Lacunary series in Q_K type spaces

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Abstract. Under mild conditions on the weight function K we characterize lacunary series in $Q_K(p, q)$ spaces, where $Q_K(p, q)$ spaces are Q_K type spaces of functions analytic in the unit disk.

1. Introduction

The Green function in the unit disk Δ with singularity at $a \in \Delta$ is given by $g(z, a) = -\log |\varphi_a(z)|$, where $\varphi_a(z) = (a - z)/(1 - \bar{a}z)$ is a Möbius transformation of Δ . For $0 < r < 1$, let $\Delta(a, r) = \{z \in \Delta : |\varphi_a(z)| < r\}$ be the pseudo-hyperbolic disk with center $a \in \Delta$ and radius r .

Let $K : [0, \infty) \rightarrow [0, \infty)$ be a right-continuous and nondecreasing function. For $0 < p < \infty, -2 < q < \infty$, A function f analytic in Δ belongs to the space $Q_K(p, q)$ if

$$\|f\|_K^p = \sup_{a \in \Delta} \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \infty,$$

where $dA(z)$ is the Euclidean area element on Δ .

An analytic function f belongs to $Q_{K,0}(p, q)$ if $f \in Q_K(p, q)$ and satisfies

$$\lim_{|a| \rightarrow 1} \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) = 0.$$

We assume throughout the paper that

$$(1.1) \quad \int_0^1 (1 - r^2)^q K\left(\log \frac{1}{r}\right) r dr < \infty.$$

Otherwise, $Q_K(p, q)$ only contains constant functions (cf. [7]). It is easy to see that $Q_K = Q_K(2, 0)$. If $K(t) = t^s$, then $Q_K(p, q) = F(p, q, s)$. The space $F(p, q, s)$ was introduced and investigated by R. Zhao in [10].

In order to state the main theorem in the paper, we need an auxiliary function φ_K defined by

$$\varphi_K(s) = \sup_{0 \leq t \leq 1} K(st)/K(t), \quad 0 < s < \infty.$$

The following condition has played a crucial role in the study of Q_K type spaces during the last few years:

$$(1.2) \quad \int_1^\infty \frac{\varphi_K(s)}{s^2} ds < \infty.$$

The characterization of the lacunary series in Q_K spaces was given in [8]. In this paper we obtain a characterization of the lacunary series in general Q_K type spaces, that is the following result:

Theorem 1.1. *Theorem 1.1 Let $0 < p < \infty$, $-1 < q < \infty$. If K satisfies condition (1.2) and $f(z) = \sum_{k=0}^\infty a_k z^{n_k} \in H(\Delta)$ has Hadamard gaps, that is, $n_{k+1}/n_k \geq \lambda > 1$ for all $k \in \mathbb{N}$, then the following statements are equivalent:*

- (1) $f \in Q_K(p, q)$;
- (2) $f \in Q_{K,0}(p, q)$;
- (3) $\sum_{k=0}^\infty n_k^{p-q-1} |a_k|^p K\left(\frac{1}{n_k}\right) < \infty$.

2. Preliminaries on weight function

In what follows we say $f \lesssim g$ (for two functions f and g) if there is a constant C such that $f \leq Cg$. We say $f \approx g$ (that is, f is comparable with g) whenever $g \lesssim f \lesssim g$.

Before proving the Theorem 1.1, we need some lemmas.

Lemma 2.1. *If K satisfies condition (1.2), then K has the following properties:*

- (1) $K(2t) \approx K(t)$ on $(0, \infty)$;
- (2) K is concave on $(0, \infty)$;
- (3) $K(t)/t$ is non-increasing on $(0, \infty)$.

The above results can be found in Proposition 4 of [8].

