Bounded holomorphic projections for exponentially decreasing weights

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Abstract. We construct generalized Bergman projections on a large class of weighted $L^\infty$–spaces. The examples include exponentially decreasing weights on the unit disc and complex plane.

1. Introduction

The classical Bergman projection $P : f \mapsto \int_D f(\zeta)(1 - z\bar{\zeta})^{-2}dA(\zeta)$, where $dA$ is the normalized 2-dimensional Lebesgue measure on the open unit disc $\mathbb{D}$ of the complex plane, is a bounded operator on $L^p(\mathbb{D}) = L^p(\mathbb{D}, dA)$ for

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$1 < p < \infty$; it projects the space onto its closed subspace of analytic functions. In the case $p = 1$, any projection

$$P_\alpha : f \mapsto (\alpha + 1) \int_{\mathbb{D}} \frac{f(\zeta)(1 - |\zeta|^2)^{\alpha}}{(1 - z\bar{\zeta})^{1+\alpha}} dA(\zeta), \quad \alpha > 0,$$

has the same role, but for $p = \infty$, no analogous bounded projection exists. See [8] for these classical facts. However, with weighted sup–norms the situation is different. It is easy to see that for every $\beta > 0$, the projection $P_\alpha$ is bounded for $\alpha + 1 > \beta > 0$ on the space $L^\infty_v(\mathbb{D})$ of measurable functions bounded with respect to the weighted sup–norm

$$\|f\|_v := \text{ess sup}_{z \in \mathbb{D}} v(z)|f(z)|, \quad \text{where } v(z) := (1 - |z|)^\beta.$$

Other Bergman–type projections and also non–radial weights were recently considered in [2]. For more results, see [1], [5], [7], [8].

Not much is known about bounded projections in the case of weighted $L^p$–norms $(\int |f|^p v^p dA)^{1/p}$, if the weight function decreases rapidly as a function of the boundary distance, e.g. $v(z) := \exp(-1/1 - |z|^2))$, $z \in \mathbb{D}$. In this case, the orthogonal projection of the space $L^2_v(\mathbb{D})$ is known to be unbounded in the spaces $L^p_v(\mathbb{D})$ for $p \neq 2$, and it seems that no bounded projection from $L^2_v(\mathbb{D})$ onto its subspace of analytic functions is known.

However, the first named author proved recently in [6] that for a large class of weights satisfying a condition $(B)$, see below, the space $H^\infty_v$ (the subspace of $L^\infty_v(\mathbb{D})$ consisting of analytic functions) is isomorphic as a Banach space to $\ell^\infty$. This implies, by [4], p.105, the existence of a bounded projection from $L^\infty_v$ onto $H^\infty_v$. The exponential weight mentioned above satisfies the condition $(B)$.

It is of course of interest to find out concrete formulas for such projections. This task is carried out in the present paper. We construct “canonical” bounded projections from $L^\infty_v$ onto the subspace $H^\infty_v$ of analytic functions for weights satisfying the condition $(B)$. It is possible to use these projections to create a satisfactory theory of Toeplitz operators on these weighted spaces, see [3].

The projections will have series representations. To describe them shortly in the case of $\mathbb{D}$, we shall associate to a given weight an increasing sequence $(s_n)_{n=1}^\infty$, $0 < s_n < 1$, and a sequence $(T_n)_{n=1}^\infty$ of finite rank operators, which are just multipliers on the sequence space of Fourier coefficients. (They essentially arise from two Cesàro summations.) If $f : \mathbb{D} \to \mathbb{C}$ is a continuous function, we form for each radius $r \in [0,1]$ the Fourier–coefficients $f_k(r)$ of
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\[ f(z) = \sum_{k=\infty}^{\infty} f_k(r)e_k(z), \]

which makes sense at least in \( L^2 \). Here \( e_k(z) := r^{|k|}e^{ik\varphi} \) for \( z = re^{i\varphi} \). The projection \( P_C \) is then formally given by

\[ P_C f(z) = \sum_{n=1}^{\infty} T_n \sum_{k \geq 0} f_k(s_n)e_k(z) \]

\[ = \sum_{k \geq 0} \left( \sum_{n=1}^{\infty} t_{nk}f_k(s_n) \right) z^k. \]

