A criterion of weak compactness for operators on subspaces of Orlicz spaces

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Abstract. We give a criterion of weak compactness for the operators on the Morse-Transue space M^Ψ , the subspace of the Orlicz space L^Ψ generated by L^∞ .

1. Introduction and Notation

In 1975, C. Niculescu established a characterization of weakly compact operators T from $\mathcal{C}(S)$, where S is a compact space, into a Banach space Z ([14, 15], see [3] Theorem 15.2 too): $T: \mathcal{C}(S) \rightarrow Z$ is weakly compact if and only if there exists a Borel probability measure μ on S such that, for every $\epsilon > 0$, there exists a constant $C(\epsilon) > 0$ such that:

$$\|Tf\| \leq C(\epsilon) \|f\|_{L^1(\mu)} + \epsilon \|f\|_\infty, \quad \forall f \in \mathcal{C}(S).$$

The same kind of result was proved by H. Jarchow for $\mathbb{C}^*$ -algebras in [7], and by the first author for $A(\mathbb{D})$ and H^∞ (see [11]). The criterion for H^∞

played a key role to give an elementary proof of the equivalence between weak compactness and compactness for composition operators on H^∞ .

Beside these spaces, one natural class of Banach spaces is the class of Orlicz spaces L^Ψ . Unfortunately, we shall see that the above criterion is in general not true for Orlicz spaces. However, it remains true when we restrict ourselves to subspaces of the Morse-Transue space M^Ψ . This space is the closure of L^∞ in the Orlicz space L^Ψ .

In this paper, we first give a characterization of the operators from a subspace of M^Ψ which fix no copy of c_0 . When the complementary function of Ψ satisfies Δ_2 , that gives a criterion of weak compactness. If moreover Ψ satisfies a growth condition, that we call Δ^0 , the criterion has a more usable formulation, analogous to those described above.

As in the case of H^∞ (but this is far less elementary), this new version obtained for subspaces of Morse-Transue spaces (Theorem 4), combined with a study of generalized Carleson measures, may be used to prove the equivalence between weak compactness and compactness for composition operators on Hardy-Orlicz spaces (see [13]), when Ψ satisfies Δ^0 .

However, we think that this characterization has an intrinsic interest for Orlicz spaces, and will be useful not only for composition operators (see Remark 5 at the end of the paper).

In this note, we shall consider Orlicz spaces defined on a probability space $(\Omega, \mathbb{P})$, that we shall assume non purely atomic.

By an Orlicz function, we shall understand that $\Psi: [0, \infty] \rightarrow [0, \infty]$ is a non-decreasing convex function such that $\Psi(0) = 0$ and $\Psi(\infty) = \infty$. To avoid pathologies, we shall assume that we work with an Orlicz function Ψ having the following additional properties: Ψ is continuous at 0, strictly convex (hence *strictly* increasing), and such that

$$\frac{\Psi(x)}{x} \xrightarrow{x \rightarrow \infty} \infty.$$

This is essentially to exclude the case of $\Psi(x) = ax$. The Orlicz space $L^\Psi(\Omega)$ is the space of all (equivalence classes of) measurable functions $f: \Omega \rightarrow \mathbb{C}$ for which there is a constant $C > 0$ such that

$$\int_{\Omega} \Psi\left(\frac{|f(t)|}{C}\right) d\mathbb{P}(t) < +\infty$$

and then $\|f\|_\Psi$ (the *Luxemburg norm*) is the infimum of all possible constants C such that this integral is ≤ 1 .

To every Orlicz function is associated the complementary Orlicz function $\Phi = \Psi^*: [0, \infty] \rightarrow [0, \infty]$ defined by:

$$\Phi(x) = \sup_{y \geq 0} (xy - \Psi(y)).$$

The extra assumptions on Ψ ensure that Φ is itself strictly convex.

Throughout this paper, we shall assume, except explicit mention of the contrary, that the *complementary* Orlicz function satisfies the Δ_2 condition ($\Phi \in \Delta_2$), *i.e.*, for some constant $K > 0$, and some $x_0 > 0$, we have:

$$\Phi(2x) \leq K \Phi(x), \quad \forall x \geq x_0.$$

This is usually expressed by saying that Ψ satisfies the ∇_2 condition ($\Psi \in \nabla_2$). This is equivalent to say that for some $\beta > 1$ and $x_0 > 0$, one has $\Psi(x) \leq \Psi(\beta x)/(2\beta)$ for $x \geq x_0$, and that implies that $\frac{\Psi(x)}{x} \xrightarrow{x \rightarrow \infty} \infty$. In particular, this excludes the case $L^\Psi = L^1$.

When Φ satisfies the Δ_2 condition, L^Ψ is the dual space of L^Φ .

We shall denote by M^Ψ the closure of L^∞ in L^Ψ . Equivalently (see [16], page 75), M^Ψ is the space of (classes of) functions such that:

$$\int_{\Omega} \Psi\left(\frac{|f(t)|}{C}\right) d\mathbb{P}(t) < +\infty, \quad \forall C > 0.$$

This space is the *Morse-Transue space* associated to Ψ , and $(M^\Psi)^* = L^\Phi$, isometrically if L^Φ is provided with the Orlicz norm, and isomorphically if it is equipped with the Luxemburg norm (see [16], Chapter IV, Theorem 1.7, page 110).

