On the degree of compactness of embeddings between weighted modulation spaces

Aicke Hinrichs*, Iwona Piotrowska and Mariusz Piotrowski

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Abstract. The paper investigates the asymptotic behaviour of entropy and approximation numbers of compact embeddings between weighted modulation spaces.

1. Introduction

Over the past twenty years, entropy and approximation numbers of operators in function spaces has become very popular subject of research. There is a large body of results on asymptotic behaviour of entropy and approximation numbers of compact embeddings between spaces of Besov and Triebel–Lizorkin type. These results have found widespread applications to spectral theory of differential and pseudo-differential

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operators. The best general reference here is [4]. A very recent account of the theory has been given in [7]. In the case of compact embeddings of weighted modulation spaces however only very few results describing the behaviour of entropy or approximation numbers are known. The only published result known to us is due to H. Rauhut [8], where the upper bounds for entropy and approximation numbers of embeddings between radial modulation spaces are presented. In this paper we study the behaviour of entropy and approximation numbers of the compact embeddings

$$(1) \quad \text{id} : M_{p_1 q_1}^{s_1}(\mathbb{R}^d) \rightarrow M_{p_2 q_2}^{s_2}(\mathbb{R}^d).$$

Here $M_{pq}^s(\mathbb{R}^d)$ denotes a weighted modulation spaces. For the definition of the underlying function spaces we refer the reader to Subsection 2.2.

We now describe briefly the contents of the paper. In the subsequent section we set up notation and terminology and summarize basic facts on weighted modulation spaces. In particular, we describe the Fourier analytical approach to define these spaces. We follow the monograph [6, Section 12.3] to present the discretization procedure by means of Wilson bases. In Section 3 we discuss the compactness of the embedding (1). The statement of Theorem 3.2 with $p_1 = p_2$ and $q_1 = q_2$ in (1) is known, see [1, 3]. However, to our best knowledge, the characterization of compactness of (1), as presented in Theorem 3.2, is new. In Section 4 two-sided estimates of entropy and approximation numbers of embedding (1) in the case $p_1 = q_1$ and $p_2 = q_2$ are given.

2. Preliminaries

2.1. Notation and conventions. For two real sequences (a_n) and (b_n) , the symbol $a_n \lesssim b_n$ (or $a_n \gtrsim b_n$) means that $a_n \leq c b_n$ (or $c a_n \geq b_n$) for all $n \in \mathbb{N}$ and some $c > 0$. If $a_n \lesssim b_n$ and $a_n \gtrsim b_n$, then we write $a_n \sim b_n$ and say that (a_n) and (b_n) are equivalent. For $p \in [1, \infty]$, the conjugate number p' is defined by $1/p + 1/p' = 1$ with the convention that $1/\infty = 0$. For any $k \in \mathbb{Z}^d$, we write $|k| = |k_1| + \dots + |k_d|$. Let $B(x, r)$ denote the open ball in $\mathbb{R}^d$ with radius $r > 0$ centered at $x \in \mathbb{R}^d$. Let $\mathcal{S}(\mathbb{R}^d)$ stand for the Schwartz space of all complex-valued rapidly decreasing C^∞ functions on $\mathbb{R}^d$. Further, we denote by $\mathcal{S}'(\mathbb{R}^d)$ its topological dual, the space of all tempered distributions.

2.2. Modulation spaces. Throughout, the usual translation operator is denoted by $T_x f(t) = f(t - x)$, whereas translation in the phase space, the so-called modulation, is given by $M_\omega f(t) = e^{2\pi i t \cdot \omega} f(t)$. The Short-Time

Fourier Transform (STFT) of a function $f \in L_2(\mathbb{R}^d)$ with respect to a window $g \in L_2(\mathbb{R}^d)$ is

$$\begin{aligned} V_g f(x, \omega) &= \langle f, M_\omega T_x g \rangle_{L_2(\mathbb{R}^d)} \\ &= \int_{\mathbb{R}^d} e^{-2\pi i \omega \cdot t} \overline{g(t-x)} f(t) dt. \end{aligned}$$

We extend the STFT to $\mathcal{S}'(\mathbb{R}^d)$ by understanding the bracket $\langle \cdot, \cdot \rangle$ denoting previously the inner product in $L_2(\mathbb{R}^d)$ as a dual pairing between an element from $\mathcal{S}'(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$. For more information on the STFT the reader is referred to [6].

We are now in a position to introduce the function spaces.