The convergence of this series will be proven below. For each fixed \( n \), there are only finitely many nonzero numbers \( t_{nk} \); they are the coefficients of the multiplier operators \( T_n \) see (6). Also, given a fixed degree \( k \), there exist at most 2 nonzero \( t_{nk} \). Actually \( 0 \leq t_{nk} \leq 1 \), and, for an increasing sequence \( (m_n)_{n=1}^{\infty} \), for a fixed \( n \), the value of \( t_{nk} \) grows linearly from 0 at \( k \equiv m_{n-1} \) to 1 at \( k \equiv m_n \), and then decreases back to 0 at \( k \equiv m_{n+1} \). This interplay of the degree \( m_n \) on one hand, and the radius \( s_n \) on the other hand, is of essential importance for the proof of the boundedness of the projection \( P_C \).

If \( f \) is just an \( L^\infty \)-function, the point evaluations of the coefficients \( f_k \) have to be replaced by integral means.

It is probably difficult to find integral kernels in terms of elementary functions for \( P_C \). One of the reasons is that the definition of the numbers \( s_n \) and \( m_n \) is not completely trivial.

2. Preliminaries, notations

We consider function spaces defined on \( \Omega \) which is either \( \mathbb{D} \) or \( \mathbb{C} \). By a weight we mean a continuous, radial function \( v : \Omega \to ]0, \infty[ \), which is also strictly decreasing as \( r := |z| \) increases. We assume that \( \lim_{r \to 1} v(r) = 0 \), if \( \Omega = \mathbb{D} \), and \( \lim_{r \to \infty} r^m v(r) = 0 \) for any \( m \geq 0 \), if \( \Omega = \mathbb{C} \). We define the spaces

\[ L_v^\infty := \left\{ f : \Omega \to \mathbb{C} \text{ measurable} \mid \|f\|_v := \text{ess sup}_{z \in \Omega} v(z)|f(z)| < \infty \right\} \]

\[ h_v := h_v^\infty := \{ f : \Omega \to \mathbb{C} \text{ harmonic} \mid \|f\|_v < \infty \} \]

\[ H_v := H_v^\infty := \{ f \in h_v \mid f \text{ holomorphic} \} \]
We also denote, for functions \( f \) on \( \mathbb{D} \) or \( \mathbb{C} \),

\[
M_\infty(f, r) := \sup_{|z|=r} |f(z)|
\]

We shall need some notations and results from [6]. In [6], the first named author showed the boundedness of the Riesz projection \( R : \sum_{k=-\infty}^{\infty} f_k e_k \mapsto \sum_{k\geq 0} f_k z^k \) from \( hv \) onto \( Hv \) provided the following condition \((B)\) holds for the weight \( v \):

**Definition 1.** The weight \( v \) satisfies the condition \((B)\), if

\[
\forall b_1 > 1 \exists b_2 > 1 \exists c > 0 \ \forall m, n > 0 \quad \left( \frac{r_m}{r_n} \right)^m \frac{v(r_m)}{v(r_n)} \leq b_1 \quad \text{and} \quad m, n, |m - n| \geq c \implies \left( \frac{r_n}{r_m} \right)^n \frac{v(r_n)}{v(r_m)} \leq b_2
\]

Note that \( m \) and \( n \) need not be integers. Here the number \( r_n > 0 \) denotes the global maximum point of the function \( r \mapsto r^n v(r) \). It is easy to see that \( r_n < r_m \) for \( n < m \) and that \( r_n \) tends to the radius of the domain, as \( n \to \infty \).

From now on we fix a weight \( v \) having the property \((B)\).