We have $M^\Psi = L^\Psi$ if and only if Ψ satisfies the Δ_2 condition, and L^Ψ is reflexive if and only if both Ψ and Φ satisfy the Δ_2 condition. When the complementary function $\Phi = \Psi^*$ of Ψ satisfies it (but Ψ does not satisfy this Δ_2 condition, to exclude the reflexive case), we have (see [16], Chapter IV, Proposition 2.8, page 122, and Theorem 2.11, page 123):

$$(*) \quad (L^\Psi)^* = (M^\Psi)^* \oplus_1 (M^\Psi)^\perp,$$

or, equivalently, $(L^\Psi)^* = L^\Phi \oplus_1 (M^\Psi)^\perp$, isometrically, with the Orlicz norm on L^Φ .

For all the matter about Orlicz functions and Orlicz spaces, we refer to [16], or to [9].

2. Main result

Our goal in this section is the following criterion of weak compactness for operators. We begin with:

Theorem 1. *Let Ψ be an arbitrary Orlicz function, and let X be a subspace of the Morse-Transue space M^Ψ . Then an operator $T: X \rightarrow Y$ from X into a Banach space Y fixes no copy of c_0 if and only if for each $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$(1) \quad \|Tf\| \leq \left[C_\varepsilon \int_\Omega \Psi\left(\varepsilon \frac{|f|}{\|f\|_\Psi}\right) d\mathbb{P} + \varepsilon \right] \|f\|_\Psi, \quad \forall f \in X.$$

Recall that saying that T fixes a copy of c_0 means that there exists a subspace X_0 of X isomorphic to c_0 such that T realizes an isomorphism between X_0 and $T(X_0)$.

Before proving that, we shall give some consequences. First, we have

Corollary 2. *Assume that the complementary function of Ψ has Δ_2 ($\Psi \in \nabla_2$). Then for every subspace X of M^Ψ , and every operator $T: X \rightarrow Y$, T is weakly compact if and only if it satisfies (1).*

Proof. When the complementary function of Ψ has Δ_2 , one has the decomposition (*), which means that M^Ψ is M -ideal in its bidual (see [6, Chapter III]); this result was first shown by D. Werner [17] (see also [6, Chapter III, Example 1.4 (d), page 105]) by a different way, using the ball intersection property; note that in these references, it is moreover assumed that Ψ does not satisfy the Δ_2 condition, but if it satisfies it, the space L^Ψ is reflexive, and so the result is obvious. But every subspace X of a Banach space which is M -ideal of its bidual has Pełczyński's property (V) ([4, 5]; see also [6, Chapter III, Theorem 3.4]), which means that operators from X are weakly compact if and only if they fix no copy of c_0 . $\square$

With Ψ satisfying the following growth condition, the characterization (1) takes on a more usable form.

Definition 3. *We say that the Orlicz function Ψ satisfies the Δ^0 condition if for some $\beta > 1$*

$$\lim_{x \rightarrow +\infty} \frac{\Psi(\beta x)}{\Psi(x)} = +\infty.$$

This growth condition is a strong negation of the Δ_2 condition and it implies that the complementary function $\Phi = \Psi^*$ of Ψ satisfies the Δ_2 condition.

Note that in the following theorem, we cannot content ourselves with $\Psi \notin \Delta_2$ (i.e. $\limsup_{x \rightarrow +\infty} \Psi(\beta x)/\Psi(x) = +\infty$), instead of $\Psi \in \Delta^0$ (see

Remark 3 in Section 3). An interesting question is whether the condition $\Psi \in \Delta^0$ is actually necessary for this characterization.

Theorem 4. *Assume that Ψ satisfies the Δ^0 condition, and let X be a subspace of M^Ψ . Then every linear operator T mapping X into some Banach space Y is weakly compact if and only if for some (and then for all) $1 \leq p < \infty$ and all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$(W) \quad \|T(f)\| \leq C_\varepsilon \|f\|_p + \varepsilon \|f\|_\Psi, \quad \forall f \in X.$$

Remark 5. This theorem extends [12] Theorem II.1. As in the case of $\mathbb{C}^*$ -algebras (see [3], Notes and Remarks, Chap. 15), there are miscellaneous applications of such a characterization.

Remark 6. Contrary to the Δ_2 condition where the constant 2 may be replaced by any constant $\beta > 1$, in this Δ^0 condition, the constant β cannot be replaced by another, as the following example shows.

Example 7. There exists an Orlicz function Ψ such that

$$(2) \quad \lim_{x \rightarrow +\infty} \frac{\Psi(5x)}{\Psi(x)} = +\infty,$$

but

$$(3) \quad \liminf_{x \rightarrow +\infty} \frac{\Psi(2x)}{\Psi(x)} < +\infty.$$

Indeed, let $(c_n)_n$ be an increasing sequence of positive numbers such that $\lim_{n \rightarrow \infty} \frac{c_{n+1}}{c_n} = +\infty$, take $\psi(t) = c_n$ for $t \in (4^n, 4^{n+1}]$ and $\Psi(x) = \int_0^x \psi(t) dt$. Then (2) is verified. On the other hand, if $x_n = 2 \cdot 4^n$, one has $\Psi(x_n) \geq c_n 4^n$, and $\Psi(2x_n) \leq c_n 4^{n+1}$, so we get (3).

Before proving Theorem 4, let us note that it has the following straightforward corollary.

Corollary 8. *Let X be like in Theorem 4, and assume that $\mathcal{F}$ is a family of operators from X into a Banach space Y with the following property: there exists a bounded sequence $(g_n)_n$ in X such that $\lim_{n \rightarrow \infty} \|g_n\|_1 = 0$ and such that an operator $T \in \mathcal{F}$ is compact whenever*

$$\lim_{n \rightarrow \infty} \|Tg_n\| = 0.$$

Then every weakly compact operator in $T \in \mathcal{F}$ is actually compact.