Definition 2.1. Fix a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$ and let $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$. Then the *modulation space* $M_{p,q}^s(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ for which the norm

$$(2) \quad \|f\|_{M_{p,q}^s(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, \omega)|^p (1 + |x| + |\omega|)^{sp} dx \right)^{q/p} d\omega \right)^{1/q}.$$

is finite. If $p = \infty$ or $q = \infty$ then the usual modification is required. If $p = q$, then we write M_p^s instead of $M_{p,p}^s$.

Remark 2.2. The above definition coincides with [6, Definition 11.3.1]. We refer the interested reader to this monograph, where the so-called submultiplicative moderate weights are considered. These spaces are independent of the particular choice of the window function g . Different window functions g define equivalent norms on $M_{p,q}^s(\mathbb{R}^d)$. It is also clear that equivalent weights yield the same spaces and equivalent norms. For instance, we may consider equivalent weight functions of the form $(1 + |x|^2 + |\omega|^2)^{s/2}$ or $(1 + (|x|^2 + |\omega|^2)^{1/2})^s$ instead of $(1 + |x| + |\omega|)^s$. An alternative way to introduce modulation spaces is in terms of a bounded partition of unity related to the congruent decomposition of $\mathbb{R}^d$. We work with the uniform covering (Q_k) of $\mathbb{R}^d$ using cubes given by $Q_0 = \{x \in \mathbb{R}^d : -1/2 \leq x_i < 1/2, i = 1, \dots, d\}$ and $Q_k = k + Q_0$, $k \in \mathbb{Z}^d$. Furthermore, let $\phi \in \mathcal{S}(\mathbb{R}^d)$ with

$$\phi(x) = 1 \text{ for } x \in B(0, \sqrt{d}/2) \text{ and } \phi(x) = 0 \text{ for } x \notin B(0, \sqrt{d})$$

and denote

$$\phi_k(x) = \phi(x - k), \quad k \in \mathbb{Z}^d.$$

We put $\varphi_k(x) = \phi_k(x) / \sum_{k \in \mathbb{Z}^d} \phi_k(x)$ for all $x \in \mathbb{R}^d$. Then, since

$$\sum_{k \in \mathbb{Z}^d} \varphi_k(x) = 1,$$

the sequence (φ_k) forms an uniform resolution of unity in $\mathbb{R}^d$. It can be easily seen that $|\varphi_k(x)| \geq c$ for all $x \in Q_k$ and $\text{supp}(\varphi_k) \subset B(k, \sqrt{d})$. Moreover, it can be shown that there is a constant $c_m > 0$ such that $|D^\alpha \varphi_k(x)| \leq c_m$ for all $|\alpha| \leq m$. It turns out that the modulation space $M_{p,q}^s(\mathbb{R}^d)$ is the collection of all tempered distributions f for which

$$\|f\|_{M_{pq}^s(\mathbb{R}^d)} = \left(\sum_{k \in \mathbb{Z}^d} \|(1 + |\cdot| + |k|)^s (\varphi_k \widehat{f})^\vee|_{L_p(\mathbb{R}^d)}\|^q \right)^{1/q}$$

is finite. The proof of the equivalence proceeds along the same lines as the proof of the same statement for the weights which depend only on the frequency variable ω , as presented in [5] and therefore will be omitted.

2.3. Wilson bases in modulation spaces. We are now ready to present a discrete characterization of modulation spaces. We start by introducing the so-called Wilson basis. Let $g \in M_1^s(\mathbb{R})$, $s \geq 0$ with $\|g\|_{L_2(\mathbb{R})} = 1$ and $g(x) = \overline{g}(-x)$ be such that the system $(M_m T_{k/2} g)_{m,k \in \mathbb{Z}}$ forms a tight Gabor frame of redundancy 2. A Wilson system generated from g consists of the functions

$$w_{k,n} = c_n T_{k/2} (M_n + (-1)^{k+n} M_{-n}) g, \quad (k, n) \in \mathbb{Z} \times \mathbb{N}_0,$$

with $c_0 = 1$ and $c_n = 1/\sqrt{2}$ if $n \neq 0$, and $w_{2k+1,0} = 0$. Then the above defined functions form an orthonormal basis for $L_2(\mathbb{R})$, see [6, Theorem 8.5.1]. To extend these bases to the multivariate case, we make use of the standard tensor product construction. Given an orthonormal Wilson basis $(w_{k,n})_{k \in \mathbb{Z}, n \in \mathbb{N}_0}$ for $L_2(\mathbb{R})$, the basis functions on $L_2(\mathbb{R}^d)$ are given by

$$w_{k,n}(x) = \prod_{j=1}^d w_{k_j, n_j}(x_j),$$

where $k \in \mathbb{Z}^d, n \in \mathbb{N}_0^d$. It turns out that $(w_{k,n})$ is an orthonormal basis for $L_2(\mathbb{R}^d)$.