We soon define (in Theorem 1 below) an increasing sequence \((m_n)_{n=1}^{\infty}\) of real numbers. Given such a sequence, we shall define the operators \( T_n \) acting on harmonic functions \( f(z) := \sum_{k \in \mathbb{Z}} f_k e_k(z) \) by

\[
T_n f := \sum_{m_{n-1} < |k| \leq m_n} \frac{|k| - [m_{n-1}]}{[m_n] - [m_{n-1}]} f_k e_k + \sum_{m_n < |k| \leq m_{n+1}} \frac{[m_{n+1}] - |k|}{[m_{n+1}] - [m_n]} f_k e_k
\]

\[
=: \sum_{k \in \mathbb{Z}} t_{nk} f_k e_k
\]

(6)

Here \([a]\) denotes the largest integer \( \leq a \). So the operator \( T_n \) is a multiplier on the space of Fourier–coefficients. Given a sequence \((m_n)_{n=1}^{\infty}\), we also denote

\[
s_n := r_{m_n}.
\]

We collect in the following theorem everything that is needed from [6]. Recall that \( v \) is assumed to satisfy \((B)\).
Theorem 1. There are numbers $0 < m_1 < m_2 < \ldots$ and constants $d_1, d_2 > 0$ such that for any $f \in \mathcal{H}$ we have

$$d_1 \sup_n M_\infty(T_n f, s_n) v(s_n) \leq \|f\|_v \leq d_2 \sup_n M_\infty(T_n f, s_n) v(s_n)$$

and

$$d_1 M_\infty(T_n f, s_n) v(s_n) \leq \|T_n f\|_v \leq d_2 \left( \sup_k \|T_k\| \right) M_\infty(T_n f, s_n) v(s_n)$$

for all $n$.

The operators $T_n$ are uniformly bounded with respect to $M_\infty(\cdot,1)$. Finally, the Riesz projection $R : \mathcal{H} \rightarrow \mathcal{H}_v$ is bounded with respect to $\|\cdot\|_v$.

Remark 1. (i) Our operators $T_n$ are equal to $V_{m_{n+1},m_n} - V_{m_n,m_{n-1}}$ of [6], see (3.1) in the reference. They have the properties $f = \sum_n T_n f$ for every trigonometric polynomial $f$, and

$$T_n T_m = 0$$

for $|n - m| \geq 2$.

(ii) For any series $\sum_{k=0}^{\infty} f_k e_k$, $f_k \in \mathbb{C}$, for any $n$, only finitely many summands of $\sum_{k=0}^{\infty} T_n f_k e_k$ are nonzero. (Hence there are absolutely no convergence problems with the latter series.)

(iii) Theorem 1 implies that $\sum_{k=0}^{\infty} f_k e_k$ is the Taylor series (converging at least uniformly on compacta of $\mathbb{D}$) of an analytic function $f \in \mathcal{H}_v$ if the coefficients $f_k$ are such that

$$\sup_n M_\infty\left( \sum_{k=0}^{\infty} T_n f_k e_k, s_n \right) v(s_n) = \sup_n M_\infty\left( \sum_{k=0}^{\infty} t_{nk} f_k e_k, s_n \right) v(s_n) < \infty.$$  

Proof of Theorem 1. We obtain the numbers $m_n$ from Lemma 5.1 of [6]. In particular, one of the quotients

$$\left( \frac{s_n}{s_{n+1}} \right)^{m_n} \frac{v(s_n)}{v(s_{n+1})} \quad \text{or} \quad \left( \frac{s_{n+1}}{s_n} \right)^{m_{n+1}} \frac{v(s_{n+1})}{v(s_n)}$$

is equal to the constant $b > 0$ of the reference. Condition (B) yields a constant $d > b$ (independent of $n$) such that both quotients are numbers in the interval $[b, d]$. Now, Proposition 5.2. of [6] yields constants $c_1, c_2 > 0$.
such that
\[
\begin{align*}
    c_1 \sup_n \sup_{s_{n-1} \leq r \leq s_{n+1}} M_\infty(T_n f, r) v(r) \\
    \leq \|f\|_v \leq c_2 \sup_n \sup_{s_{n-1} \leq r \leq s_{n+1}} M_\infty(T_n f, r) v(r).
\end{align*}
\]
(12)

This already implies the first inequality of (8). Using [6], Proposition 4.1, we see that either
\[m_{n+1} - m_n \leq c \quad \text{for all } n \quad (c \text{ as in condition } (B)) \] or there are \(\eta, \kappa > 0\), independent of \(n\), such that
\[\eta \leq \frac{m_{n+1} - m_n}{m_n - m_{n-1}} \leq \kappa.\]

Lemma 3.3.(c) of [6] then tells us that, in any case, the operators \(T_n\) are uniformly bounded with respect to \(M_\infty(\cdot, 1)\). (If \(m_{n+1} - m_n \leq c\), then \(\text{rank } T_n \leq c\).