Lemma 2.2. *Let $\alpha \geq 1, \beta > 0$. If K satisfies (1.2), then*

$$\int_0^1 r^{\alpha-1} (\log \frac{1}{r})^{\beta-1} K(\log \frac{1}{r}) dr \approx (\frac{1}{\alpha})^\beta K(\frac{1}{\alpha}).$$

Proof. Let

$$I = \int_0^1 r^{\alpha-1} (\log \frac{1}{r})^{\beta-1} K(\log \frac{1}{r}) dr.$$

By a change of variables we have

$$I = \int_0^\infty e^{-\alpha t} t^{\beta-1} K(t) dt.$$

Denote

$$I_1 = \int_{\frac{1}{\alpha}}^\infty e^{-\alpha t} t^{\beta-1} K(t) dt$$

and

$$I_2 = \int_0^{\frac{1}{\alpha}} e^{-\alpha t} t^{\beta-1} K(t) dt.$$

By Lemma 2.1, we have

$$(2.1) \quad I_1 \leq \frac{K(\frac{1}{\alpha})}{\frac{1}{\alpha}} \int_{\frac{1}{\alpha}}^\infty e^{-\alpha t} t^\beta dt = K(\frac{1}{\alpha}) (\frac{1}{\alpha})^\beta \int_1^\infty e^{-s} s^\beta ds.$$

Since $K(t)$ is non-decreasing on $(0, \infty)$, we have

$$(2.2) \quad I_2 \leq K(\frac{1}{\alpha}) \int_0^{\frac{1}{\alpha}} e^{-\alpha t} t^{\beta-1} dt \leq K(\frac{1}{\alpha}) (\frac{1}{\alpha})^\beta \int_0^1 e^{-s} s^{\beta-1} ds.$$

By (2.1) and (2.2), we have

$$(2.3) \quad \int_0^1 r^{\alpha-1} (\log \frac{1}{r})^{\beta-1} K(\log \frac{1}{r}) dr \leq C(\beta) (\frac{1}{\alpha})^\beta K(\frac{1}{\alpha}),$$

where $C(\beta)$ is a constant which only depends on β .

On the other hand, recall that $K(t)$ is non-decreasing on $(0, \infty)$. Then

$$I \geq \int_{\frac{1}{\alpha}}^{\infty} e^{-\alpha t} t^{\beta-1} K(t) dt \geq K\left(\frac{1}{\alpha}\right) \int_{\frac{1}{\alpha}}^{\infty} e^{-\alpha t} t^{\beta-1} K(t) dt = C(\beta) \left(\frac{1}{\alpha}\right)^{\beta} K\left(\frac{1}{\alpha}\right).$$

We get the desired result and complete the proof. $\square$

Lemma 2.3. *Let $0 < \gamma \leq 1$ and $\eta(r) = \sum_{n=0}^{\infty} 2^{n\gamma} r^{2^n}$, $0 \leq r < 1$, then*

$$\eta(r) \leq 2\Gamma(\gamma) \left(\log \frac{1}{r}\right)^{-\gamma}.$$

The above result can be found in [5].

Lemma 2.4.

$$\frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta}, a) d\theta = \begin{cases} \log \frac{1}{|a|}, & 0 < r \leq |a|, \\ \log \frac{1}{r}, & |a| < r \leq 1. \end{cases}$$

By Jensen's formula, we can directly obtain the above result.

Theorem 2.5. *Let $f(x) = \sum_{n=1}^{\infty} a_n x^n$ with $a_n \geq 0$. If K satisfies condition (1.2) and $\alpha > 0, p > 0$, then*

$$(2.4) \quad \int_0^1 (1-x)^{\alpha-1} f(x)^p K\left(\log \frac{1}{x}\right) dx \approx \sum_{n=0}^{\infty} 2^{-n\alpha} t_n^p K\left(\frac{1}{2^n}\right),$$

where $I_n = \{k : 2^n \leq k < 2^{n+1}\}$, $t_n = \sum_{k \in I_n} a_k$, $n \in \mathbb{N}$.