In the forthcoming paper [13], we prove, using a generalization of the notion of Carleson measure, that a composition operator $C_\phi: H^\Psi \rightarrow H^\Psi$

(H^Ψ is the space of analytic functions on the unit disk $\mathbb{D}$ of the complex plane whose boundary values are in $L^\Psi(\partial\mathbb{D})$, and $\phi: \mathbb{D} \rightarrow \mathbb{D}$ is an analytic self-map) is compact whenever

$$\lim_{r \rightarrow 1^-} \sup_{|\xi|=1} \Psi^{-1}(1/(1-r)) \|C_\phi(u_{\xi,r})\|_\Psi = 0,$$

where

$$u_{\xi,r}(z) = \left(\frac{1-r}{1-\bar{\xi}rz} \right)^2, \quad |z| < 1,$$

and we have

$$\lim_{r \rightarrow 1^-} \sup_{|\xi|=1} \Psi^{-1}(1/(1-r)) \|C_\phi(u_{\xi,r})\|_\Psi = 0$$

when C_ϕ is weakly compact and $\Psi \in \Delta^0$.

Though the situation does not fit exactly as in Corollary 8 (not because of the space H^Ψ , which is not a subspace of M^Ψ : we actually work in $HM^\Psi = H^\Psi \cap M^\Psi$ since $u_{\xi,r} \in HM^\Psi$, but because of the fact that we ask a uniform limit for $|\xi| = 1$), the same ideas allow us to get, when Ψ satisfies the condition Δ^0 , that C_ϕ is compact if and only if it is weakly compact.

Proof of Theorem 4. Assume that we have (W). We may assume that $p > 1$, since if (W) is satisfied for some $p \geq 1$, it is satisfied for all $q \geq p$. Moreover, we may assume that $L^\Psi \xrightarrow{j} L^p$ since Ψ satisfies condition Δ^0 (since we have: $\lim_{x \rightarrow +\infty} \frac{\Psi(x)}{x^r} = +\infty$, for every $r > 0$). Then $T[(1/C_\varepsilon)B_{L^p} \cap (1/\varepsilon)B_X] \subseteq 2B_Y$. Taking the polar of these sets, we get $T^*(B_{Y^*}) \subseteq (2C_\varepsilon)B_{j^*[(L^p)^*]} + (2\varepsilon)B_{X^*}$, for every $\varepsilon > 0$. By a well-known lemma of Grothendieck, we get, since $B_{j^*[(L^p)^*]}$ is weakly compact, that $T^*(B_{Y^*})$ is relatively weakly compact, *i.e.* T^* , and hence also T , is weakly compact.

Conversely, assume in Theorem 4 that T is weakly compact. We are going to show that (W) is satisfied with $p = 1$ (hence for all finite $p \geq 1$). Let $\varepsilon > 0$. Since the Δ^0 condition implies that the complementary function of Ψ satisfies the Δ_2 condition, Corollary 2 implies that, when $\|f\|_\Psi = 1$

$$\|Tf\| \leq C_{\varepsilon/2} \int_{\Omega} \Psi((\varepsilon/2)|f|) d\mathbb{P} + \varepsilon/2.$$

As Ψ satisfies the Δ^0 condition, there is some $\beta > 1$ such that $\frac{\Psi(x)}{\Psi(\beta x)} \rightarrow 0$ as $x \rightarrow \infty$; hence, with $\kappa = \varepsilon/2C_{\varepsilon/2}$, there exists some $x_\kappa > 0$ such that $\Psi(x) \leq \kappa\Psi(\beta x)$ for $x \geq x_\kappa$. By the convexity of Ψ , one has

$\Psi(x) \leq \frac{\Psi(x_\kappa)}{x_\kappa}x =: K_\kappa x$ for $0 \leq x \leq x_\kappa$. Hence, for every $x \geq 0$ $\Psi(x) \leq \kappa\Psi(\beta x) + K_\kappa x$. It follows that, for $f \in X$, with $\|f\|_\Psi = 1$

$$\int_{\Omega} \Psi((\varepsilon/2)|f|) d\mathbb{P} \leq \kappa \int_{\Omega} \Psi(\beta(\varepsilon/2)|f|) d\mathbb{P} + K_\kappa \frac{\varepsilon}{2} \|f\|_1 \leq \kappa + K_\kappa \frac{\varepsilon}{2} \|f\|_1$$

if we have chosen $\varepsilon \leq 2/\beta$. Hence

$$\|Tf\| \leq C_{\varepsilon/2} \left(\kappa + K_\kappa \frac{\varepsilon}{2} \|f\|_1 \right) + \frac{\varepsilon}{2} = C_{\varepsilon/2} \frac{\varepsilon}{2} K_\kappa \|f\|_1 + \left(C_{\varepsilon/2} \kappa + \frac{\varepsilon}{2} \right) = C'_\varepsilon \|f\|_1 + \varepsilon,$$

which is (W). $\square$

Remark 9. The sufficient condition is actually a general fact, which is surely well known (see [11], Theorem 1.1, for a similar result, and [3], Theorem 15.2 for $\mathcal{C}(K)$; see also [8], page 81), and has close connection with interpolation (see [2], Proposition 1), but we have found no reference, and so we shall state it separately without proof (the proof follows that given in [3], page 310).

Proposition 10. *Let $T: X \rightarrow Y$ be an operator between two Banach spaces. Assume that there is a Banach space Z and a weakly compact map $j: X \rightarrow Z$ such that: for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$\|Tx\| \leq C_\varepsilon \|jx\|_Z + \varepsilon \|x\|_X, \quad x \in X.$$

Then T is weakly compact.