Using Lemma 3.1. of [6] we see that here
\[
\sup_{s_{n-1} \leq r \leq s_{n+1}} M_\infty(T_n f, r) v(r) \leq 2bd M_\infty(T_n f, s_n) v(s_n).
\]

Hence, the second inequality of (8) follows.

The first inequality in (9) is a triviality. Concerning the second, we use (10) and (12) to obtain for a fixed \(n\)
\[
\|T_n f\|_v \leq 2bd^2 c_2 (M_\infty(T_{n-1} T_n f, s_{n-1}) v(s_{n-1}), \quad M_\infty(T_n^2 f, s_n) v(s_n), \quad M_\infty(T_{n+1} T_n f, s_{n+1}) v(s_{n+1})) \\
\leq 2bd^2 c_2 (\sup_k \|T_k\|) \sup_k (M_\infty(T_n f, s_{n-1}) v(s_{n-1}), \quad M_\infty(T_n f, s_n) v(s_n), \quad M_\infty(T_n f, s_{n+1}) v(s_{n+1})) \\
\leq 4bd^4 c_2 (\sup_k \|T_k\|) M_\infty(T_n f, s_n) v(s_n)
\]

In the last inequality we used Lemma 3.1 of [6] again.

The boundedness of the Riesz projection is proven in Proposition 6.8 of [6].

\[\square\]

3. Main result with examples

Assume that a weight \(v\) on \(\Omega = \mathbb{D} \text{ or } \mathbb{C}\), satisfying \((B)\), is given; let the sequences \((m_n)_{n=1}^\infty\) and \((s_n)_{n=1}^\infty\) and also the multiplier sequences \((t_{nk})_{k=1}^\infty\) be as above.
Given a continuous function $f$ on $\Omega$, let $f_k(r)$ be the $k$th Fourier coefficient of $f|_{r\partial \mathbb{D}}$, i.e.

$$f_k(r) = \frac{1}{2\pi} \int_{0}^{2\pi} f(re^{i\varphi}) r^{-|k|} e^{-ik\varphi} d\varphi. \quad (13)$$

Hence, for $z \in \mathbb{D}$ and $r = |z|$, we have

$$f(z) = \sum_k f_k(r) e_k(z) \quad (14)$$
at least in the $L^2$-sense on every $r\partial \mathbb{D}$. We define for all $n$ the operators

$$W_n f = W_n \sum_k f_k e_k := \sum_k t_{nk} f_k(s_n) e_k \quad (15)$$

Notice that the sum on the right hand side has only finitely many terms. We define the projection

$$P_C f := \sum_{n=1}^{\infty} RW_n f = \sum_{k \geq 0} \sum_{n=1}^{\infty} t_{nk} f_k(s_n) e_k, \quad (16)$$

provided the series in (16) converges at least uniformly on compact subsets of $\mathbb{D}$. (It is then the Taylor series of an analytic function.) This is true, if we can show that $\|P_C f\|_v < \infty$, see Theorem 2 below.

