Q_p-spaces on bounded symmetric domains

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Abstract. We generalize the theory of Q_p spaces, introduced on the unit disc in 1995 by Aulaskari, Xiao and Zhao, to bounded symmetric domains in C^d, as well as to analogous Möbius-invariant function spaces and Bloch spaces defined using higher order derivatives; the latter generalization contains new results even in the original context of the unit disc.

1. Introduction

Let D be the unit disc in the complex plane C. For −∞ < p < ∞, a holomorphic function f is said to belong to the space Q_p if

\begin{equation}
\sup_{x \in D} \int_D |f'(z)|^2 \left(1 - \frac{|x - z|^2}{1 - x \overline{z}}\right)^p \frac{dz}{|1 - x \overline{z}|^2} < \infty,
\end{equation}

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the square root of the last quantity being, by definition, the (semi)norm in $Q_p$. Here $dz$ denotes the Lebesgue area measure. Since any Moebius map $\phi$ (i.e. biholomorphic self-map of $D$) is of the form $\phi(z) = \frac{x - z}{1 - \bar{x}z}$, with $|\epsilon| = 1$ and $x \in D$, the quantity (1) can be rewritten as

$$\sup_{\phi \in \text{Aut}(D)} \int_D |f(z)|^2 (1 - |\phi(z)|^2)^p \, dz$$

$$= \sup_{\phi \in \text{Aut}(D)} \int_D \Delta|f|^2(z) (1 - |\phi(z)|^2)^p \, dz$$

$$= \sup_{\phi \in \text{Aut}(D)} \int_D (\overline{\Delta}|f|^2)(z) (1 - |\phi(z)|^2)^p \, d\mu(z)$$

$$= \sup_{\phi \in \text{Aut}(D)} \int_D \overline{\Delta}|f \circ \phi(z)|^2 (1 - |z|^2)^p \, d\mu(z),$$

where $\overline{\Delta} = (1 - |z|^2)^2 \frac{\partial^2}{\partial z \partial \overline{z}}$ and $d\mu(z) = \frac{dz}{(1 - |z|^2)^2}$ are the $\text{Aut}(D)$-invariant Laplacian and the $\text{Aut}(D)$-invariant measure on $D$, respectively, and $\text{Aut}(D)$ stands for the group of all Moebius maps. (Note that we are using the normalization $\Delta = \frac{1}{4} \partial^2 \partial \overline{z}$ for the Euclidean Laplacian, which differs from the usual one by a factor of 4.) From the last formula it is apparent that $f \in Q_p$ implies $f \circ \phi \in Q_p$ and $f$ and $f \circ \phi$ have the same norm in $Q_p$, for all $\phi \in \text{Aut}(D)$. That is, the space $Q_p$ is Moebius invariant.

The spaces $Q_p$ were introduced in 1995 by Aulaskari, Xiao and Zhao [5], who showed that

$$p > 1 \implies Q_p = B,$$

the Bloch space,

$$p = 1 \implies Q_p = BMOA,$$

$$0 \leq p_1 < p_2 \leq 1 \implies Q_{p_1} \subsetneq Q_{p_2},$$

$$p = 0 \implies Q_p = D,$$

the Dirichlet space,

$$p < 0 \implies Q_p = \{\text{constants}\}.$$ (2)

Thus the $Q_p$ spaces provide a whole range of M"obius-invariant function spaces on $D$ lying strictly between the Dirichlet space on the one hand, and $BMOA$ and the Bloch space

$$B = \left\{ f \text{ holomorphic on } D : \sup_{z \in D} (1 - |z|^2)|f'(z)| < \infty \right\}$$

on the other hand.

The $Q_p$ spaces subsequently attracted a lot of attention; see e.g. the book by Xiao [17] and the references therein. They were generalized to the
unit ball $B^d \subset \mathbb{C}^d$ in 1998 by Ouyang, Yang and Zhao [12]:

$$(3) \quad f \in Q_p(B^d) \iff \sup_{\phi \in \text{Aut}(B^d)} \int_{B^d} \tilde{\Delta}|f \circ \phi|^2 (1 - \|z\|^2)^{pd} d\mu(z) < \infty,$$

where $\tilde{\Delta}$ and $d\mu$ denote the $\text{Aut}(B^d)$-invariant Laplacian and the $\text{Aut}(B^d)$-invariant measure on $B^d$, respectively, $\text{Aut}(B^d)$ being the group of all biholomorphic self-maps of $B^d$. Again, these spaces are $\text{Aut}(B^d)$-invariant, and it was proved in [12] that

$$(4) \quad \begin{array}{c}
p > 1 \implies Q_p = \mathcal{B}(B^d), \quad \text{the Bloch space,} \\
p = 1 \implies Q_p = \text{BMOA}(B^d), \\
\frac{d-1}{d} < p_1 < p_2 \leq 1 \implies Q_{p_1} \subsetneq Q_{p_2}, \\
p \leq \frac{d-1}{d} \implies Q_p = \{ \text{constants} \}.
\end{array}$$

Note that, in contrast to the disc, for $d > 1$ the Dirichlet space does not turn up as one of the $Q_p$’s, though in all other cases the situation is the same as for $\mathbb{D}$.

Other generalizations include $Q_p$ spaces on smoothly bounded strictly pseudoconvex domains [1] or the $F(p, q, s)$ spaces of Rättyä and Zhao [13], [18]. In this paper, we will consider a generalization in another direction. Note that the definitions (1) and (3) involve the invariant Laplacian $\tilde{\Delta}$, the invariant measure $d\mu$, and the quantity $1 - \|z\|^2$, whose power $(1 - \|z\|^2)^{-d-1}$ is at the same time the density of $d\mu$ with respect to $dz$ as well as — up to a constant factor — the Bergman kernel $K(z, z)$ of $B^d$. Our generalization concerns the context where all of these ingredients still prevail — namely, the bounded symmetric domains.