Theorem 2.3. *Let $(w_{k,n})_{k \in \mathbb{Z}^d, n \in \mathbb{N}_0^d}$ be a Wilson system in d -dimensions. Then*

$$(3) \|f\|_{M_{pq}^s(\mathbb{R}^d)}^* = \left(\sum_{n \in \mathbb{N}_0^d} \left(\sum_{k \in \mathbb{Z}^d} (1 + |k|/2 + |n|)^{sp} |\langle f, w_{k,n} \rangle|^p \right)^{q/p} \right)^{1/q}$$

is an equivalent norm on $M_{pq}^s(\mathbb{R}^d)$.

The proof of this result may be found in [6, Proposition 12.3.8]. In view of the above theorem, we associate to a modulation space the following sequence space.

Definition 2.4. Let $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$. We define

(4)

$$m_{pq}^s = \left\{ \lambda = (\lambda_{k,n})_{k \in \mathbb{Z}^d, n \in \mathbb{N}_0^d} : \lambda_{k,n} \in \mathbb{C}, \right. \\ \left. \|\lambda\|_{m_{p,q}^s} = \left(\sum_{n \in \mathbb{N}_0^d} \left(\sum_{k \in \mathbb{Z}^d} (1 + |k|/2 + |n|)^{sp} |\lambda_{k,n}|^p \right)^{q/p} \right)^{1/q} < \infty \right\}.$$

Again, if $p = q$, then we write m_p^s instead of $m_{p,p}^s$. For simplicity, we write $m_{pq}^0 = m_{pq}$.

A rephrasing of Theorem 2.3 in the context of the mixed norm sequence spaces is that the mapping assigning to a function $f \in M_{pq}^s(\mathbb{R}^d)$ a sequence $(\langle f, w_{kn} \rangle)_{k \in \mathbb{Z}^d, n \in \mathbb{N}_0^d}$ is an isomorphism between $M_{pq}^s(\mathbb{R}^d)$ and m_{pq}^s .

3. Compact embeddings of weighted modulation spaces

In this section we clarify under which restrictions on the parameters an embedding between weighted modulation spaces is compact. We start by quoting a known result which is due to M. Dörfler et al. [3, Theorem 5], and independently to P. Boggiatto and J. Toft in [1, Theorem 2.2]. Remark that it was shown for a more general class of the so-called moderate weights, but we present it only for our weight function.

Proposition 3.1. Let $1 \leq p, q \leq \infty$. Then the embedding

$$\text{id} : M_{pq}^{s_1}(\mathbb{R}^d) \rightarrow M_{pq}^{s_2}(\mathbb{R}^d)$$

is compact if, and only if, $s_1 - s_2 > 0$.

We are interested in describing the compactness of the embeddings between modulation spaces with nonequal integrability parameters p, q .

Theorem 3.2. *Let $1 \leq p_1 \leq p_2 \leq \infty$, $1 \leq q_1, q_2 \leq \infty$, and let $s_1, s_2 \in \mathbb{R}$. Then the embedding*

$$(5) \quad \text{id} : M_{p_1 q_1}^{s_1}(\mathbb{R}^d) \rightarrow M_{p_2 q_2}^{s_2}(\mathbb{R}^d)$$

is compact if, and only if,

$$(6) \quad \frac{s_1 - s_2}{d} > \max \left\{ 0, \frac{1}{q_2} - \frac{1}{q_1} \right\}.$$

Proof. *Step 1:* We first reduce our problem (5) to the sequence spaces. Note that by virtue of Theorem 2.3, the compactness of (5) is equivalent to saying that the embedding

$$\text{id} : m_{p_1 q_1}^{s_1} \rightarrow m_{p_2 q_2}^{s_2}$$

is compact. Next, we show that the weights can be "shifted" from the target space to the start space. To see this, we remark that the mapping L given by

$$(7) \quad L : \lambda_{k,n} \longrightarrow \lambda_{k,n} (1 + |k|/2 + |n|)^{s_2} \quad k \in \mathbb{Z}^d, n \in \mathbb{N}_0^d$$

provides an isometry of $m_{p_1 q_1}^{s_1}$ onto $m_{p_1 q_1}^{s_1 - s_2}$. Furthermore, L^{-1} yields an isometry of $m_{p_2 q_2}^{s_2}$ onto $m_{p_2 q_2}^{s_2}$. The situation is illustrated in the following diagram.