Note that, by the Davis-Figiel-Johnson-Pełczyński factorization theorem, we may assume that Z is reflexive. We may also assume that j is injective, because $\ker j \subseteq \ker T$, so T induces a map $\tilde{T}: X/\ker j \rightarrow Y$ with the same property as T . Indeed, if $jx = 0$, then $\|Tx\| \leq \varepsilon \|x\|$ for every $\varepsilon > 0$, and hence $Tx = 0$.

Proof of Theorem 1. Assume first that T fixes a copy of c_0 . There are hence some $\delta > 0$ and a sequence $(f_n)_n$ in X equivalent to the canonical basis of c_0 such that $\|f_n\|_\Psi = 1$ and $\|Tf_n\| \geq \delta$. In particular, there is some $M > 0$ such that, for every choice of $\varepsilon_n = \pm 1$

$$\left\| \sum_{n=1}^N \varepsilon_n f_n \right\|_\Psi \leq M, \quad N \geq 1.$$

Let $(r_n)_n$ be a Rademacher sequence. We have, first by Khintchine's inequality, then by Jensen's inequality and Fubini's Theorem

$$\begin{aligned}
\int_{\Omega} \Psi\left(\frac{1}{M\sqrt{2}}\left(\sum_{n=1}^N |f_n|^2\right)^{1/2}\right) d\mathbb{P} &\leq \int_{\Omega} \Psi\left[\frac{1}{M} \int_0^1 \left|\sum_{n=1}^N r_n(t)f_n\right| dt\right] d\mathbb{P} \\
&\leq \int_{\Omega} \int_0^1 \Psi\left[\frac{1}{M} \left|\sum_{n=1}^N r_n(t)f_n\right|\right] dt d\mathbb{P} \\
&= \int_0^1 \int_{\Omega} \Psi\left[\frac{1}{M} \left|\sum_{n=1}^N r_n(t)f_n\right|\right] d\mathbb{P} dt \leq 1.
\end{aligned}$$

The monotone convergence Theorem gives then

$$\int_{\Omega} \Psi\left(\frac{1}{M\sqrt{2}}\left(\sum_{n=1}^{\infty} |f_n|^2\right)^{1/2}\right) d\mathbb{P} \leq 1.$$

In particular, $\sum_{n=1}^{\infty} |f_n|^2$ is finite almost everywhere, and hence $f_n \rightarrow 0$ almost everywhere. Since $\Psi\left(\frac{1}{M\sqrt{2}}\left(\sum_{n=1}^{\infty} |f_n|^2\right)^{1/2}\right) \in L^1$, by the above inequalities, Lebesgue's dominated convergence Theorem gives

$$\int_{\Omega} \Psi\left(\frac{|f_n|}{M\sqrt{2}}\right) d\mathbb{P} \xrightarrow{n \rightarrow \infty} 0.$$

But that contradicts (1) with $\varepsilon \leq 1/M\sqrt{2}$ and $\varepsilon < \delta$, since $\|Th_n\| \geq \delta$.

The converse follows from the following lemma.

Lemma 11. *Let X be a subspace of M^{Ψ} , and let $(h_n)_n$ be a sequence in X , with $\|h_n\|_{\Psi} = 1$ for all $n \geq 1$, and such that, for some $M > 0$*

$$\int_{\Omega} \Psi(|h_n|/M) d\mathbb{P} \xrightarrow{n \rightarrow \infty} 0.$$

Then $(h_n)_n$ has a subsequence equivalent to the canonical basis of c_0 .

Indeed, if condition (1) is not satisfied, there exist some $\varepsilon_0 > 0$ and functions $h_n \in X$ with $\|h_n\|_{\Psi} = 1$ such that $\|Th_n\| \geq 2^n \int_{\Omega} \Psi(\varepsilon_0 |h_n|) d\mathbb{P} + \varepsilon_0$. That implies that $\int_{\Omega} \Psi(\varepsilon_0 |h_n|) d\mathbb{P}$ tends to 0, so Lemma 11 ensures that $(h_n)_n$ has a subsequence, which we shall continue to denote by $(h_n)_n$, equivalent to the canonical basis of c_0 . Then $(Th_n)_n$ is weakly unconditionally Cauchy. Since $\|Th_n\| \geq \varepsilon_0$, $(Th_n)_n$ has, by Bessaga-Pelczyński's Theorem, a further subsequence equivalent to the canonical basis of c_0 . It is then obvious that T realizes an isomorphism between the spaces generated by these subsequences. $\square$

Proof of Lemma 11. The proof uses the idea of the construction made in the proof of Theorem II.1 in [12], which it generalizes, but with some additional details.

By the continuity of Ψ , there exists $a > 0$ such that $\Psi(a) = 1$. Then, since Ψ is increasing, we have, for every $g \in L^\infty$, $\int_\Omega \Psi\left(a \frac{|g|}{\|g\|_\infty}\right) d\mathbb{P} \leq 1$, and so $\|g\|_\Psi \leq (1/a) \|g\|_\infty$. Now, choose, for every $n \geq 1$, positive numbers $\alpha_n < a/2^{n+2}$ such that $\Psi(\alpha_n/2M) \leq 1$.