Recall that a bounded domain $\Omega \subset \mathbb{C}^d$ is called symmetric if for any $x \in \Omega$ there exists $s_x \in \text{Aut}(\Omega)$ such that $s_x \circ s_x = \text{id}$ and $x$ is an isolated fixed-point of $s_x$. One calls $s_x$ the geodesic symmetry at $x$. A bounded symmetric domain is irreducible if it is not biholomorphic to a Cartesian product of another two nontrivial bounded symmetric domains. Any such domain can be realized as (i.e. is biholomorphic to) one which is circular with respect to the origin and convex. Its Bergman kernel $K(x, y)$ is then of the form $\text{const} \cdot h(x, y)^{-p}$, where $p$ is a positive integer, called the genus of $\Omega$, and $h(x, y)$ is an irreducible polynomial, holomorphic in $x$ and $\overline{y}$, such that $h(0, 0) = 1$. The measure $d\mu(z) = h(z, z)^{-p} dz = \text{const} \cdot K(z, z) dz$ on $\Omega$ is invariant under biholomorphic self-maps, i.e. $d\mu(\phi(z)) = d\mu(z)$ for all $\phi \in \text{Aut}(\Omega)$, the group of all biholomorphic self-maps of $\Omega$ (called Moebius transformations). Finally, a (linear) differential operator $L$ on $\Omega$ is called
invariant if

\[ L(f \circ \phi) = (Lf) \circ \phi \]

for any \( f \in C^\infty(\Omega) \) and any \( \phi \in \text{Aut}(\Omega) \).

Assume that \( L \) is an invariant differential operator such that

\[ L|f|^2 \geq 0 \quad \text{for any } f \text{ holomorphic on } \Omega \]

and let \(-\infty < \nu < \infty\). Then we define the \((L-)\)Bloch space

\[ B_L := \left\{ f \text{ holomorphic on } \Omega : \sup_{\Omega} L|f|^2 < \infty \right\}, \]

and the \(Q_{\nu,L}\)-space

\[ Q_{\nu,L} := \left\{ f \text{ holomorphic on } \Omega : \sup_{\phi \in \text{Aut}(\Omega)} \int_{\Omega} L|f \circ \phi|^2 h^\nu \, d\mu < \infty \right\}, \]

the square roots of the indicated suprema being, by definition, the semi-norms in \( B_L \) and \( Q_{\nu,L} \). Clearly, both \( B_L \) and \( Q_{\nu,L} \) are Moebius invariant.

Note that since \( 0 < h(z,z) \leq 1 \forall z \in \Omega \), we have \( Q_{\nu_1} \subset Q_{\nu_2} \) continuously if \( \nu_1 < \nu_2 \).

For the unit disc and the unit ball, one has \( h(z,z) = 1 - \|z\|^2 \), and taking for \( L \) the invariant Laplacian, (6) and (7) reduce to the definitions of the ordinary Bloch space and \( p\) spaces, respectively (the latter with \( \nu = pd \)).

Our goal in this paper is to provide counterparts, for general irreducible bounded symmetric domains, of the characterizations (2) and (4). In more detail, our results are the following.

First of all, we characterize the invariant differential operators \( L \) satisfying (5). It turns out that there exists a basis \( \Delta_m \) of the vector space of all invariant differential operators such that \( L = \sum_m l_m \Delta_m \) satisfies (5) if and only if \( l_m \geq 0 \forall m \). Here \( m \) runs through the set of all signatures, i.e. tuples \( m_1, \ldots, m_r \) of integers such that \( m_1 \geq m_2 \geq \cdots \geq m_r \geq 0 \), \( r \) being the rank of \( \Omega \). It follows that

\[ B_L = \bigcap_{m : l_m > 0} B_m \quad \text{and} \quad Q_{\nu,L} = \bigcap_{m : l_m > 0} Q_{\nu,m}, \]

with the norm in \( B_L \) equivalent to \( \max_m l_m \cdot \| \cdot \|_{B_m} \), and similarly for \( Q_{\nu,L} \); here, for the sake of brevity, we have introduced the shorthand \( B_m, Q_{\nu,m} \) for \( B_{\Delta_m} \) and \( Q_{\nu,\Delta_m} \). This reduces the study of \( B_L \) and \( Q_{\nu,L} \) to \( B_m \) and \( Q_{\nu,m} \), respectively.
For $m = (1,0,\ldots,0)$, the operator $\Delta_m$ reduces to the ordinary invariant Laplacian $\tilde{\Delta}$, and $B_m$ coincides with the Bloch space introduced by Timoney [15].

For $\Omega$ a domain of tube type with $s := \frac{d}{r}$ an integer and $m = (s,\ldots,s)$, the Bloch space $B_m$ was studied by the first author [3] in connection with Hankel operators on the top quotient of the composition series. (See Section 2 below the various definitions.)

Our first main result is then that for $\nu > p - 1$, we have a full analogue of the first lines in (2) and (4), namely,

\[ Q_{\nu,m} = B_m, \text{ with equivalent norms.} \]

Further, the Bloch space $B_m$ depends only on the height $q(m)$ of the signature $m$, i.e.

\[ B_m = B_n \text{ if } q(m) = q(n) \text{ (with equivalent norms);} \]

where $a$ is the characteristic multiplicity of $\Omega$. (See again Section 2 below for the various definitions.)

Note that (9) and (10) give new results even in the original context of the unit disc $D$: for instance, for any $k \geq 1$, the Bloch space seminorm is equivalent to the square root of

\[ \sup_{x \in D} \Delta^k|f \circ \phi_x|^2(0) = \sup_{x \in D} |(f \circ \phi_x)^{(k)}(0)|^2, \]

where $\phi_x(z) = \frac{z - x}{1 - x \bar{z}}$. Similarly for the unit ball.

For $\nu \leq p - 1$, the situation turns out to be more subtle. The spaces $Q_{\nu,m}$ always contain the set

\[ N_m := \{ f \text{ holomorphic on $\Omega$ : } \Delta_m|f|^2 \equiv 0 \}. \]

(Again, as with $B_m$, this set in fact depends not on $m$ but only on the height $q(m)$.) We say that $Q_{\nu,m}$ is trivial if $Q_{\nu,m} = N_m$. This is always the case if $\nu < 0$. It may happen that $Q_{\nu,m}$ is trivial even for all $\nu \leq p - 1$: for instance, this is the case for $m = (1,0,\ldots,0)$ — that is, when $\Delta_m$ is just the invariant Laplacian — for any irreducible bounded symmetric domain $\Omega$ of rank $> 1$ (i.e. not biholomorphic to the ball $B^d$); this is in sharp contrast with (2) and (4). On the other hand, it may happen that
$Q_{p,m}$ is nontrivial for all $\nu \geq 0$: this is the case, for instance, for tube type domains $\Omega$ and $m = (s, \ldots, s)$, if $s = d/r$ is an integer.