$$\begin{array}{ccc} m_{p_1 q_1}^{s_1} & \xrightarrow{\text{id}} & m_{p_2 q_2}^{s_2} \\ \downarrow L & & \uparrow L^{-1} \\ m_{p_1 q_1}^{s_1 - s_2} & \xrightarrow{\text{id}} & m_{p_2 q_2}^{s_2} \end{array}$$

Hence, we may consider the compactness of the embedding

$$(8) \quad \text{id} : m_{p_1 q_1}^s \rightarrow m_{p_2 q_2}^{s_2}$$

with $s = s_1 - s_2$.

Step 2: We first present the "if" part of the proof. So we assume that (6) holds. Since this condition contains two cases, the proof will be divided into two substeps.

Substep 2.1: For the case when $\max\{0, 1/q_2 - 1/q_1\} = 0$ on the right hand side of (6), i.e. $q_1 \leq q_2$, the compactness the embedding (5) is an obvious consequence of the ideal property for compact operators. We simply combine Proposition 3.1 with the known continuous embeddings between modulation spaces. The needed continuity result has been known long before. We wish to mention the pioneering work of H. Feichtinger [5],

see Proposition 6.5. For the simple proof of the necessity of (6) to have continuity of (5), we refer the reader to [6, Theorem 12.2.2].

Substep 2.2: It remains to assume that $\max\{0, 1/q_2 - 1/q_1\} = 1/q_2 - 1/q_1$, i.e. $q_1 > q_2$. We choose $s_3 \in \mathbb{R}$ such that $s_1 > s_3 > s_2$ and $s_3 - s_2 > d/q_2 - d/q_1$. We factorize the embedding (5) in the following way

$$M_{p_1 q_1}^{s_1} \xrightarrow{\text{id}_1} M_{p_1 q_1}^{s_3} \xrightarrow{\text{id}_2} M_{p_1 q_2}^{s_2} \xrightarrow{\text{id}_3} M_{p_2 q_2}^{s_2}.$$

Under the assumption on s_1 and s_3 and by Proposition 3.1 we have that id_1 is compact. Plainly, the identity id_3 is continuous. Thus, we are left with the task of proving the continuity of id_2 . By Theorem 2.3 we have to show that

$$(9) \quad \|\lambda |m_{p_1 q_2}^{s_2}\| \leq c \|\lambda |m_{p_1 q_1}^{s_3}\|.$$

By the triangle inequality applied to the left hand side of (9), we divide it into two summands in the following way,

$$(10) \quad \begin{aligned} & \left(\sum_{n \in \mathbb{N}_0^d} \left(\sum_{k \in \mathbb{Z}^d} \left(1 + \frac{|k|}{2} + |n| \right)^{s_2 p_1} |\lambda_{k,n}|^{p_1} \right)^{q_2/p_1} \right)^{1/q_2} \\ & \leq \left(\sum_{n \in \mathbb{N}_0^d} \left(\sum_{|k| < |n|} |n|^{s_2 p_1} |\lambda_{k,n}|^{p_1} \right)^{q_2/p_1} \right)^{1/q_2} \\ & \quad + \left(\sum_{n \in \mathbb{N}_0^d} \left(\sum_{|k| \geq |n|} |k|^{s_2 p_1} |\lambda_{k,n}|^{p_1} \right)^{q_2/p_1} \right)^{1/q_2}. \end{aligned}$$

We put

$$a_n = \left(\sum_{|k| < |n|} |\lambda_{k,n}|^{p_1} \right)^{1/p_1}$$

Hence, for the first summand in (10), by applying the Hölder inequality, we obtain

$$\begin{aligned} & \left(\sum_{n \in \mathbb{N}_0^d} |n|^{s_2 q_2} \left(\sum_{|k| < |n|} |\lambda_{k,n}|^{p_1} \right)^{q_2/p_1} \right)^{1/q_2} = \left(\sum_{n \in \mathbb{N}_0^d} |n|^{s_2 q_2} a_n^{q_2} \right)^{1/q_2} \\ & = \left(\sum_{n \in \mathbb{N}_0^d} |n|^{s_3 q_2} a_n^{q_2} |n|^{(s_2 - s_3) q_2} \right)^{1/q_2} \end{aligned}$$

$$\leq \left(\sum_{n \in \mathbb{N}_0^d} |n|^{s_3 p_1} a_n^{q_1} \right)^{1/q_1} \left(\sum_{n \in \mathbb{N}_0^d} |n|^{(s_2 - s_3) q_2 q_1 / (q_1 - q_2)} \right)^{\frac{q_1 - q_2}{q_1 q_2}} \leq c \|\lambda\| m_{p_1 q_1}^{s_3}$$