Proof. Let $r_n = 1 - 2^{-n}$, $n = 1, 2, \dots$, then $r_n^{2^n-1} \geq 1/e$. By Lemma 2.1 we have that

$$\begin{aligned} & \int_0^1 \left(\sum_{n=1}^{\infty} a_n r^n\right)^p (1-r)^{\alpha-1} K\left(\log \frac{1}{r}\right) dr \\ & \geq \int_{\frac{1}{2}}^1 \left(\sum_{n=0}^{\infty} t_n r^{2^{n+1}-1}\right)^p (1-r)^{\alpha-1} K\left(\log \frac{1}{r}\right) dr \\ & \gtrsim \sum_{n=0}^{\infty} t_n^p (r_{n+1}^{2^{n+1}-1})^p \int_{r_{n+1}}^{r_{n+2}} (1-r)^{\alpha-1} K\left(\log \frac{1}{r}\right) dr \\ & \gtrsim \sum_{n=0}^{\infty} K\left(\frac{1}{2^n}\right) t_n^p 2^{-n\alpha}. \end{aligned}$$

The last inequality holds because of $\log(1/r) \geq 1-r$.

Conversely, we firstly suppose that $p > 1$. Let $\gamma = \min\{1, \alpha/p\}$ and $\eta(r) = \sum_{k=0}^{\infty} 2^{k\gamma} r^{2k}$, $0 \leq r < 1$. Then by Jensen's inequality we have

$$\left(\sum_{n=0}^{\infty} a_n r^n \right)^p \leq \left(\sum_{n=0}^{\infty} t_n r^{2^n} \right)^p \lesssim (\eta(r))^{p-1} \sum_{n=0}^{\infty} 2^{n\gamma(1-p)} r^{2^n} |t_n|^p.$$

Now it follows from Lemma 2.1, Lemma 2.2 and Lemma 2.3, the inequality $r(1-r)^{\alpha-1} \leq (\log(1/r))^{\alpha-1}$, $0 \leq r < 1$. A direct calculation shows that

$$(2.5) \quad \int_0^1 \left(\sum_{n=1}^{\infty} a_n r^n \right)^p (1-r)^{\alpha-1} K\left(\log \frac{1}{r}\right) dr \lesssim \sum_{n=0}^{\infty} 2^{-n\alpha} K\left(\frac{1}{2^n}\right) \left(\sum_{I_n} a_k \right)^p.$$

Secondly, suppose that $p = 1$. By Lemma 2.2 it is easy to see that

$$(2.6) \quad \begin{aligned} & \int_0^1 \left(\sum_{n=1}^{\infty} a_n r^n \right) (1-r)^{\alpha-1} K\left(\log \frac{1}{r}\right) dr \\ & \leq \int_0^1 \left(\sum_{n=0}^{\infty} t_n r^{2^n} \right) (1-r)^{\alpha-1} K\left(\log \frac{1}{r}\right) dr \\ & \leq \sum_{n=0}^{\infty} t_n \int_0^1 r^{2^n} (1-r)^{\alpha-1} K\left(\log \frac{1}{r}\right) dr \\ & \leq \sum_{n=0}^{\infty} 2^{-n\alpha} t_n K\left(\frac{1}{2^n}\right). \end{aligned}$$

Lastly, if $p < 1$ we have

$$(2.7) \quad \begin{aligned} & \int_0^1 \left(\sum_{n=1}^{\infty} a_n r^n \right)^p (1-r)^{\alpha-1} K\left(\log \frac{1}{r}\right) dr \\ & \leq \int_0^1 \left(\sum_{n=0}^{\infty} t_n r^{2^n} \right)^p (1-r)^{\alpha-1} K\left(\log \frac{1}{r}\right) dr \\ & \leq \sum_{n=0}^{\infty} t_n^p \int_0^1 r^{p2^n} (1-r)^{\alpha-1} K\left(\log \frac{1}{r}\right) dr \\ & \lesssim \sum_{n=0}^{\infty} 2^{-n\alpha} t_n^p K\left(\frac{1}{2^n}\right). \end{aligned}$$

From (2.5), (2.6) and (2.7) we obtain the desired result and the proof is complete. $\square$