We are going to construct inductively a subsequence $(f_n)_n$ of $(h_n)_n$, a sequence of functions $g_n \in L^\infty$ and two sequences of positive numbers β_n and $\varepsilon_n \leq \min\{1/2^{n+1}, M/2^{n+1}\}$, such that, for every $n \geq 1$

(i) if we set $M_1 = 1$ and, for $n \geq 2$

$$M_n = \max \left\{ 1, \Psi \left(\frac{\|g_1\|_\infty + \dots + \|g_{n-1}\|_\infty}{2M} \right) \right\},$$

then $M_n \beta_n \leq 1/2^{n+1}$;

(ii) $\|f_n\|_\Psi = 1$;

(iii) $\|f_n - g_n\|_\Psi \leq \varepsilon_n$, with ε_n such that $\beta_n \Psi(\alpha_n/2\varepsilon_n) \geq 2$;

(iv) $\mathbb{P}(\{|g_n| > \alpha_n\}) \leq \beta_n$;

(v) $\|\check{g}_n\|_\Psi \geq 1/2$, with $\check{g}_n = g_n \mathbb{1}_{\{|g_n| > \alpha_n\}}$.

We shall give only the inductive step, since the starting one unfolds identically. Suppose hence that the functions $f_1, \dots, f_{n-1}$, $g_1, \dots, g_{n-1}$ and the numbers $\beta_1, \dots, \beta_{n-1}$ and $\varepsilon_1, \dots, \varepsilon_{n-1}$ have been constructed. Choose then $\beta_n > 0$ such that $M_n \beta_n \leq 1/2^{n+1}$. Note that $M_n \geq 1$ implies that $\beta_n \leq 1/2^{n+1}$. Since $\int_\Omega \Psi(|h_k|/M) d\mathbb{P} \rightarrow 0$ as $n \rightarrow \infty$, we can find $f_n = h_{k_n}$ such that $\|f_n\|_\Psi = 1$, and moreover

$$\mathbb{P}(\{|f_n| > \alpha_n/2\}) \leq \frac{1}{\Psi(\alpha_n/2M)} \int_\Omega \Psi\left(\frac{|f_n|}{M}\right) d\mathbb{P} \leq \frac{\beta_n}{2}.$$

Take now $\varepsilon_n \leq \min\{1/2^{n+1}, M/2^{n+1}\}$ such that $0 < \varepsilon_n \leq \alpha_n/2\Psi^{-1}(2/\beta_n)$ and $g_n \in L^\infty$ such that $\|f_n - g_n\|_\Psi \leq \varepsilon_n$. Then, since

$$\mathbb{P}(\{|f_n - g_n| > \alpha_n/2\}) \Psi\left(\frac{\alpha_n}{2\varepsilon_n}\right) \leq \int_\Omega \Psi\left(\frac{|f_n - g_n|}{\varepsilon_n}\right) d\mathbb{P} \leq 1,$$

we have

$$\begin{aligned} \mathbb{P}(\{|g_n| > \alpha_n\}) &\leq \mathbb{P}(\{|f_n| > \alpha_n/2\}) + \mathbb{P}(\{|f_n - g_n| > \alpha_n/2\}) \\ &\leq \frac{\beta_n}{2} + \frac{1}{\Psi(\alpha_n/2\varepsilon_n)} \leq \beta_n. \end{aligned}$$

To end the construction, it remains to note that

$$\begin{aligned}\|f_n - \check{g}_n\|_\Psi &\leq \|f_n - g_n\|_\Psi + \|\check{g}_n - g_n\|_\Psi \leq \varepsilon_n + \frac{1}{a}\|\check{g}_n - g_n\|_\infty \\ &\leq \frac{1}{2^{n+1}} + \frac{\alpha_n}{a} \leq \frac{1}{2^n} \leq \frac{1}{2}\end{aligned}$$

and so

$$\|\check{g}_n\|_\Psi \geq \|f_n\|_\Psi - \|f_n - \check{g}_n\|_\Psi \geq 1 - \frac{1}{2} = \frac{1}{2}.$$

This ends the inductive construction.

Consider now

$$\check{g} = \sum_{n=1}^{+\infty} |\check{g}_n|.$$

Set $A_n = \{|g_n| > \alpha_n\}$ and, for $n \geq 1$

$$B_n = A_n \setminus \bigcup_{j>n} A_j.$$

We have $\mathbb{P}(\limsup A_n) = 0$, because

$$\sum_{n \geq 1} \mathbb{P}(A_n) \leq \sum_{n \geq 1} \beta_n \leq \sum_{n \geq 1} \frac{1}{2^n} < +\infty.$$

Now $\check{g}$ vanishes out of $\bigcup_{n \geq 1} B_n \cup (\limsup A_n)$ and we have

$$\begin{aligned}\int_{B_n} \Psi\left(\frac{|\check{g}_n|}{2M}\right) d\mathbb{P} &\leq \int_{\Omega} \Psi\left(\frac{|g_n|}{2M}\right) d\mathbb{P} \\ &\leq \int_{\Omega} \Psi\left(\frac{|g_n - f_n|}{2M} + \frac{|f_n|}{2M}\right) d\mathbb{P} \\ &\leq \frac{1}{2} \int_{\Omega} \Psi\left(\frac{|g_n - f_n|}{M}\right) d\mathbb{P} + \frac{1}{2} \int_{\Omega} \Psi\left(\frac{|f_n|}{M}\right) d\mathbb{P}.\end{aligned}$$

The first integral is less than ε_n/M , because $\Psi(at) \leq a\Psi(t)$ for $0 \leq a \leq 1$ and $\varepsilon_n/M \leq 1$, so that

$$\int_{\Omega} \Psi\left(\frac{|g_n - f_n|}{M}\right) d\mathbb{P} \leq \frac{\varepsilon_n}{M} \int_{\Omega} \Psi\left(\frac{|g_n - f_n|}{\varepsilon_n}\right) d\mathbb{P} \leq \frac{\varepsilon_n}{M} \leq \frac{1}{2^{n+1}}$$

since $\|f_n - g_n\|_\Psi \leq \varepsilon_n$. Since

$$\int_{\Omega} \Psi\left(\frac{|f_n|}{M}\right) d\mathbb{P} \leq \frac{\beta_n}{2} \Psi(\alpha_n/2M) \leq \beta_n/2,$$

we obtain

$$\int_{B_n} \Psi\left(\frac{|\check{g}_n|}{2M}\right) d\mathbb{P} \leq \frac{1}{2^{n+2}} + \frac{\beta_n}{4}.$$