The last series is convergent under the assumption $s_3 - s_2 > d/q_2 - d/q_1$. For the second summand in (10) we set

$$b_n = \left(\sum_{|k| \geq |n|} |k|^{s_2 p_1} |\lambda_{k,n}|^{p_1} \right)^{1/p_1} \quad \text{and} \quad c_n = \left(\sum_{|k| \geq |n|} |k|^{s_3 p_1} |\lambda_{k,n}|^{p_1} \right)^{1/p_1}.$$

We shortly discuss the behaviour of the quotient b_n/c_n . We get

$$\begin{aligned} b_n^{p_1} &= \sum_{|k| \geq |n|} |k|^{s_2 p_1} |\lambda_{k,n}|^{p_1} \\ &= \sum_{|k| \geq |n|} |k|^{(s_2 - s_3) p_1} |k|^{s_3 p_1} |\lambda_{k,n}|^{p_1} \\ &\leq \sum_{|k| \geq |n|} |n|^{(s_2 - s_3) p_1} |k|^{s_3 p_1} |\lambda_{k,n}|^{p_1} \\ &= |n|^{(s_2 - s_3) p_1} \sum_{|k| \geq |n|} |k|^{s_3 p_1} |\lambda_{k,n}|^{p_1} \\ &\leq |n|^{(s_2 - s_3) p_1} c_n^{p_1} \end{aligned}$$

By virtue of the above estimate and by the Hölder inequality we obtain.

$$\begin{aligned} \left(\sum_{n \in \mathbb{N}_0^d} b_n^{q_2} \right)^{1/q_2} &= \left(\sum_{n \in \mathbb{N}_0^d} b_n^{q_2} c_n^{-q_2} c_n^{q_2} \right)^{1/q_2} \\ &\leq \left(\sum_{n \in \mathbb{N}_0^d} c_n^{q_1} \right)^{1/q_1} \left(\sum_{n \in \mathbb{N}_0^d} \left(\frac{b_n}{c_n} \right)^{\frac{q_1 q_2}{q_1 - q_2}} \right)^{\frac{q_1 - q_2}{q_1 q_2}} \\ &\leq \|\lambda\| m_{p_1 q_1}^{s_3} \left(\sum_{n \in \mathbb{N}_0^d} |n|^{(s_2 - s_3) \frac{q_1 q_2}{q_1 - q_2}} \right)^{\frac{q_1 - q_2}{q_1 q_2}} \leq c \|\lambda\| m_{p_1 q_1}^{s_3}. \end{aligned}$$

Again taking into account our assumption $s_3 - s_2 > d/q_2 - d/q_1$, we conclude that the last series above is convergent. This finishes the proof of the first implication.

Step 3: We are now concerned with the converse implication, i.e. that (5) implies (6). We assume that the embedding (5) is compact. Then in particular the embedding (5) is continuous which means that there is a

constant $c > 0$ such that

$$(11) \quad \left(\sum_{n \in \mathbb{N}_0^d} \left(\sum_{k \in \mathbb{Z}^d} |\lambda_{k,n}|^{p_2} \right)^{q_2/p_2} \right)^{1/q_2} \\ \leq c \left(\sum_{n \in \mathbb{N}_0^d} \left(\sum_{k \in \mathbb{Z}^d} (1 + |k|/2 + |n|)^{s p_1} |\lambda_{k,n}|^{p_1} \right)^{q_1/p_1} \right)^{1/q_1}$$

for all sequences $(\lambda_{k,n}) \in m_{p_1 q_1}^s$. Define the sequence

$$(12) \quad \lambda_{0,n} := \begin{cases} \alpha_j 2^{-jd/q_2}, & \text{if } |n| \leq 2^j \\ 0 & \text{otherwise.} \end{cases}$$

After plugging this sequence into (11), we conclude that

$$\left(\sum_{j=0}^{\infty} |\alpha_j|^{q_2} \right)^{1/q_2} \leq c \left(\sum_{j=0}^{\infty} 2^{jd q_1 (1/q_1 + s/d - 1/q_2)} |\alpha_j|^{q_1} \right)^{1/q_1}.$$