In general, there exists an integer or half-integer $\rho_m$ such that

$Q_{\nu,m}$ is nontrivial $\iff \nu \geq 0$ and $\nu > p - 1 - \rho_m$.

We have $\rho_{(0,\ldots,0)} = 0$ for any $\Omega$, $\rho_m = 2$ for $\Omega = D$ and $m = (1)$, $\rho_m = 1$ for $\Omega = B^d$, $d > 1$, and $m = (1)$, $\rho_{(1,0,\ldots,0)} = 0$ for $\Omega$ not biholomorphic to $B^d$, and $\rho_{(s,\ldots,s)} = p$ for $\Omega$ a tube type domain with $s = \frac{d}{r}$ an integer.

For general $\Omega$ and $m$, the exact value of $\rho_m$ is, unfortunately, unknown.

The paper is organized as follows. In Section 2 we review various prerequisites on bounded symmetric domains. In Section 3 we establish several auxiliary results, including the characterization of invariant differential operators satisfying (5) and the proof of (8). The main results are established in Section 4. The last Section 5 contains some concluding remarks, open problems, and an additional material on certain Pieri-type coefficients.

A preliminary version of this paper, containing only a selection of the results and with the more difficult parts of their proofs omitted, appeared in the proceedings of the 13th ICFIDCAA conference [7]; the second author thanks the organizers for the invitation.

2. Bounded symmetric domains

Throughout the rest of this paper, $\Omega$ will be an irreducible bounded symmetric domain in $\mathbb{C}^d$ in its Harish-Chandra realization (i.e. a Cartan domain). As usual we denote by $G = \text{Aut}(\Omega)$ the group of all biholomorphic self-maps of $\Omega$, and by $K$ the stabilizer in $G$ of the origin $0 \in \Omega$. Then $K$ consists precisely of the unitary maps on $\mathbb{C}^d$ that preserve $\Omega$, and $\Omega$ is isomorphic to the coset space $G/K$. We further denote by $r, a, b$ and $p$ the rank, the characteristic multiplicities and the genus of $\Omega$, respectively, so that

$$p = (r - 1)a + b + 2, \quad d = \frac{r(r - 1)}{2}a + rb + r.$$ 

If $b = 0$, $\Omega$ is said to be of tube type.

Irreducible bounded symmetric domains were completely classified by E. Cartan. There are four infinite series of such domains plus two exceptional domains in $\mathbb{C}^{16}$ and $\mathbb{C}^{27}$. For future reference, we include a table with brief descriptions of these domains and with the corresponding values of $r, a, b, p$ and $d$. 


The unit balls $B^d = I_{1d}$ are the only bounded symmetric domains of rank 1, and the only bounded symmetric domains with smooth boundary.

For $x \in \Omega$, $\phi_x$ will denote the (unique) geodesic symmetry which interchanges $x$ and the origin, i.e.

$$\phi_x \circ \phi_x = \text{id}, \quad \phi_x(0) = x, \quad \phi_x(x) = 0,$$

and $\phi_x$ has only an isolated fixed-point. (In fact, $\phi_x$ has only one fixed point, namely the geodesic mid-point between 0 and $x$.) We will also use the transvections

$$\gamma_x(z) := \phi_x(-z)$$

which map the origin into $x$. Note that from the definition of $K$ it is immediate that any $\phi \in G$ is of the form $\phi = \phi_x k = \gamma_x k'$, where $k, k' \in K$ and $x \in \Omega$. (In fact $x = \phi(0)$.)

It is known that the ambient space $C^d =: Z$ possesses a structure of Jordan-Banach $^*$-triple system (or JB$^*$-triple for short) for which $\Omega$ is the open unit ball. That is, there exists a Jordan triple product

$$\{\cdot, \cdot, \cdot\} : Z \times Z \times Z \to Z, \quad x, y, z \mapsto \{x, y, z\},$$

(linear and symmetric in $x, z$ and anti-linear in $y$) such that

$$\Omega = \{z \in Z : \|\{z, z, \cdot\}\| < 1\}.$$

Moreover, if one uses the notation, for $x, y \in Z$,

$$D(x, y) = \{x, y, \cdot\} : Z \to Z, \quad Q(x) = \{x, \cdot, x\} : Z \to Z,$$
then for every $x \in \Omega$, $D(x, x)$ is Hermitian and has nonnegative spectrum, and $iD(x, x)$ is a triple derivation. The linear operator

\begin{equation}
B(x, y) = I - 2D(x, y) + Q(x)Q(y)
\end{equation}

on $Z$ is called the Bergman operator.

Two vectors $x, y \in Z$ are said to be orthogonal (in the Jordan-theoretic sense) if $D(x, y) = 0$, and a vector $v \in Z$ is called a tripotent if $\{ v, v, v \} = v$. Any maximal set $e_1, \ldots, e_r$ of pairwise orthogonal nonzero tripotents is called a Jordan frame; its cardinality $r$ is independent of the frame and equal to the rank of $\Omega$. For any tripotent $v$, the ambient space admits the Peirce decomposition

\[ Z = Z_0(v) \oplus Z_{1/2}(v) \oplus Z_1(v) \]

into the orthogonal components

\[ Z_{1/2}(v) := \{ z \in Z : D(v, v)z = \frac{i}{2}z \}. \]

(The orthogonality is only with respect to the inner product in $\mathbb{C}^d$, not in the triple-product (Jordan-theoretic) sense.) For any Jordan frame $e_1, \ldots, e_r$, we similarly have the joint Peirce decomposition

\begin{equation}
Z = \bigoplus_{0 \leq i \leq j \leq r} Z_{ij}
\end{equation}

with

\begin{equation}
Z_{ij} = \left\{ z \in Z : D(e_k, e_k)z = \frac{\delta_{ik} + \delta_{jk}}{2} \forall k = 1, \ldots, r \right\}.
\end{equation}

In terms of the Jordan triple data, the geodesic symmetries and transvections (11) are given by

\begin{align*}
\phi_x(z) &= x - B(x, x)^{1/2}B(z, x)^{-1}(z - Q(z)x) \\
&= x - B(x, x)^{1/2}(I - D(z, x))^{-1}z, \\
\gamma_x(z) &= x + B(x, x)^{1/2}(I + D(z, x))^{-1}z.
\end{align*}

Given any Jordan frame $e_1, \ldots, e_r$ — which we choose and fix once and for all from now on — any $z \in Z$ has a polar decomposition

\begin{equation}
z = k(t_1e_1 + \cdots + t_re_r)
\end{equation}
with \( k \in K \) and \( t_1 \geq t_2 \geq \cdots \geq t_r \geq 0 \); the numbers \( t_1, \ldots, t_r \), called the \textit{singular numbers} of \( z \), are determined uniquely, but \( k \) need not be (it is if all the \( t_j \) are distinct). Further, \( z \in \Omega \) if and only if \( t_1 < 1 \), \( z \in \partial \Omega \) if and only if \( t_1 = 1 \), and \( z \) belongs to the Shilov boundary \( \partial_e \Omega \) of \( \Omega \) if and only if \( t_1 = \cdots = t_r = 1 \); that is, if and only if \( z = ke \), where \( e = e_1 + \cdots + e_r \) is a \textit{maximal tripotent}.