To define a corresponding projection $P_M$ on the larger space of bounded functions we choose for every $n$ the interval $I_n := [s_n, s_n + \epsilon_n]$ with $I_n \subset [0, 1]$, if $\Omega = \mathbb{D}$, and $I_n \subset [0, \infty]$, if $\Omega = \mathbb{C}$. Here $\epsilon_n$ is fixed so small that $v(s_n) \geq v(r) \geq v(s_n)/2$ for all $r \in I_n$ (and that $s_n + \epsilon_n < s_{n+1}$). We define

$$Z_n f := Z_n \sum_k f_k e_k := \sum_k \frac{t_{nk}}{\epsilon_n} \left( \int_{I_n} f_k(s) ds \right) e_k, \quad (17)$$

and, as in (16),

$$P_M f := \sum_{n=1}^{\infty} RZ_n f = \sum_{k \geq 0} \sum_{n=1}^{\infty} \frac{t_{nk}}{\epsilon_n} \left( \int_{I_n} f_k(s) ds \right) e_k, \quad (18)$$

provided the series converges at least uniformly on compact subsets of $\mathbb{D}$.

**Theorem 2.** Let $v$ satisfy (B). Then $P_M$ is a bounded projection $L_v^\infty \to H_v$, and $P_C$ is a bounded projection from $L_v^\infty \cap C(\Omega)$ onto $H_v$.

Here $C(\Omega)$ is the space of all continuous functions $f : \Omega \to \mathbb{C}$.
**Remark 2.** Consider the case $\Omega = \mathbb{D}$. Then the condition $(B)$ is necessary and sufficient for Theorem 2. Indeed, if $(B)$ does not hold, then $Hv$ is isomorphic to $H^\infty$, by [6]. In this case $Hv$ cannot be complemented in $L^\infty_v$ and hence, in particular, $P_M$ cannot be bounded.

**Proof.** We start by showing that $P_M$ and $P_C$ are bounded. Concerning $P_M$, we want to prove first

$$M_\infty(Z_nf, s_n)v(s_n) \leq C\|T_nf\|_v$$

for $f \in L^\infty_v$. For $f(z) = \sum_k f_k(r) e_k(z)$ we have

$$M_\infty(Z_n f, s_n)v(s_n) = \sup_{\varphi \in [0, 2\pi]} \left| \frac{1}{\epsilon_n} \sum_{k \in \mathbb{Z}} t_{nk} f_k(s_n) e^{ik\varphi} \right| v(s_n)$$

$$\leq \sup_{\varphi \in [0, 2\pi]} \left| \frac{1}{\epsilon_n} \int_{I_n} \sum_{k \in \mathbb{Z}} t_{nk} f_k(s_n) e^{ik\varphi} ds \right| v(s_n)$$

$$\leq \sup_{\varphi \in [0, 2\pi]} \sup_{s \in I_n} \left| \sum_{k \in \mathbb{Z}} t_{nk} f_k(s_n) e^{ik\varphi} \right| v(s_n)$$

(20)

We obtain

$$\sup_{\varphi \in [0, 2\pi]} \left| \sum_{k \in \mathbb{Z}} t_{nk} f_k(s_n) e^{ik\varphi} \right| \leq \sup_{\varphi \in [0, 2\pi]} \left| \sum_{k \in \mathbb{Z}} t_{nk} f_k(s_n) e^{ik\varphi} \right|$$

for every $s \in I_n$. (To see this, we have $r_k \leq s$ for these $s$. We apply the maximum modulus principle to the harmonic function $h(r e^{i\varphi}) = \sum_{k \in \mathbb{Z}} t_{nk} f_k(s) r^{|k|} e^{ik\varphi}$ to get $\sup_{\varphi \in [0, 2\pi]} |h(r e^{i\varphi})| \leq \sup_{\varphi \in [0, 2\pi]} |h(s e^{i\varphi})|$.)

By the choice of the interval $I_n$, $v(s_n) \leq 2v(s)$ for $s \in I_n$. Using these we obtain (19): we bound (20) by

$$2 \sup_{s \in I_n} \sup_{\varphi \in [0, 2\pi]} \left| \sum_{k \in \mathbb{Z}} t_{nk} f_k(s_n) e^{ik\varphi} \right| v(s) \leq 2\|T_nf\|_v.$$

(21)