Thus, taking a quick look at the exponent on the right hand side of the last inequality, there is no compact embedding if

$$s_1 - s_2 \leq d/q_2 - d/q_1.$$

The second possible assertion in (6), i.e. $s_1 - s_2 > 0$, may be verified by applying the same argument as we use above with the modified sequence given by

$$\lambda_{0,n} := \begin{cases} \alpha_j 2^{-jd/q_1}, & \text{if } |n| \leq 2^j \\ 0 & \text{otherwise.} \end{cases}$$

We leave the details to the reader. $\square$

4. Entropy and approximation numbers of related embeddings

4.1. Entropy numbers. Let X and Y be Banach spaces and let $T : X \rightarrow Y$ be a bounded linear operator. Let $B_X := \{x \in X : \|x\|_X \leq 1\}$ be the unit ball in the Banach space X . Recall that an operator $T \in L(X, Y)$ is compact if for any given $\varepsilon > 0$ the image of the unit ball B_X can be covered by finitely many balls in Y of radius ε .

Definition 4.1. Let X, Y be Banach spaces and let $T \in L(X, Y)$. Then for all $k \in \mathbb{N}$, the k th dyadic entropy number $e_k(T)$ of T is defined by

$$e_k(T) = \inf \left\{ \varepsilon > 0 : T(B_X) \subset \bigcup_{j=1}^{2^{k-1}} (y_j + \varepsilon B_Y) \text{ for some } y_1, \dots, y_{2^{k-1}} \in Y \right\},$$

where B_X and B_Y denote the unit balls in X and Y , respectively.

These numbers have various elementary properties summarized in the following lemma.

Lemma 4.2. Let X, Y and Z be Banach spaces, let $S, T \in L(X, Y)$ and $R \in L(Y, Z)$.

- (i) **(Monotonicity):** $\|T\| = e_1(T) \geq e_2(T) \geq \dots \geq 0$.
- (ii) **(Additivity):** For all $j, k \in \mathbb{N}$

$$e_{j+k-1}(S+T) \leq e_j(S) + e_k(T).$$

- (iii) **(Multiplicativity):** For all $j, k \in \mathbb{N}$

$$e_{j+k-1}(RT) \leq e_j(R)e_k(T).$$

- (iv) **(Compactness):** T is compact if, and only if, $\lim_{k \rightarrow \infty} e_k(T) = 0$.

In view of the last statement in the above lemma, entropy numbers may be seen as a quantification of the notion of compactness. Instead of identities between weighted sequence spaces we may equivalently work with diagonal operators between the unweighted ones. The following estimates of entropy numbers of diagonal operator is due to B. Carl, see [2].

Proposition 4.3. Let $1 < p_1, p_2 \leq \infty$ and suppose that $\alpha > \max(1/p_2 - 1/p_1, 0)$. Then the entropy numbers of the diagonal operator $D_\sigma : \ell_{p_1} \rightarrow \ell_{p_2}$ generated by sequences $\sigma = (\sigma_n)_{n \in \mathbb{N}}$ with $\sigma_n = n^{-\alpha}$, defined by

$$D_\sigma x = (\sigma_n x_n), \quad x = (x_n)_{n \in \mathbb{N}},$$

behave asymptotically like

$$(13) \quad e_k(D_\sigma : \ell_{p_1} \rightarrow \ell_{p_2}) \sim k^{1/p_2 - 1/p_1 - \alpha}.$$

Below, we state the main result of the paper. The asymptotic formula for the behaviour of entropy numbers of embedding between weighted modulation spaces in the case when $p = q$ is proven.

Theorem 4.4. *Let $1 < p_1 < p_2 \leq \infty$ and $s_1 - s_2 > 0$ with $s_1, s_2 \in \mathbb{R}$. Then the entropy numbers of the compact embedding*

$$(14) \quad \text{id} : M_{p_1}^{s_1}(\mathbb{R}^d) \rightarrow M_{p_2}^{s_2}(\mathbb{R}^d)$$

satisfy

$$(15) \quad e_k(\text{id} : M_{p_1}^{s_1}(\mathbb{R}^d) \rightarrow M_{p_2}^{s_2}(\mathbb{R}^d)) \sim k^{1/p_2 - 1/p_1 - (s_1 - s_2)/2d}.$$