Since the Jordan triple product is invariant under \( K \) (i.e. \( \{ kx, ky, kx \} = k \{ x, y, z \} \forall k \in K \)), it is immediate from (14) that under the decomposition (13), the Bergman operator \( B(z, z) \) with \( z \) as in (16) is given by
\[
B(z, z)|_{Z_{ij}} = (1 - t_1^2)(1 - t_j^2)I|_{Z_{ij}}
\]
(17) where \( t_0 := 0 \).

There exists a unique polynomial \( h(x, y) \) on \( \mathbb{C}^d \times \mathbb{C}^d \), holomorphic in \( x \) and anti-holomorphic in \( y \), which is \( K \)-invariant, in the sense that
\[
h(kx, ky) = h(x, y) \forall k \in K,
\]
and satisfies
\[
h(z, z) = \prod_{j=1}^{r}(1 - t_j^2) \quad \text{for } z \text{ as in (16)}.
\]

It is known that \( h(x, y) \) is irreducible, of degree \( r \) in \( x \) as well as in \( y \), and \( h(x, 0) = h(0, x) = 1 \forall x \in \mathbb{C}^d \); also, \( h(x, y)p = \det B(x, y) \). Further, the measure
\[
h(z, z)^{\nu - p} \, dz
\]
is finite if and only if \( \nu > p - 1 \), and the corresponding weighted Bergman kernel — i.e. the reproducing kernel of the space of all holomorphic functions on \( \Omega \) square-integrable with respect to (18) — is equal to
\[
K_\nu(x, y) = c_\nu h(x, y)^{-\nu}
\]
for some constant \( c_\nu \). In particular, for \( \nu = p \), the ordinary (i.e. unweighted) Bergman kernel of \( \Omega \) is equal to
\[
K(x, y) = \frac{1}{\text{vol}(\Omega)} h(x, y)^{-p}.
\]

Finally, the measure
\[
d\mu(z) = \frac{dz}{h(z, z)^p} = \text{vol}(\Omega) K(z, z) \, dz
\]
is the unique (up to constant multiples) $G$-invariant measure on $\Omega$.

In the polar coordinates (16), the measures (18) assume the form

\begin{equation}
\int_{\Omega} f(z) h(z, z)^p \, d\mu(z) =
\end{equation}

\begin{equation}
c \int_{[0,1]^r} \int_K f(k(\sum_{j=1}^r t_je_j)) \prod_{j=1}^r (1 - t_j^2)^{\nu-p} \prod_{j=1}^r t_j^{2\nu+1} \prod_{1 \leq i < j \leq r} |t_i^2 - t_j^2|^a \, dk \, dt,
\end{equation}

where $dt = dt_1 \ldots dt_r$, $dk$ is the normalized Haar measure on the (compact) group $K$, and $c$ is a constant whose exact value will not be needed and which we will therefore omit in the sequel. (Alternatively, choosing $c = 1$ amounts to a special choice of the invariant measure $\mu$.)

For the sake of brevity, we will often abbreviate $h(z, z)$ just to $h(z)$ (or even to $h$) if there is no danger of confusion.

Let $P$ denote the vector space of all (holomorphic) polynomials on $C^d$. We endow $P$ with the Fock (or Fischer) inner product

\begin{equation}
\langle f, g \rangle_F := \pi^{-d} \int_{C^d} f(z) \overline{g(z)} e^{-|z|^2} \, dz
\end{equation}

\begin{equation}
= (f(\bar{\partial})g^*)(0) = (g^*(\overline{\partial})f)(0),
\end{equation}

where

$$g^*(z) := \overline{g(\bar{z})}.$$ 

This makes $P$ into a pre-Hilbert space, and the action

$$f \mapsto f \circ k, \quad k \in K,$$

is a unitary representation of $K$ on $P$. It is a deep result of W. Schmid [14] that this representation has a multiplicity-free decomposition into irreducibles

$$P = \bigoplus_m P_m$$

where $m$ ranges over all signatures, i.e. $r$-tuples $m = (m_1, m_2, \ldots, m_r) \in Z^r$ satisfying $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$. Polynomials in $P_m$ are homogeneous of degree $|m| := m_1 + m_2 + \cdots + m_r$; in particular, $P_{(0)}$ are the constants and $P_{(1)}$ the linear polynomials. Any holomorphic function thus has a decomposition $f = \sum_m f_m$, $f_m \in P_m$, which refines the usual homogeneous expansion.

Since the spaces $P_m$ are finite dimensional, they automatically possess a reproducing kernel: there exist polynomials $K_m(x, y)$ on $C^d \times C^d$, holomorphic in $x$ and $\overline{y}$, such that for each $f \in P_m$ and $y \in C^d$,
\( f(y) = \langle f, K_m(\cdot, y) \rangle_F \). In terms of any orthonormal basis \( \{ \psi_j \}_{j=1}^{d_m} \) of \( P_m \), where \( d_m := \dim P_m \), \( K_m \) is given by

\[
K_m(x, y) = \sum_{j=1}^{d_m} \psi_j(x) \overline{\psi_j(y)}.
\]

From the definition of the spaces \( P_m \) it also follows that the kernels \( K_m(x, y) \) are \( K \)-invariant.