Therefore, since $\mathbb{P}(B_n) \leq \mathbb{P}(A_n) \leq \beta_n$, we have

$$\begin{aligned} \int_{\Omega} \Psi\left(\frac{|\check{g}|}{4M}\right) d\mathbb{P} &= \sum_{n=1}^{+\infty} \int_{B_n} \Psi\left(\frac{|\check{g}|}{4M}\right) d\mathbb{P} \\ &\leq \sum_{n=1}^{+\infty} \int_{B_n} \frac{1}{2} \left[\Psi\left(\frac{\|g_1\|_{\infty} + \cdots + \|g_{n-1}\|_{\infty}}{2M}\right) + \Psi\left(\frac{|\check{g}_n|}{2M}\right) \right] d\mathbb{P} \\ &\quad \text{by convexity of } \Psi \text{ and because } \check{g}_j = 0 \text{ on } B_n \text{ for } j > n \\ &\leq \frac{1}{2} \sum_{n=1}^{+\infty} \left(M_n \beta_n + \frac{1}{2^{n+2}} + \frac{\beta_n}{4} \right) \\ &\leq \frac{1}{2} \sum_{n=1}^{+\infty} \left(\frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+2}} \right) \leq 1 \end{aligned}$$

which proves that $\check{g} \in L^{\Psi}$, and consequently that the series $\sum_{n \geq 1} \check{g}_n$ is weakly unconditionally Cauchy in L^{Ψ} :

$$\sup_{n \geq 1} \sup_{\theta_k = \pm 1} \left\| \sum_{k=1}^n \theta_k \check{g}_k \right\|_{\Psi} \leq \sup_{n \geq 1} \left\| \sum_{k=1}^n |\check{g}_k| \right\|_{\Psi} \leq \|\check{g}\|_{\Psi} \leq 4M.$$

Since $\|\check{g}_n\|_{\Psi} \geq 1/2$, $(\check{g}_n)_{n \geq 1}$ has, by Bessaga-Pelczyński's theorem, a subsequence $(\check{g}_{n_k})_{k \geq 1}$ which is equivalent to the canonical basis of c_0 . The corresponding subsequence $(f_{n_k})_{k \geq 1}$ of $(f_n)_{n \geq 1}$ remains equivalent to the canonical basis of c_0 , since

$$\sum_{n=1}^{+\infty} \|f_n - \check{g}_n\|_{\Psi} \leq \sum_{n=1}^{+\infty} \varepsilon_n + \frac{\alpha_n}{a} \leq \sum_{n=1}^{+\infty} \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} < 1$$

and the assertion follows. $\square$

3. Comments

Remark 12. Let us note that the assumption $X \subseteq M^{\Psi}$ in Theorem 4 cannot be relaxed in general. In fact, suppose that X is a subspace of L^{Ψ} containing L^{∞} , and let $\xi \in (M^{\Psi})^{\perp} \subseteq (L^{\Psi})^*$, $\xi \neq 0$. Being of rank one, ξ is trivially weakly compact. Suppose that it satisfies (W). Let $f \in X$ with norm 1, and let $\varepsilon > 0$. For t large enough and $f_t = f \mathbb{1}_{\{|f| \leq t\}}$, we have $\|f - f_t\|_2 \leq \varepsilon/C_{\varepsilon}$. Moreover, $f_t \in L^{\infty} \subseteq X$ and $\|f_t\|_{\Psi} \leq \|f\|_{\Psi} = 1$. Since

ξ vanishes on L^∞ and $f - f_t \in X$, we get

$$|\xi(f)| = |\xi(f - f_t)| \leq C_\varepsilon \|f - f_t\|_2 + \varepsilon \|f - f_t\|_\Psi \leq 3\varepsilon.$$

This implies that $\xi(f) = 0$. Since this occurs for every $\xi \in (M^\Psi)^\perp$, we get that $X \subseteq M^\Psi$ (and actually $X = M^\Psi$ since X contains L^∞).

In particular Theorem 4 does not hold for $X = L^\Psi$.

Remark 13. However, condition (W) remains true for bi-adjoint operators coming from subspaces of M^Ψ : if $T: X \subseteq M^\Psi \rightarrow Y$ satisfies the condition (W), then $T^{**}: X^{**} \rightarrow Y^{**}$ also satisfies it. Indeed, for every $\varepsilon > 0$, we get an equivalent norm $\|\cdot\|_\varepsilon$ on X by putting

$$\|f\|_\varepsilon = C_\varepsilon \|f\|_2 + \varepsilon \|f\|_\Psi.$$

Hence if $f \in X^{**}$, there exists a net $(f_\alpha)_\alpha$ of elements in X , with $\|f_\alpha\|_\varepsilon \leq \|f\|_\varepsilon$ which converges weak-star to f . Then $(Tf_\alpha)_\alpha$ converges weak-star to $T^{**}f$, and

$$\begin{aligned} \|T^{**}f\| &\leq \liminf_\alpha \|Tf_\alpha\| \leq \liminf_\alpha (C_\varepsilon \|f_\alpha\|_2 + \varepsilon \|f_\alpha\|_\Psi) \\ &= \liminf_\alpha \|f_\alpha\|_\varepsilon \leq \|f\|_\varepsilon = C_\varepsilon \|f\|_2 + \varepsilon \|f\|_\Psi. \end{aligned}$$

Hence, from Proposition 10 above, for such a T , T^{**} is weakly compact if and only if it satisfies (W). We shall use this fact in the forthcoming paper [13].