Proof. By virtue of Theorem 2.3 and basic properties of the entropy numbers, described in Lemma 4.2, we can reduce our problem to the estimate of the entropy numbers of embeddings between related sequence spaces m_p^s . In other words, we have

$$e_k(\text{id} : M_{p_1}^{s_1}(\mathbb{R}^d) \rightarrow M_{p_2}^{s_2}(\mathbb{R}^d)) \sim e_k(\text{id} : m_{p_1}^{s_1} \rightarrow m_{p_2}^{s_2})$$

with equivalence constant independent of $k \in \mathbb{N}$. Taking into account the definition of the entropy numbers together with the properties of isometries L and L^{-1} described in the proof of Theorem 3.2, Step 1, we have

$$(16) \quad e_k(\text{id} : m_{p_1}^{s_1} \rightarrow m_{p_2}^{s_2}) = e_k(\text{id} : m_{p_1}^{s_1 - s_2} \rightarrow m_{p_2}).$$

Once again put $s = s_1 - s_2$ and assume $s > 0$. We translate the identity operator $\text{id} : m_{p_1}^s \rightarrow m_{p_2}$ into a suitable diagonal operator. To do this we define the diagonal operator

$$D_\tau : m_{p_1} = \ell_{p_1}(\mathbb{Z}^d \times \mathbb{N}_0^d) \rightarrow \ell_{p_2}(\mathbb{Z}^d \times \mathbb{N}_0^d) = m_{p_2}$$

which is generated by the sequence

$$(17) \quad \tau_{k,n} = (1 + |k|/2 + |n|)^{-s} \sim (1 + |k|^2/4 + |n|^2)^{-s/2} \sim \max\{1, |k|^2, |n|^2\}^{-s/2}.$$

To use the information about entropy numbers of diagonal operators contained in Proposition 4.3, we are looking for a diagonal sequence $\sigma = (\sigma_h)$, $h \in \mathbb{N}_0$ such that for the corresponding diagonal operator the following diagram commutes.

$$\begin{array}{ccc} \ell_{p_1}(\mathbb{Z}^d \times \mathbb{N}_0^d) & \xrightarrow{D_\tau} & \ell_{p_2}(\mathbb{Z}^d \times \mathbb{N}_0^d) \\ \downarrow \kappa & & \uparrow \kappa^{-1} \\ \ell_{p_1}(\mathbb{N}_0) & \xrightarrow{D_\sigma} & \ell_{p_2}(\mathbb{N}_0) \end{array}$$

Here κ stands for a bijection of $\mathbb{Z}^d \times \mathbb{N}_0^d$ onto $\mathbb{N}_0$. At the core of our proof is a simple enumeration argument. For $h \in \mathbb{N}_0$ we define the index sets

$$I_h = \left\{ (k, n) \in \mathbb{Z}^d \times \mathbb{N}_0^d : 2^h \leq \max\{1, |k|, |n|\} \leq 2^{h+1} \right\}.$$

Let us compute the size of the set I_h . We obtain

$$\begin{aligned} |I_h| &= \left| \left\{ (k, n) : 2^h \leq \max\{1, |k|, |n|\} < 2^{h+1} \right\} \right| \\ &\sim |\{l \in \mathbb{Z}^{2d} : 2^h \leq \left(\sum_{i=1}^{2d} l_i^2 \right)^{1/2} < 2^{h+1}\}| \\ &\sim \text{vol}\left(B(0, 2^{h+1}) \setminus B(0, 2^h)\right) \sim \text{vol}(B(0, 2^h)) \sim 2^{h2d} \sim 4^{hd} \end{aligned}$$

Here $B(0, 2^h)$ stands for the ball in $\mathbb{R}^{2d}$ centered at the origin with the radius 2^h . Hence, we have approximately 4^{hd} points in I_h . Moreover, we have

$$h \sim 1 + 4^d + 4^{2d} + \dots + 4^{hd} \sim \frac{4^{(h+1)d}}{4^d} \sim 4^{hd}$$

Finally we have that $\sigma_h \sim h^{-s/2d}$. By virtue of Proposition 4.3 we obtain

$$e_k(D_\tau) \sim k^{1/p_2 - 1/p_1 - (s_1 - s_2)/(2d)}.$$

This finishes the proof. $\square$

4.2. Approximation numbers. In this subsection we present the corresponding result on the behaviour of approximation numbers of compact embedding (1). We start by recalling needed definitions and facts on approximation numbers.

