1}{12n}, & 0 \leq t \leq \frac{1}{2}, \\ \leq \frac{t}{n}, \frac{1}{2} < t \leq 1, & (n = 1, 2, 3, \dots) \end{cases}$$

In fact, let

$$a = 1, \quad E = c_0 = \{u = (u_1, u_2, \dots, u_n, \dots) : u_n \rightarrow 0\}$$

with the norm be defined by $\|u\| = \sup_n |u_n|$ and

$$P = \{u = (u_1, u_2, \dots, u_n, \dots) \in c_0 : u_n \geq 0, \quad n = 1, 2, 3, \dots\}.$$

Then P is a normal cone in E and (23) can be regarded as an initial value problem of form (1) in E , where

$$\begin{aligned} u_0 &= (0, \dots, 0, \dots), & J &= [0, 1], & k(t, s) &= e^{-ts}, \\ u &= (u_1, u_2, \dots, u_n, \dots), & v &= (v_1, v_2, \dots, v_n, \dots), \\ f &= (f_1, f_2, \dots, f_n, \dots) \end{aligned}$$

with

$$(24) \quad f_n(t, u, v) = \frac{1}{3n}[(t - u_n)^3 + u_{n+1}^4] - \frac{1}{2}v_n + \frac{1}{\sqrt{n}}v_{2n}^2,$$

$$(25) \quad m = 1, t_1 = \frac{1}{2}, \quad I_1(u) = -u.$$

Clearly, $f \in C[J \times E \times E, E]$ and $I_1 \in C[E, E]$. Let

$$v_0(t) = (0, \dots, 0, \dots), \quad t \in J,$$

$$w_0(t) = \begin{cases} \left(t + \frac{1}{12}, \dots, \frac{t}{n} + \frac{1}{12n}, \dots\right), & 0 \leq t \leq \frac{1}{2}, \\ \left(t, \dots, \frac{t}{n}, \dots\right), & \frac{1}{2} < t \leq 1. \end{cases}$$

Then we have

$$v_0, w_0 \in PC[J, E] \cap C^1[J', E] \quad \left(J' = J \setminus \left\{\frac{1}{2}\right\} = \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]\right)$$

and $v_0(t) \leq w_0(t)$ for all $t \in J$ and so

$$v_0(0) = w_0(0) = (0, \dots, 0, \dots),$$

$$\Delta v_0|_{t=\frac{1}{2}} = (0, \dots, 0, \dots) = -v_0\left(\frac{1}{2}\right),$$

$$\Delta w_0|_{t=\frac{1}{2}} = \left(-\frac{3}{12}, \dots, -\frac{3}{12n}, \dots\right) \geq -w_0\left(\frac{1}{2}\right),$$

$$v'_0(t) = (0, \dots, 0, \dots), \quad t \in J,$$

$$w'_0(t) = (1, \dots, \frac{1}{n}, \dots), \quad t \in J, \quad t \neq \frac{1}{2},$$

$$f_n(t, v_0(t), (Tv_0)(t)) = \frac{t^3}{3n} \geq 0, \quad t \in J \quad (n = 1, 2, 3, \dots).$$

In addition, if $0 \leq t \leq \frac{1}{2}$, then we have

$$\begin{aligned}
f_n(t, w_0(t), (Tw_0)(t)) &= \frac{1}{3n} \left[\left(t - \frac{t}{n} - \frac{1}{12n} \right)^3 + \left(\frac{t}{n+1} + \frac{1}{12n+12} \right)^4 \right] \\
&\quad - \frac{1}{2} \left(\int_0^{\frac{1}{2}} e^{-ts} \left(\frac{s}{n} + \frac{1}{12n} \right) ds \right) \\
&\quad + \frac{1}{\sqrt{n}} \left[\int_0^{\frac{1}{2}} e^{-ts} \left(\frac{s}{2n} + \frac{1}{24n} \right) ds \right]^2 \\
&\leq \frac{1}{3n} \left[\frac{1}{8} + \left(\frac{7}{12(n+1)} \right)^4 \right] \\
&\quad + \frac{1}{\sqrt{n}} \left[\int_0^t e^{-ts} \left(\frac{s}{2n} + \frac{1}{24n} \right) ds \right]^2 \\
&\leq \frac{1}{3n} \left[\frac{1}{8} + \left(\frac{7}{12(n+1)} \right)^4 \right] + \frac{1}{4n^2 \sqrt{n}} \frac{12}{77} \\
&\leq \frac{1}{3n} \cdot \frac{1}{2} + \frac{1}{4n^2} \cdot \frac{12}{77} < \frac{1}{n} \quad (n = 1, 2, 3, \dots).
\end{aligned}$$