3. Lacunary series in Q_K type spaces

Theorem 3.1. *Let $0 < p < \infty, -1 < q < \infty$. If K satisfies condition (1.2) and $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ has Hadamard gaps, that is, $n_{k+1}/n_k \geq \lambda > 1$ for all $k > 0$, then f belongs to $Q_K(p, q)$ if and only if*

$$(3.1) \quad \sum_{k=1}^{\infty} n_k^{p-q-1} |a_k|^p K\left(\frac{1}{n_k}\right) < \infty.$$

Proof. Suppose that $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ is a lacunary series in $Q_K(p, q)$. Without loss of generality, we assume that $n_k \geq 1$. If f has Hadamard gaps, then $M_p(r, f') \approx M_2(r, f')$ (cf. [12]), where

$$M_p^p(r, f') = \frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^p d\theta.$$

Since $f \in Q_K(p, q)$, by Theorem 2.5 we have

$$\begin{aligned} \infty &> \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q K\left(\log \frac{1}{|z|}\right) dA(z) \\ &= \int_0^1 M_p^p(r, f') (1 - r^2)^q K\left(\log \frac{1}{r}\right) r dr \\ &\gtrsim \int_0^1 \left(\sum_{k=1}^{\infty} n_k^2 |a_k|^2 r^{2(n_k-1)} \right)^{\frac{p}{2}} (1 - r^2)^q K\left(\log \frac{1}{r}\right) r dr \\ &\gtrsim \sum_{k=1}^{\infty} 2^{-k(q+1)} t_k^{\frac{p}{2}} K\left(\frac{1}{2^k}\right), \end{aligned}$$

where $t_k = \sum_{n_j \in I_k} n_j^2 |a_j|^2$. The Taylor series of f has at most $[\log_{\lambda} 2] + 1$ terms $a_j z^{n_j}$ when $n_j \in I_k$ for $k \geq 1$. By Hölder's inequality, we note that $t_k^{\frac{p}{2}} \gtrsim \sum_{n_j \in I_k} n_j^p |a_j|^p$. Then

$$\sum_{k=0}^{\infty} n_k^{p-q-1} |a_k|^p K\left(\frac{1}{2^k}\right) \lesssim \sum_{k=0}^{\infty} 2^{-k(q+1)} \left(\sum_{n_j \in I_k} n_j^p |a_j|^p \right) K\left(\frac{1}{2^k}\right) < \infty.$$

Next suppose that condition (3.1) holds. Since K satisfies condition (1.2), K is concave by Lemma 2.1. By Jensen's inequality and Lemma 2.4

we obtain that

$$\frac{1}{2\pi} \int_0^{2\pi} K(g(re^{i\theta}, a) d\theta \leq K \left(\frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta}, a) d\theta \right) \leq K \left(\log \frac{1}{r} \right).$$

Given $a \in \Delta$, by Theorem 2.5 we have

$$\begin{aligned} I(a) &= \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ &\leq \int_0^1 \left(\sum_{k=1}^{\infty} n_k |a_k| r^{n_k-1} \right)^p (1 - r^2)^q \left\{ \frac{1}{2\pi} \int_0^{2\pi} K(g(re^{i\theta}, a)) d\theta \right\} r dr \\ &\leq \int_0^1 \left(\sum_{k=1}^{\infty} n_k |a_k| r^{n_k-1} \right)^p (1 - r^2)^q K \left(\log \frac{1}{r} \right) r dr \\ &\lesssim \sum_{n=0}^{\infty} 2^{-n(1+q)} t_n^p K \left(\frac{1}{2^n} \right) \\ &\lesssim \sum_{k=1}^{\infty} n_k^{p-q-1} |a_k|^p K \left(\frac{1}{n_k} \right) \\ &< \infty, \end{aligned}$$

where $t_n = \sum_{n_k \in I_n} n_k |a_k|$. The proof is complete. $\square$

Theorem 3.2. *Let $0 < p < \infty$, $-1 < q < \infty$. If K satisfies condition (1.2) and $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ has Hadamard gaps, that is, $n_{k+1}/n_k \geq \lambda > 1$ for all $k > 0$, then $f \in Q_K(p, q)$ if and only if $f \in Q_{K,0}(p, q)$.*