Definition 4.5. Let X, Y be Banach spaces and let $T \in L(X, Y)$. Then for all $k \in \mathbb{N}$, the k th approximation number $a_k(T)$ of T is defined by

$$a_k(T) = \inf \left\{ \|T - A\| : A \in L(X, Y), \text{rank}(A) < k \right\},$$

where $\text{rank}(T)$ denotes the dimension of the range $T(X) = \{T(x) : x \in X\}$.

In the following lemma we present some well-known properties of approximation numbers, see [4, Section 1.3].

Lemma 4.6. Let X, Y and Z be Banach spaces and let $S, T \in L(X, Y)$ and $R \in L(Y, Z)$.

(i) **(Monotonicity):** $\|T\| = a_1(T) \geq a_2(T) \geq \dots \geq 0$.

(ii) **(Additivity):** For all $j, k \in \mathbb{N}$

$$a_{j+k-1}(S+T) \leq a_j(S) + a_k(T).$$

(iii) **(Multiplicativity):** For all $j, k \in \mathbb{N}$

$$a_{j+k-1}(RT) \leq a_j(R)a_k(T).$$

(iv) **(Rank property):**

$$a_n(T) = 0 \text{ if, and only if, } \text{rank } T < n.$$

The subsequent result on approximation numbers of diagonal operators may be found in [9, Proposition 1].

Proposition 4.7. Let $1 \leq p_0 < p_1 \leq \infty$, $(p_0, p_1) \neq (1, \infty)$, and α be a positive real number. We put $\frac{1}{t} = \frac{1}{\min\{p_0', p_1\}}$. Let D_α be a diagonal operator generated by the sequence $\sigma_n = n^{-\alpha}$. Then for all $k \in \mathbb{N}$ the estimate

$$a_k(D_\alpha : \ell_{p_0} \hookrightarrow \ell_{p_1}) \sim k^{-\beta}$$

holds, where

$$\beta = \begin{cases} \alpha & \text{if } 1 \leq p_0 < p_1 \leq 2 \text{ or } 2 \leq p_0 < p_1 \leq \infty \\ \alpha + \frac{1}{2} - \frac{1}{t} & \text{if } 1 \leq p_0 < 2 < p_1 \leq \infty, (p_0, p_1) \neq (1, \infty) \\ \text{and } \alpha > \frac{1}{t}, \\ \alpha \frac{t}{2} & \text{if } 1 \leq p_0 < 2 < p_1 \leq \infty, (p_0, p_1) \neq (1, \infty) \\ \text{and } \alpha \leq \frac{1}{t}. \end{cases}$$

The main result in this subsection reads as follows.

Proposition 4.8. Let $1 \leq p_1 < p_2 \leq \infty$ and $s_1 - s_2 > 0$ for $s_1, s_2 \in \mathbb{R}$. Then for the compact embedding

$$\text{id} : M_{p_1}^{s_1}(\mathbb{R}^d) \hookrightarrow M_{p_2}^{s_2}(\mathbb{R}^d)$$

holds

$$a_k(\text{id}) \sim k^{-\beta},$$

where

$$\beta = \begin{cases} \frac{s_1 - s_2}{2d} & \text{if } 1 \leq p_1 < p_2 \leq 2 \text{ or } 2 \leq p_0 < p_1 \leq \infty \\ \frac{s_1 - s_2}{2d} + \frac{1}{2} - \frac{1}{t} & \text{if } 1 \leq p_1 < 2 < p_2 \leq \infty, (p_1, p_2) \neq (1, \infty) \\ & \text{and } \frac{s_1 - s_2}{2d} > \frac{1}{t}, \\ \frac{(s_1 - s_2)t}{4d} & \text{if } 1 \leq p_1 < 2 < p_2 \leq \infty, (p_1, p_2) \neq (1, \infty) \\ & \text{and } \frac{s_1 - s_2}{2d} \leq \frac{1}{t}. \end{cases}$$

Proof. The proof follows easily from Proposition 4.7 and the proof of the Theorem 4.4. $\square$

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Mathematisches Institut, FSU Jena
Ernst-Abbe-Platz 2
D-07737 JENA
Germany
(E-mail : hinrichs@minet.uni-jena.de)

University of Bremen
Center of Industrial Mathematics
Postfach 33 0440
D-28334 Bremen
Germany
(E-mail : iwona@math.uni-bremen.de)

Faculty of Mathematics
University of Vienna
Nordbergstrasse 15
A-1090 Wien
Austria
(E-mail : mariusz.piotrowski@univie.ac.at)

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