If $\frac{1}{2} < t \leq 1$, we have

$$\begin{aligned}
f_n(t, w_0(t), (Tw_0)(t)) &= \frac{1}{3n} \left[\left(t - \frac{t}{n} \right)^3 + \left(\frac{t}{n+1} \right)^4 \right] \\
&\quad - \frac{1}{2n} \left[\int_0^{\frac{1}{2}} \left(s + \frac{1}{12} \right) e^{-ts} ds + \int_{\frac{1}{2}}^t e^{-ts} s ds \right] \\
&\quad + \frac{1}{\sqrt{n}} \left(\int_0^{\frac{1}{2}} e^{-ts} \left(\frac{s}{2n} + \frac{1}{24} \right) ds + \int_{\frac{1}{2}}^t e^{-ts} \frac{s}{n} ds \right) \\
&\leq \frac{1}{3n} - \frac{1}{6n} \left[\int_0^{\frac{1}{2}} \left(s + \frac{1}{12} \right) ds + \int_{\frac{1}{2}}^t s ds \right] \\
&\quad + \frac{1}{\sqrt{n}} \left(\int_0^{\frac{1}{2}} \left(\frac{s}{2n} + \frac{1}{24} \right) ds + \int_{\frac{1}{2}}^t \frac{s}{n} ds \right)^2 \\
&\leq \frac{1}{3n} - \frac{7}{48n} + \frac{1}{6n^2} < \frac{1}{n} \quad (n = 1, 2, 3, \dots).
\end{aligned}$$

Thus v_0 and w_0 satisfy the condition (H_1) . On the other hand, if $t \in J$. $v_0(t) \leq \bar{u} \leq u \leq w_0(t)$ and $(Tv_0)(t) \leq \bar{v} \leq v \leq (Tw_0)(t)$, then we have

$$\begin{aligned}
0 &\leq \bar{u}_n \leq u_n \\
&\leq \begin{cases} \frac{t}{n} + \frac{1}{12n}, & 0 \leq t \leq \frac{1}{2}, \\ \frac{t}{n}, & \frac{1}{2} < t \leq 1, \quad (n = 1, 2, 3, \dots), \end{cases} \\
0 &\leq \bar{v}_n \leq v_n \\
&\leq \begin{cases} \frac{1}{2n} \left(t^2 + \frac{t}{6} \right) \leq \frac{1}{2n} \left(t^2 + \frac{1}{12} \right), & 0 \leq t \leq \frac{1}{2}, \\ \frac{1}{2n} \left(t^2 + \frac{1}{12} \right), & \frac{1}{2} < t \leq 1, \quad (n = 1, 2, 3, \dots), \end{cases}
\end{aligned}$$

and so, from (24), it follows that

$$\begin{aligned}
f_n(t, u, v) - f_n(t, \bar{u}, \bar{v}) &= \frac{1}{3n} [(t - u_n)^3 + u_{n+1}^4] - \frac{1}{2} v_n \\
&\quad + \frac{1}{\sqrt{n}} v_{2n}^2 - \frac{1}{3n} [(t - \bar{u}_n)^3 + \bar{u}_{n+1}^4] + \frac{1}{2} \bar{v}_n - \frac{1}{\sqrt{n}} \bar{v}_{2n}^2 \\
&\geq \frac{1}{3n} [(t - u_n)^3 - (t - \bar{u}_n)^3] - \frac{1}{2} (v_n - \bar{v}_n) \geq -\frac{1}{n} (u_n - \bar{u}_n) - \frac{1}{2} (v_n - \bar{v}_n) \\
&\geq -(u_n - \bar{u}_n) - \frac{1}{2} (v_n - \bar{v}_n) \quad (n = 1, 2, 3, \dots),
\end{aligned}$$

where we have used the following:

$$\frac{\partial}{\partial s} (t - s)^3 = -3(t - s)^2 \geq -3, \quad 0 \leq s \leq t \leq 1.$$

Thus the condition (H_2) is satisfied, where $M = 1$ and $N = \frac{1}{2}$. It follows from (25) that (H_3) is satisfied, where $L_1 = 1$. Let $f = g + h$, where $g = (g_1, g_2, \dots, g_n, \dots)$, $h = (h_1, h_2, \dots, h_n, \dots)$ and

$$g_n(t, u, v) = \frac{1}{3n} [(t - u_n)^3 + u_{n+1}^4] + \frac{1}{\sqrt{n}} v_{2n}^2, \quad h_n(t, u, v) = -\frac{1}{2} v_n.$$

Hence, by imitating the proof of Example 4.1.1 in [6], we can prove that

$$\alpha(g(J, B, B^*)) = 0, \quad \alpha(h(J, B, B^*)) = \frac{1}{2} \alpha(B^*)$$

for any bounded sets $B, B^* \subset E$ and so

$$\alpha(f(J, B, B^*)) \leq \frac{1}{2}\alpha(B^*)$$

for any bounded set $B, B^* \subset E$. Thus the condition (H_4) is satisfied and, by Corollary 4, the conclusion holds.

Remark 3. Clearly, $\max\{k(t, s) : (t, s) \in D\} = 1, \max\{\Delta t_i : i = 0, 1\} = \frac{1}{2}$. Let $l_0 = \max\{k(t, s) : (t, s) \in D\}, \tau = \max\{\Delta t_i : i = 0, 1\}$ and $M_0 = M^{-1}(e^{Ma} - 1)$, then, we have

$$M_0 = M^{-1}(e^{Ma} - 1) = e - 1 > 0,$$

$$M_0 e^{-Ma} = M^{-1}(1 - e^{-Ma}) = 1 - e^{-1} > 0,$$

and so

$$Nl_0M_0\tau = \frac{1}{2}(e - 1) > 0 = \frac{1 - L_1}{1 + 1 - L_1},$$

$$Nl_0M_0ae^{-Ma} + L_1 = \frac{1}{2}(1 - e^{-1}) + 1 > 1.$$

Therefore, the conclusion of our example can not be obtained from the results of [5], [6] and [9]. This also shows that our results are different from the results in [5], [6] and [9].

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