Let $f \in L^\infty_v$ be arbitrary. Notice that $Z_n f$ is always of the form $T_ng$ for some $g \in Hv$: we can write the right hand side of (17) as

$$T_n \sum_{|k| \leq N_n} \frac{1}{\epsilon_n} \int_{I_n} f_k(s) ds e_k.$$
for an $N_n \in \mathbb{N}$ large enough. We thus can use (10) to obtain

$$
\|P_M f\|_v \leq \|R\| \cdot \|\sum_n Z_n f\|_v
$$

$$
\leq d_2 \|R\| \sup_n M_\infty (T_n Z_n f + T_n Z_{n+1} f, s_n) v(s_n)
$$

$$
\leq d_2 \|R\| (\sup_k \|T_k\|) \sup_n \left( M_\infty (Z_n f, s_n) v(s_n) + M_\infty (Z_{n+1} f, s_n) v(s_n) \right).
$$

This is bounded by a $v$-dependent constant times $\sup_n \|Z_n f\|_v$. Since $Z_n f = T_n g$ for some $g \in hv$, the second inequality of Theorem 1, (9), applies and yields, together with (19), the bound

$$
C \sup_n M_\infty (Z_n f, s_n) v(s_n) \leq C' \sup_n \|T_n f\|_v \leq C' (\sup_n \|T_n\|) \|f\|_v.
$$

By Remark 1, (iii), the right hand side of (18) is the Taylor series of an analytic function and hence $P_M f$ is well-defined.

The proof for $P_C$ is easier, since the analogue of (19) is quite trivial:

$$
M_\infty (W_n f, s_n) v(s_n) = \sup_{|z|=s_n} \left| \sum_k \left[ m_n + 1 \right] t_{n,k} f_k(z) e_k(z) \right| v(z) \leq \|T_n f\|_v.
$$

The rest of the proof goes as that for $P_M$.

2°. We show that $P_M$ and $P_C$ are projections. To this end fix $f \in L^\infty_v$ and fix $n$. Put $h = RZ_n f$. Then we have $h = \sum_{k=[m_n+1]}^{[m_{n+1}]} h_k e_k$ for certain constants $h_k$ and hence $Z_n h = T_n h$ for any $n$. We obtain

$$
P_M h = \sum_{k=1}^{\infty} RT_k h
$$

$$
= T_{n-1} h + T_n h + T_{n+1} h
$$

$$
= (V_{m_n+2, m_{n+1}} - V_{m_n-1, m_{n-2}}) h = h
$$

This proves that $P_M$ is a projection. The proof for $P_C$ is the same, replacing $Z_n$ by $W_n$. \hfill \Box

**Example 1.** If $\Omega := \mathbb{D}$ and $v(z) = \exp(-1/(1-r))$, then

$$
r_m := 1 - \frac{2}{1 + \sqrt{4m + 1}} \approx 1 - \frac{1}{\sqrt{m}}.
$$
and the sequence \((m_n)_{n=1}^\infty\) is defined by \(m_{n+1} := m_n + \mathcal{O}(m_n^{3/4})\) for all \(n\). We refer to Example 2.3 of [6].

**Example 2.** If \(\Omega := \mathbb{C}\) and \(v(z) = \exp(-e^r)\), then the number \(r_m\) is the solution of the equation \(xe^x = m\), i.e. \(x + \log x = \log m\), hence, \(r_m\) equals \(\log m + \) very small corrections. The sequence \((m_n)_{n=1}^\infty\) is obtained from \(m_{n+1} := m_n + \mathcal{O}(\sqrt{m}\log m)\). For details, see [6], Example 2.1.

**Example 3.** If \(\Omega := \mathbb{C}\) and \(v(z) = \exp(-(\log r)^\rho)\), \(|z| \geq 1\), \(\rho \geq 2\) fixed, and \(v(z) = 1\) for \(|z| < 1\) then

\[
(24) \quad r_m := \exp((n/\rho)^{1/(\rho-1)})
\]

and the sequence \((m_n)_{n=1}^\infty\) is defined by \(m_{n+1} := m_n + \beta m_n^{(\rho-2)/(\rho-1)}\) for a positive constant \(\beta\). See [6], Example 2.2.