It is a consequence of Schur’s lemma from representation theory that for any \( K \)-invariant inner product \( \langle \cdot, \cdot \rangle \) on \( P \), \( P_m \) and \( P_n \) are orthogonal if \( m \neq n \), while on each \( P_m \), \( \langle \cdot, \cdot \rangle \) is proportional to \( \langle \cdot, \cdot \rangle_F \). In particular, for the inner product

\[
\langle f, g \rangle_\nu := c_\nu \int_\Omega f(z) \overline{g(z)} h(z, z)^\nu d\mu(z) \quad (\nu > p-1),
\]

(with \( c_\nu \) as in (19)) we have, for any \( f_m \in P_m \) and \( g_n \in P_n \),

\[
\langle f_m, g_n \rangle_\nu = \frac{\langle f_m, g_n \rangle_F}{(\nu)_m} \tag{23}
\]

(cf. [8]), where \( (\nu)_m \) is the generalized Pochhammer symbol

\[
(\nu)_m := (\nu)_1 (\nu - \frac{a}{2})_2 \ldots (\nu - r - 1)_{m_r};
\]

here

\[
(\nu)_k := \nu(\nu + 1) \ldots (\nu + k - 1) \quad \left( = \frac{\Gamma(\nu + k)}{\Gamma(\nu)} \text{ if } \nu \neq 0, -1, -2, \ldots \right)
\]

is the ordinary Pochhammer symbol.

A consequence of the relation (23) is the Faraut-Koranyi formula

\[
h(x, y)^{-\nu} = \sum_m (\nu)_m K_m(x, y) \tag{24}
\]

relating the reproducing kernels \( K_\nu \) from (19) and \( K_m \) from (22).

For a signature \( m \), consider the function

\[
\nu \mapsto (\nu)_m, \quad \nu \in \mathbb{C}.
\]
Let $q(m)$ (the height of the signature $m$) be the multiplicity of zero of this function at $\nu = 0$:
\[
q(m) := \text{card} \left\{ j : m_j > \frac{j-1}{2} a \in \mathbb{Z} \right\}.
\]
Denote by $q$ the maximum possible value of $q(m)$; that is,
\[
q = \begin{cases} 
  r & a \text{ even}, \\
  \left[ \frac{r+1}{2} \right] & a \text{ odd}.
\end{cases}
\]
For $-1 \leq \ell \leq q$, let
\[
M_\ell = \{ f = \sum_m f_m \text{ holomorphic} : f_m = 0 \text{ if } q(m) > \ell \}.
\]
Thus, in particular,
\[
(25) \quad M_{-1} \subset M_0 \subset M_1 \subset \cdots \subset M_q,
\]
\[
M_{-1} = \{ 0 \}, \quad M_0 = \{ \text{constants} \}, \quad M_q = \{ \text{all holomorphic functions} \}.
\]
The sequence (25) is known as the composition series of $\Omega$. It is a deep result of Ørsted (in the special case of $\Omega = I_{nn}$) and Faraut and Koranyi (in general) that
\[
(26) \quad \text{each } M_\ell \text{ is } G\text{-invariant}
\]
and that for any $G$-invariant space $E$ of holomorphic functions on $\Omega$ on which the action $f \mapsto f \circ k$ of $K$ is strongly continuous,
\[
(27) \quad E \setminus M_{\ell-1} \neq \emptyset \implies \mathcal{P} \cap M_\ell \subset E.
\]
In other words, if $E$ is not wholly contained in $M_{\ell-1}$, then $E$ contains every $\mathcal{P}_m$ with $q(m) = \ell$.

Standard references for the material in this section are [2], [10], [8], [9], or [16].

3. Invariant differential operators and some convolutions

Recall that we have called a (linear) differential operator $L$ on $\Omega$ invariant if
\[
L(f \circ \phi) = (Lf) \circ \phi \quad \forall \phi \in G = \text{Aut}(\Omega).
\]
It is well known that on the unit disc, invariant differential operators are precisely the polynomials of the invariant Laplacian \( \tilde{\Delta} = (1 - |z|^2)^2 \Delta \); the same is true for \( B^d \). For a general Cartan domain, the situation is more complicated: namely, there exist \( r \) commuting algebraically independent differential operators \( \Delta_1, \ldots, \Delta_r \), where \( r \) is the rank, which can be chosen to have orders \( 2, 4, \ldots, 2r \), respectively, such that the algebra of all invariant differential operators consists precisely of all polynomials in \( \Delta_1, \ldots, \Delta_r \). In particular, the monomials \( \Delta_1^{n_1} \ldots \Delta_r^{n_r} \) form a linear basis of all invariant differential operators. However, often it is much more convenient to use another basis, the construction of which we now describe.

For any invariant differential operator \( L \), let \( L_0 \) be the (non-invariant) constant-coefficient linear differential operator obtained upon freezing the coefficients of \( L \) at the origin; that is, \( Lf(0) =: L_0 f(0) \). From the invariance of \( L \) it follows that

\[
k \in G, \ k0 = 0 \implies L_0(f \circ k) = (L_0 f) \circ k
\]

(i.e. \( L_0 \) is \( K \)-invariant) and

\[
L_f(z) = L_0(f \circ \phi_z)(0).
\]

Conversely, if \( L_0 \) is a \( K \)-invariant constant-coefficient differential operator, then the recipe (28) clearly defines an invariant differential operator \( L \) on \( \Omega \). Thus there is a one-to-one correspondence between invariant linear differential operators on \( \Omega \) and \( K \)-invariant linear constant-coefficient differential operators on \( C^d \).

Further, any constant-coefficient linear differential operator \( L_0 \) can be written in the form \( L_0 = p(\partial, \overline{\partial}) \) for some polynomial \( p \) on \( C^d \times C^d \). It is not difficult to see that such operator is \( K \)-invariant if and only if the polynomial \( p \) is \( K \)-invariant, in the sense that \( p(x, \overline{y}) = p(kx, \overline{ky}) \) for all \( x, y \in C^d \) and \( k \in K \). Combining this with the observation in the preceding paragraph, we thus see that the recipe

\[
p(x, \overline{y}) \mapsto L_p, \quad L_p f(x) := p(\partial, \overline{\partial})(f \circ \phi_x)(0) = p(\partial, \overline{\partial})(f \circ \gamma_x)(0)
\]

sets up a one-to-one correspondence between invariant differential operators on \( \Omega \) and \( K \)-invariant sesqui-holomorphic polynomials on \( C^d \times C^d \).