Remark 14. In Theorem 4, we cannot only assume that $\Psi \notin \Delta_2$, instead of $\Psi \in \Delta^0$, as the following example shows. It also shows that in Corollary 2, we cannot obtain condition (W) instead of condition (1).

Example 15. Let us define

$$\psi(t) = \begin{cases} t & \text{for } 0 \leq t < 1, \\ (k!)(k+2)t - k!(k+1)! & \text{for } k! \leq t \leq (k+1)!, \quad k \geq 1, \end{cases}$$

($\psi(k!) = (k!)^2$ for every integer $k \geq 1$ and ψ is linear between $k!$ and $(k+1)!$), and $\Psi(x) = \int_0^x \psi(t) dt$. Since $t^2 \leq \psi(t)$ for all $t \geq 0$, one has $x^3/3 \leq \Psi(x)$ for all $x \geq 0$. Then

$$\Psi(2 \cdot n!) \geq \int_{n!}^{2 \cdot n!} \psi(t) dt = n!(n+2) \frac{3}{2} (n!)^2 - (n!)^2 (n+1)! = (n!)^3 \left(\frac{n}{2} + 2 \right),$$

whereas

$$\Psi(n!) = \int_0^{n!} \psi(t) dt \leq (n!)^2 n! = (n!)^3;$$

hence

$$\frac{\Psi(2.n!)}{\Psi(n!)} \geq \frac{n}{2} + 2,$$

and so

$$\limsup_{x \rightarrow +\infty} \frac{\Psi(2x)}{\Psi(x)} = +\infty,$$

which means that $\Psi \notin \Delta_2$. On the other hand, for every $\beta > 1$

$$\Psi(n!/\beta) \geq \frac{1}{3} \left(\frac{n!}{\beta} \right)^3 = \frac{(n!)^3}{3\beta^3},$$

so

$$\frac{\Psi(n!)}{\Psi(n!/\beta)} \leq \frac{(n!)^3}{(n!)^3/3\beta^3} = 3\beta^3;$$

hence

$$\liminf_{x \rightarrow +\infty} \frac{\Psi(2x)}{\Psi(x)} \leq 3\beta^3,$$

and $\Psi \notin \Delta^0$ (actually, this will follow too from the fact that Theorem 4 is not valid for this Ψ).

Moreover, the conjugate function of Ψ satisfies the condition Δ_2 . Indeed, since ψ is convex, one has $\psi(2u) \geq 2\psi(u)$ for all $u \geq 0$, and hence:

$$\Psi(2x) = \int_0^{2x} \psi(t) dt = 2 \int_0^x \psi(2u) du \geq 2 \int_0^x 2\psi(u) du = 4\Psi(x),$$

and as it was seen in the Introduction that means that $\Psi \in \nabla_2$.

Now, we have $x^3/3 \leq \Psi(x)$ for all $x \geq 0$; therefore $\|\cdot\|_3 \leq 3^{1/3}\|\cdot\|_\Psi$. In particular, we have an inclusion map $j: M^\Psi \hookrightarrow L^3$, which is, of course, weakly compact. Nevertheless, assuming that $\mathbb{P}$ is diffuse, condition (W) is not verified by j , when $\varepsilon < 1$. Indeed, as we have seen before, one has $\Psi(n!) \leq (n!)^3$. Hence, if we choose a measurable set A_n such that $\mathbb{P}(A_n) = 1/\Psi(n!)$, we have

$$\|\mathbb{1}_{A_n}\|_\Psi = \frac{1}{\Psi^{-1}(1/\mathbb{P}(A_n))} = \frac{1}{n!};$$

whereas

$$\|\mathbb{1}_{A_n}\|_3 = \mathbb{P}(A_n)^{1/3} = \frac{1}{\Psi(n!)^{1/3}} \geq \frac{1}{n!}$$

and

$$\|\mathbb{1}_{A_n}\|_2 = \mathbb{P}(A_n)^{1/2} \leq \left[\frac{3}{(n!)^3} \right]^{1/2} = \frac{\sqrt{3}}{(n!)^{3/2}}.$$

If condition (W) were true, we should have, for every $n \geq 1$

$$\frac{1}{n!} \leq C_\varepsilon \frac{\sqrt{3}}{(n!)^{3/2}} + \varepsilon \frac{1}{n!},$$

that is $\sqrt{n!} \leq \sqrt{3} \frac{C_\varepsilon}{1-\varepsilon}$, which is of course impossible for n large enough.

Remark 16. In the case of the whole space M^Ψ , we can give a direct proof of the necessity in Theorem 4. Indeed, suppose that $T: M^\Psi \rightarrow X$ is weakly compact. Then $T^*: X^* \rightarrow L^\Phi = (M^\Psi)^*$ is weakly compact, and so the set $K = T^*(B_{X^*})$ is relatively weakly compact.