Proof. In fact, sufficiency is obvious because of $Q_{K,0}(p, q) \subset Q_K(p, q)$. Now we will prove necessity of Theorem 3.2. Suppose that the lacunary series f belongs to $Q_K(p, q)$. We must show that $I(a) \rightarrow 0$ as $|a| \rightarrow 1$, where

$$I(a) = \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z).$$

From the proof of Theorem 3.1, we know that $f \in Q_K(p, q)$ implies that

$$\int_0^1 \left(\sum_{k=1}^{\infty} n_k |a_k| r^{n_k-1} \right)^p (1 - r)^q K \left(\log \frac{1}{r} \right) r dr < \infty.$$

For any $\epsilon > 0$, there is a $\delta \in (0, 1)$ such that

$$\int_{\delta}^1 \left(\sum_{k=1}^{\infty} n_k |a_k| r^{n_k-1} \right)^p (1 - r)^q K \left(\log \frac{1}{r} \right) dr < \epsilon.$$

We may as well assume that $K(0+0) = 0$ if K satisfies condition (1.2). Then we choose a such that $1 > |a| > \delta$. By Lemma 2.4 and Theorem 3.1 we have

$$\begin{aligned}
& \int_0^\delta \left(\sum_{k=1}^\infty n_k |a_k| r^{n_k-1} \right)^p (1-r)^q r dr \int_0^{2\pi} K(g(z, a)) d\theta \\
& \leq \int_0^\delta \left(\sum_{k=1}^\infty n_k |a_k| r^{n_k-1} \right)^p (1-r)^q K\left(\log \frac{1}{|a|}\right) dr \\
& = K\left(\log \frac{1}{|a|}\right) \int_0^\delta \left(\sum_{k=1}^\infty n_k |a_k| r^{n_k-1} \right)^p (1-r)^q dr \\
& \leq \frac{K\left(\log \frac{1}{|a|}\right)}{K\left(\log \frac{1}{\delta}\right)} \int_0^\delta \left(\sum_{k=1}^\infty n_k |a_k| r^{n_k-1} \right)^p (1-r)^q K\left(\log \frac{1}{r}\right) dr \\
& \leq \frac{K\left(\log \frac{1}{|a|}\right)}{K\left(\log \frac{1}{\delta}\right)} \sum_{k=1}^\infty n_k^{p-q-1} |a_k|^p K\left(\frac{1}{n_k}\right).
\end{aligned}$$

Since ϵ is arbitrary, we conclude that $I(a) \rightarrow 0$ as $|a| \rightarrow 1$. So $f \in Q_{K,0}(p, q)$ and the proof is complete. $\square$

We have obtained Theorem 1.1 from Theorem 3.1 and 3.2. Two special cases are worth mentioning. If $p = 2, q = 0$, we get a characterization of lacunary series in Q_K spaces which was given in [8].

Corollary 3.3. *If K satisfies condition (1.2) and $f(z) = \sum_{k=0}^\infty a_k z^{n_k} \in H(\Delta)$ has Hadamard gaps, that is, $n_{k+1}/n_k \geq \lambda > 1$ for all $k > 0$, then the following statement are equivalent:*

- (1) $f \in Q_K$;
- (2) $f \in Q_{K,0}$;
- (3) $\sum_{k=0}^\infty n_k |a_k|^2 K\left(\frac{1}{n_k}\right) < \infty$.

If we choose $K(t) = t^s$, then we have the following result which was given in [10].

Corollary 3.4. *Let $0 < p < \infty, -1 < q < \infty$ and $0 < s < 1$. If K satisfies condition (1.2) and $f(z) = \sum_{k=0}^\infty a_k z^{n_k} \in H(\Delta)$ has Hadamard gaps, that is, $n_{k+1}/n_k \geq \lambda > 1$ for all $k > 0$, then the following statement are equivalent:*

- (1) $f \in F(p, q, s)$;
- (2) $f \in F_0(p, q, s)$;
- (3) $\sum_{k=0}^\infty n_k^{p-q-s-1} |a_k|^p < \infty$.

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