**Example 1.** Since \( K \) consists of unitary maps, the simplest \( K \)-invariant polynomial (apart from the constants) is \( p(x, \overline{y}) = \langle x, y \rangle \). Then \( p(\partial, \overline{\partial}) = \sum_{j=1}^{d} \partial_j \overline{\partial}_j = \Delta \), and the corresponding invariant differential operator is

\[
Lf(x) = \Delta(f \circ \phi_x)(0).
\]
This operator is called the invariant Laplacian on $\Omega$: it coincides with the Laplace-Beltrami operator with respect to the Bergman metric on $\Omega$. Note that for $f$ holomorphic,

$$L|f|^2(x) = \sum_{j=1}^{d} \left| \frac{\partial(f \circ \phi_x)(0)}{\partial z_j} \right|^2 = \|\partial(f \circ \phi_x)(0)\|^2$$

is the norm-squared of what we might call the invariant holomorphic gradient of $f$.

We have seen in the preceding section that for each signature $\mathbf{m}$, the reproducing kernel $K_m(x, y)$ of the Peter-Weyl space $\mathcal{P}_m$ is a $K$-invariant polynomial on $\mathbb{C}^d \times \mathbb{C}^d$. By the discussion above, $K_m$ therefore defines an invariant differential operator

$$\Delta_m f(x) := K_m(\partial, \partial)(f \circ \phi_x)(0).$$

Proposition 2. The polynomials $K_m(x, y)$ form a basis of the space of all $K$-invariant sesqui-holomorphic polynomials on $\mathbb{C}^d \times \mathbb{C}^d$. Consequently, the operators $\Delta_m$ form a basis for the space of all invariant differential operators on $\Omega$.

Proof. Any polynomial $p(x, \overline{y})$ on $\mathbb{C}^d \times \mathbb{C}^d$ is uniquely determined by its restriction to $\Omega \times \Omega$ and, hence (by holomorphy), by its restriction to the Shilov boundary $\partial_e \Omega \times \partial_e \Omega$ of $\Omega \times \Omega$; that is, by its values $p(k_1e, \overline{k_2e})$ where $e$ is a fixed maximal tripotent and $k_1, k_2 \in K$. By $K$-invariance, $p(k_1e, \overline{k_2e}) = p(k_2^{-1}k_1e, \overline{e})$, so $p$ is actually uniquely determined by its values $p(ke, \overline{e})$ for $k \in K$. Now $f(x) := p(x, \overline{x})$ is a holomorphic polynomial on $\mathbb{C}^d$, and $f(lx) = p(lx, \overline{l}) = p(lx, \overline{e}) = p(x, \overline{e}) = f(x)$ for any $l \in K$ which fixes $e$; that is, letting $L$ stand for the stabilizer of $e$ in $K$, $f(x)$ is $L$-invariant. If $f = \sum_{m} f_m$ is the Peter-Weyl decomposition of $f$, it follows that each $f_m$ is also $L$-invariant. However, it is known [8, Theorem 2.1] that the only $L$-invariant polynomial in $\mathcal{P}_m$, up to constant multiples, is $K_m(\cdot, e)$. Thus $f = \sum_{m} c_m K_m(\cdot, e)$ for some constants $c_m \in \mathbb{C}$, which implies (tracing back the arguments from the beginning of this proof) that $p(x, \overline{y}) = \sum_{m} c_m K_m(x, y)$.

The uniqueness of the $c_m$ is obvious. □

The following result makes it clear why the basis $\Delta_m$ is very appropriate for our applications to the $Q_p$-spaces.

Proposition 3. An invariant differential operator

$$L = \sum_{m} l_m \Delta_m$$
satisfies $L|f|^2 \geq 0$ for all holomorphic $f$ if and only if

$$l_m \geq 0 \quad \forall m.$$  

**Proof.** From (22) and (30) we see that for any $f$ holomorphic,

$$\Delta_m |f|^2(x) = \sum_j |\psi_j(\partial)(f \circ \phi_x)(0)|^2 \geq 0.$$  

Thus $l_m \geq 0 \forall m$ implies $L|f|^2 \geq 0$.

On the other hand, if $f = \sum_n f_n$ then

$$\Delta_m |f|^2(0) = \sum_j |\psi_j f(0)|^2 = \sum_j |\langle \psi_j, f^* \rangle_F|^2 = \|f_m^*\|^2_F = \|f_m\|^2_F.$$  

Thus if $l_m < 0$ for some $m$, then $L|f_m|^2(0) < 0$ for any nonzero $f_m \in \mathcal{P}_m$. □

Recall that for any $L$ as in the last proposition and $\nu \in \mathbb{R}$, we have defined the $L$-Bloch space and the $Q_{\nu,L}$-space, respectively, by

$$\mathcal{B}_L = \{ f \text{ holomorphic on } \Omega : \sup_\Omega L|f|^2 < \infty \},$$

$$Q_{\nu,L} = \{ f \text{ holomorphic on } \Omega : \sup_{\phi \in \text{Aut}(\Omega)} \int_\Omega L|f \circ \phi|^2 h^\nu \, d\mu < \infty \},$$

the square roots of the indicated suprema being, by definition, the semi-norms in these Moebius invariant spaces. We have also agreed to denote, for brevity, $\mathcal{B}_L$ and $Q_{\nu,L}$ simply by $\mathcal{B}_m$ and $Q_{\nu,m}$ if $L = \Delta_m$.

**Corollary 4.** For any $L$ as in the preceding proposition and $\nu \in \mathbb{R}$,

$$\mathcal{B}_L = \bigcap_{m : l_m > 0} \mathcal{B}_m, \quad Q_{\nu,L} = \bigcap_{m : l_m > 0} Q_{\nu,m},$$

with the norm in $\mathcal{B}_L$ equivalent to $\max_m l_m \| \cdot \|_{\mathcal{B}_m}$, and similarly for $Q_{\nu,L}$.

**Proof.** Immediate from the fact that there can be only finitely many $m$ for which $l_m \neq 0$, and the fact that

$$\|f\|^2_{\mathcal{B}_L} = \sup_\Omega \sum_m l_m \Delta_m |f|^2$$
satisfies, on the one hand,
\[ \| f \|_{B_L}^2 \geq l_m \sup_{m} \Delta_m | f |^2 = l_m \| f \|_{B_m}^2 \]
for each \( m \), hence also
\[ \| f \|_{B_L}^2 \geq \left( \min_{m : l_m > 0} l_m \right) \left( \max_{m : l_m > 0} \| f \|_{B_m}^2 \right) ; \]
and on the other hand
\[ \| f \|_{B_L}^2 \leq \left( \sum_{m} l_m \right) \sup_{m : l_m > 0} \Delta_m | f |^2 \]
\[ = \left( \sum_{m} l_m \right) \left( \max_{m : l_m > 0} \| f \|_{B_m}^2 \right) . \]
Similarly for \( Q_{\nu,L} \).