Since Φ satisfies the Δ^0 condition, it follows from [1] (Corollary 2.9) that K has equi-absolutely continuous norms. Hence, for every $\varepsilon > 0$, we can find $\delta_\varepsilon > 0$ such that:

$$m(A) \leq \delta_\varepsilon \quad \Rightarrow \quad \|g\mathbb{1}_A\|_\Phi \leq \varepsilon/2, \quad g \in T^*(B_{X^*}).$$

But (the factor $1/2$ appears because we use the Luxemburg norm on the dual, and not the Orlicz norm: see [16], Proposition III.3.4)

$$\begin{aligned} \sup_{g \in T^*(B_{X^*})} \|g\mathbb{1}_A\|_\Phi &\geq \frac{1}{2} \sup_{u \in B_{X^*}} \sup_{\|f\|_\Psi \leq 1} | \langle f, (T^*u)\mathbb{1}_A \rangle | \\ &= \frac{1}{2} \sup_{u \in B_{X^*}} \sup_{\|f\|_\Psi \leq 1} \left| \int f(T^*u)\mathbb{1}_A \, dm \right| \\ &= \frac{1}{2} \sup_{u \in B_{X^*}} \sup_{\|f\|_\Psi \leq 1} | \langle T(f\mathbb{1}_A), u \rangle | = \frac{1}{2} \sup_{\|f\|_\Psi \leq 1} \|T(f\mathbb{1}_A)\|; \end{aligned}$$

so

$$m(A) \leq \delta_\varepsilon \quad \Rightarrow \quad \sup_{\|f\|_\Psi \leq 1} \|T(f\mathbb{1}_A)\| \leq \varepsilon.$$

Now, we have

$$m(|f| \geq \|f\|_2/\delta_\varepsilon) \leq \frac{\delta_\varepsilon}{\|f\|_2} \int |f| \, dm = \frac{\delta_\varepsilon}{\|f\|_2} \|f\|_1 \leq \delta_\varepsilon;$$

hence, with $A = \{|f| \geq \|f\|_2/\delta_\varepsilon\}$, we get, for $\|f\|_\Psi \leq 1$:

$$\|Tf\| \leq \|T(f\mathbb{1}_A)\| + \|T(f\mathbb{1}_{A^c})\| \leq \varepsilon + \|T\| \frac{\|f\|_2}{\delta_\varepsilon}$$

since $|f\mathbb{1}_{A^c}| \leq \|f\|_2/\delta_\varepsilon$ implies $\|f\mathbb{1}_{A^c}\|_\Psi \leq \|f\mathbb{1}_{A^c}\|_\infty \leq \|f\|_2/\delta_\varepsilon$.

Remark 17. Conversely, E. Lavergne ([10]) recently uses our Theorem 4 to give a proof of the above quoted result of J. Alexopoulos ([1], Corollary 2.9), and uses it to show that, when $\Psi \in \Delta^0$, then the reflexive

subspaces of L^Φ (where Φ is the conjugate of Ψ) are closed for the L^1 -norm.

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References

- [1] J. Alexopoulos, *De la Vallée–Poussin's theorem and weakly compact sets in Orlicz spaces*, Quaestiones Math., 17 (1994), 231–248.
- [2] B. Beauzamy, *Propriétés géométriques des espaces d'interpolation, Séminaire Maurey-Schwartz 1974–1975: Espaces L_p , applications radonifiantes et géométrie des espaces de Banach*, Exp. No. XIV, 17 pp., Centre Math., École Polytech., Paris, 1975.
- [3] J. Diestel, H. Jarchow and A. Tonge, *Absolutely Summing Operators*, Cambridge studies in advanced mathematics, 43, Cambridge University Press, 1995.
- [4] G. Godefroy and P. Saab, *Quelques espaces de Banach ayant les propriétés (V) ou (V^*) de A. Pełczyński*, C.R. Acad. Sci. Paris, Série A, 303 (1986), 503–506.
- [5] G. Godefroy and P. Saab, *Weakly unconditionally convergent series in M -ideals*, Math. Scand., 64 (1989), 307–318.
- [6] P. Harmand, D. Werner and W. Werner, *M -ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Math., 1547, Springer, 1993.
- [7] H. Jarchow, *On Weakly Compact Operators on $\mathbf{C}^*$ -Algebras*, Math. Ann., 273 (1986), 341–343.
- [8] H. Jarchow and U. Matter, *On weakly compact operators on $\mathcal{C}(K)$ -spaces*, 80–88, Proceedings of the Missouri Conference held in Columbia, USA, June 24–29 1984, N. Kalton and E. Saab, Eds., Lecture Notes in Math., 1166, 1985.
- [9] M. A. Krasnosel'skiĭ and Ya. B. Rutickiĭ, *Convex functions and Orlicz spaces (translation)*, P. Noordhoff Ltd., Groningen, 1961.
- [10] E. Lavergne, *Reflexive subspaces of some Orlicz spaces*, Colloq. Math., 113 (2008), 333–340.
- [11] P. Lefèvre, *Some characterizations of weakly compact operators on H^∞ and on the disk algebra. Application to composition operators*, Journal of Operator Theory, 54 (2005), 229–238.

- [12] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, *Some translation-invariant Banach function spaces which contain c_0* , *Studia Math.*, 163 (2004), 137–155.
- [13] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, *Composition operators on Hardy-Orlicz spaces*, *Memoirs Amer. Math. Soc.*, to appear.
- [14] C. P. Niculescu, *Absolute continuity and weak compactness*, *Bull. Amer. Math. Soc.*, 81 (1975), 1064–1066.
- [15] C. P. Niculescu, *Absolute continuity in Banach space theory*, *Rev. Roum. Math. Pures Appl.*, 24 (1979), 413–422.
- [16] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, *Pure and Applied Mathematics* 146, Marcel Dekker, Inc., 1991.
- [17] D. Werner, *New classes of Banach spaces which are M -ideals in their biduals*, *Math. Proc. Cambridge Philos. Soc.*, 111 (1992), 337–354.

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