We finish our paper by the remark that in the case \(\Omega = \mathbb{D}\) the condition \((B)\) can be described in a somewhat easier way. We have

**Proposition 1.** Let \(\Omega = \mathbb{D}\). Then the following are equivalent:

(i) Condition \((B)\)

(ii) \(\forall b_1 > 1 \exists b_2 > 1 \forall m, n \geq 1\)

\[
\left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)} \leq b_1 \Rightarrow \left(\frac{r_n}{r_m}\right)^n \frac{v(r_n)}{v(r_m)} \leq b_2
\]

**Proof.** We only have to show that condition \((B)\) implies (ii). To this end let \(b_1 > 1\) and \(m, n \geq 1\) be such that

\[
\left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)} \leq b_1
\]

We show that there exists \(b_2\), independent of \(m\) and \(n\), such that \((r_n/r_m)^{n-m} \leq b_2\). This implies (ii). Indeed, then we obtain

\[
\left(\frac{r_n}{r_m}\right)^n \frac{v(r_n)}{v(r_m)} = \left(\frac{r_n}{r_m}\right)^{n-m} \frac{r_m^n v(r_n)}{r_m^n v(r_m)} \leq \left(\frac{r_n}{r_m}\right)^{n-m} \leq b_2.
\]

Let \(c = c(b_1)\) be the constant of condition \((B)\). Recall, that here we have \(r_k < 1\) for all \(k\). We consider several cases.

If \(m \geq c, n \geq c\) and \(|m - n| \geq c\) then \((B)\) implies the existence of a constant \(b_3\), independent of \(m\) and \(n\), with

\[
\left(\frac{r_n}{r_m}\right)^n \frac{v(r_n)}{v(r_m)} \leq b_3
\]
and hence \((r_n/r_m)^{n-m} \leq b_1 b_3\).

If \(|m - n| < c\) then \((r_n/r_m)^{n-m} \leq r_1^{-c}\) where the right-hand side is independent of \(m\) and \(n\).

Fix \(N\) such that \(r_{c+N} > r_c\). If \(m \leq c + N\) and \(n \leq c + N\) then similarly \((r_n/r_m)^{n-m} \leq r_1^{-2c-2N}\).

Now consider the case \(n < c < c + N \leq m\). We have
\[
\left( \frac{r_{c+N}}{r_c} \right)^{m-c-N} \leq \left( \frac{r_{c+N}}{r_n} \right)^{m-c-N} \frac{v(r_{c+N})}{v(r_n)} \frac{r_n}{r_m} \frac{v(r_m)}{v(r_n)} \leq b_1.
\]

Since \(\lim_{k \to \infty} \left( \frac{r_{c+N}}{r_c} \right)^{k-c-N} = \infty\) we find \(k_0\), independent of \(m\) and \(n\), such that \(\left( \frac{r_{c+N}}{r_c} \right)^{k-c-N} > b_1\) for all \(k \geq k_0\). Hence \(m \leq k_0\) and we obtain
\[
\left( \frac{r_n}{r_m} \right)^{n-m} = \left( \frac{r_m}{r_n} \right)^{m-n} \leq b_1 \left( \frac{1}{r_1} \right)^{k_0}.
\]

Finally, assume \(m < c < c + N \leq n\). Since \(\lim_{l \to \infty} v(r_l) = 0\) we find \(l_0\) such that \(r_j^l v(r_l) \leq v(r_l) < v(r_c)/b_1\) for all \(j > 0\) and all \(l \geq l_0\). Since
\[
\frac{r_j^l v(r_c)}{r_m v(r_m)} \leq \frac{\mu^{e-m} \frac{v(r_m)}{r_m v(r_n)}}{\frac{v(r_n)}{r_{n} v(r_n)}} \leq b_1
\]
we have \(r_j^l v(r_c)/b_1 \leq r_m v(r_n)\). Hence \(n \leq l_0\) and we obtain \((r_n/r_m)^{n-m} \leq r_1^{-l_0}\).

Now put \(b_2 = \max (b_1 b_3, r_1^{-\max(2c+2N,l_0,k_0)})\). Then we have in any case \((r_n/r_m)^{n-m} \leq b_2\). 

\[\Box\]

References


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