The next proposition will be useful on several occasions later on. Note that the integral there is nothing but the value at \( \phi \in G \) of the convolution \( h^\rho * h^\nu \) of the two functions \( h^\rho, h^\nu \) on \( G \) (upon lifting them from \( \Omega \cong G/K \) to \( G \)); thus the proposition gives a characterization of the pairs \( \rho, \nu \) for which \( h^\rho * h^\nu \) is bounded.

**Proposition 5.** For \( \rho, \nu \in \mathbb{R} \), the supremum
\[ \sup_{\phi \in G \cap \Omega} \int h(\phi(z))^\rho h(z)^\nu d\mu(z) \]
is finite if and only if
\[ \nu \geq 0, \rho \geq 0, \text{ and } \rho + \nu > p - 1. \]

**Proof.** For \( \phi = \text{id} \), the integral becomes
\[ \int_{\Omega} h(z)^{\rho + \nu} d\mu(z), \]
which we know from Section 2 to be finite if and only if \( \nu + \rho > p - 1 \). Thus the supremum is certainly infinite if \( \nu + \rho \leq p - 1 \).

Next, assume that \( \nu + \rho > p - 1 \) and, say, \( \rho < 0 \). Then \( \nu > p - 1 \), so that \( d\mu_\nu(z) := h(z)^\nu d\mu(z) \) is a finite measure. Let \( e \) be a maximal tripotent and \( 0 < t < 1 \). It was shown in [4, Section 4] that as \( t \uparrow 1 \), \( \gamma_t e(z) \to e \) for any \( z \in \Omega \), and, hence, \( h(\gamma_t e(z))^\rho \to +\infty \). For each \( N > 0 \), denote temporarily \( f_N(t, z) := \min\{N, h(\gamma_t e(z))^\rho\} \). Then \( f_N(t, z) \to N \ \forall z \in \Omega \)
as \( t \nearrow 1 \), so by the Lebesgue Dominated Convergence Theorem

\[
\int_{\Omega} f_N(t, z) \, d\mu(z) \to N \mu_\nu(\Omega).
\]

Since \( \int_\Omega h(\gamma t e^Z(z))^{\rho} \, d\mu_\nu(z) \geq \int_\Omega f_N(t, z) \, d\mu_\nu(z) \) for any \( N \), it follows that

\[
\lim_{t \nearrow 1} \int_{\Omega} h(\gamma t e^Z(z))^{\rho} \, d\mu_\nu(z) = +\infty.
\]

Thus the supremum (31) is infinite in this case as well.

Owing to the invariance of the measure \( d\mu \), the integral in (31) remains unchanged if \( \phi \) is replaced by \( \phi^{-1} \) and \( \rho \) and \( \nu \) are interchanged. It follows that the supremum is infinite also if \( \nu + \rho > p - 1 \) and \( \nu < 0 \).

Thus it only remains to show that (31) is finite if \( \nu \geq 0 \), \( \rho \geq 0 \) and \( \rho + \nu > p - 1 \).

However, then by the Hölder inequality

\[
\int_{\Omega} h(\phi(z))^{\rho} h(z)^\nu \, d\mu(z)
\]

\[
\leq \left( \int_{\Omega} h(\phi(z))^{\rho+\nu} \, d\mu(z) \right)^{\frac{\rho}{\rho+\nu}} \left( \int_{\Omega} h(z)^{\rho+\nu} \, d\mu(z) \right)^{\frac{\nu}{\rho+\nu}}
\]

\[
= \left( \int_{\Omega} h(z)^{\rho+\nu} \, d\mu(z) \right)^{\frac{\rho}{\rho+\nu}} \left( \int_{\Omega} h(z)^{\rho+\nu} \, d\mu(z) \right)^{\frac{\nu}{\rho+\nu}}
\]

\[
= \mu_{\rho+\nu}(\Omega) < \infty,
\]

completing the proof. \( \square \)

Remark 6. An alternative way of proving that (31) is infinite if \( \rho + \nu > p - 1 \) but, say, \( \nu < 0 \) is by noting that

\[
\int_{\Omega} h(\phi(z))^{\rho} h(z)^\nu \, d\mu(z) = c_{\nu}^{-1} h(x)^\rho _2F_1(\rho, \rho + \nu; x, x),
\]

where \( x = \phi^{-1}(0) \) and \( _2F_1 \) is the Faraut-Koranyi-Yan hypergeometric function (cf. Section 4 of [8]). It is known that \( _2F_1(\alpha, \beta; \gamma; x, x) \approx h(x)^{\gamma - \alpha - \beta} \) if \( \alpha + \beta - \gamma > \frac{1}{2}a \); since \( \rho + \nu > p - 1 > \frac{1}{2}a \), we thus see that the last integral is \( \approx h(x)^\nu \) and, consequently, unbounded on \( \Omega \).

We conclude this section by describing the simplest Bloch and \( Q_\nu \)-spaces.

Proposition 7. Let \( L = I \), the identity operator. Then

\[
B_L = H^\infty(\Omega), \quad \text{the space of bounded analytic functions},
\]
while

\[
Q_{\nu,I} = \begin{cases} H^\infty, & \text{if } \nu > p - 1, \\ \{0\}, & \text{if } \nu \leq p - 1. \end{cases}
\]

**Proof.** Recall that

\[
B_I = \{ f \text{ holomorphic on } \Omega : \sup_\Omega |f|^2 < \infty \},
\]

\[
Q_{\nu,I} = \{ f \text{ holomorphic on } \Omega : \sup_{\phi \in G} \int_\Omega |f \circ \phi|^2 h^\nu d\mu < \infty \}.
\]

The assertion concerning \( B_I \) is thus trivial. For \( Q_{\nu,I} \), we know by the Ørsted-Faraut-Koranyi theorem (27) that whenever \( Q_{\nu,I} \) does not reduce to \( \{0\} \), then it contains the function constant one. On the other hand, by the last proposition (with \( \rho = 0 \) ), \( 1 \in Q_{\nu,I} \) if and only if \( \nu > p - 1 \).

**Example 8.** For \( L = \Delta_{(1,0,\ldots,0)} = \tilde{\Delta} \), the invariant Laplacian on \( \Omega \), we have by (29)

\[
B_L = \{ f \text{ holomorphic on } \Omega : \sup_{\phi \in G} \| \partial(f \circ \phi)(0) \| < \infty \},
\]

and \( Q_{\nu,L} \) consists of all holomorphic functions \( f \) on \( \Omega \) for which

\[
\sup_{\phi \in G} \int_\Omega \| \partial(f \circ \phi)(z) \|^2 h(z)^\nu d\mu(z) < \infty.
\]

The space \( B_L \) is the Bloch space studied by Timoney [15]. We will see in Theorem 18 below that unless \( \Omega \) is (biholomorphic to) the unit disc \( D \) or the unit ball \( B^d \), \( Q_{\nu,L} \) coincides with \( B_L \) for \( \nu > p - 1 \), and reduces to the constant functions for \( \nu \leq p - 1 \).

**Example 9.** Let \( \Omega \) be a tube type domain for which \( s := \frac{d}{r} \) is an integer, and let \( L = \Delta_{m} \) where \( m = (s, \ldots, s) =: (s^r) \). It is known that in this case the space \( P_{(s^r)} \) is one-dimensional and consists of multiples of \( N(z)^s \), where \( N \), the Jordan determinant polynomial (also called the Koecher norm), is a polynomial of degree \( r \); the kernel \( K_m \) is given by \( (s)_m K_m(x,y) = N(x)^s N(y)^s \); and for any \( f \) holomorphic, \( N(\partial)^s (f \circ \phi_x)(0) = (-1)^d h(x)^s N(\partial)^s f(x) \). Hence,

\[
\Delta_{(s^r)} |f|^2 = h^p |N(\partial)^s f|^2
\]

and

\[
B_{(s^r)} = \{ f \text{ holomorphic on } \Omega : h^p |N(\partial)^s f|^2 \text{ is bounded} \}.
\]

This is the so-called top quotient Bloch space studied by the first author in connection with generalized Hankel operators [3]. (The terminology comes
from the fact that the associated Bloch seminorm vanishes on $\mathcal{M}_{q-1}$, so that $\mathcal{B}_{(s^r)}$ really “lives” on the top quotient $\mathcal{M}_{q}/\mathcal{M}_{q-1}$ of the composition series.) It is the maximal $\text{Aut}(\Omega)$-invariant space of holomorphic functions on $\Omega$.

Further, for $\nu = 0$ we have, by the invariance of $d\mu$,

$$Q_{0,(s^r)} = \left\{ f \text{ holomorphic on } \Omega : \int_{\Omega} |N(\partial)^s f(z)|^2 \, dz < \infty \right\},$$

which is, by definition, the generalized Dirichlet space of $\Omega$. It is the unique $\text{Aut}(\Omega)$-invariant Hilbert space of holomorphic functions on $\Omega$ (modulo $\mathcal{M}_{q-1}$).

We remark that the definitions (6) and (7) of $\mathcal{B}_L$ and $Q_{\nu,L}$ are special cases of a more general construction of Moebius invariant spaces, which goes as follows. Let $X$ be any Banach space of holomorphic functions on $\Omega$ with the property that $f \in X$ and $\phi \in \text{Aut}(\Omega)$ imply $f \circ \phi \in X$. We define $M(X)$ to be the space of all $f \in X$ for which $\|f\|_{M(X)} := \sup_{\phi \in \text{Aut}(\Omega)} \|f \circ \phi\|_X < \infty$. Then $M(X)$ is $\text{Aut}(\Omega)$-invariant. Of course, one can replace here “Banach space” with “semi-Banach space” (i.e. complete semi-normed space). This construction is very basic and generalizes the spaces defined by (6) and (7). Notice that if $X$ is already $\text{Aut}(\Omega)$-invariant then $M(X) = X$. Finally, the composition $f \mapsto f \circ \phi$ can be replaced by the weighted action $f \mapsto (\det \phi')^{\nu/p}(f \circ \phi)$, with some fixed real parameter $\nu$, which leads to weighted analogues of all the above Moebius-invariant spaces (in particular, to “weighted” analogues of Bloch and $Q_{\nu}$ spaces). The authors hope to return to this topic in future.

4. Main results

Recall that we have defined, for a signature $m$ and a real number $\nu$,

$$\mathcal{B}_m = \{ f \text{ holomorphic on } \Omega : \sup_{\Omega} \Delta_m |f|^2 < \infty \},$$

$$Q_{\nu,m} = \left\{ f \text{ holomorphic on } \Omega : \sup_{\phi \in G} \int_{\Omega} \Delta_m |f \circ \phi|^2 \, h^\nu \, d\mu < \infty \right\},$$

the square roots of the indicated quantities being the seminorms in these spaces.

**Lemma 10.** The involution

$$f \mapsto f^*, \quad f^*(z) := \overline{f(z)},$$

(32)
maps each $\mathcal{P}_m$ into itself.

**Proof.** Since $\mathcal{P}_m$ is spanned by $K_m(\cdot,y), y \in \mathbb{C}^d$, the image $\mathcal{P}_m^*$ of $\mathcal{P}_m$ under (32) is spanned by $K_m(\cdot,\overline{y})^*, y \in \mathbb{C}^d$. Thus it is enough to show that $K_m(\cdot,\overline{y})^* = K_m(\cdot,y)$ — that is, that $K_m(\overline{y},\overline{x}) = K_m(x,y)$ for all $x,y$. As both sides are holomorphic in $x$ and $\overline{y}$, and any such function is uniquely determined by its restriction to the diagonal $x = y$ [6, Proposition II.4.7], it is in turn enough to show that $K_m(\overline{z},\overline{z}) = K_m(z,z)$ $\forall z \in \mathbb{C}^d$.

However, an examination of the list of Cartan’s domains in the table in Section 2 reveals that they are all preserved by complex conjugation; hence, so are the stabilizer subgroup $K$ and the Jordan triple product $\{\cdot,\cdot,\cdot\}$.

It follows that $\tau_1,\ldots,\tau_r$ is a Jordan frame whenever $e_1,\ldots,e_r$ is, and that $\overline{\tau} = \overline{K(t_1\tau_1 + \cdots + t_r\tau_r)} = k(t_1e_1 + \cdots + t_re_r)$. Since $K$ acts transitively on the set of all Jordan frames, there must exist $k' \in K$ such that $k'\overline{\tau} = ke_j \forall j$, i.e. $k'\overline{\tau} = z$. By $K$-invariance, this implies that $K_m(\overline{\tau},\overline{\tau}) = K_m(k'\overline{\tau},k'\overline{\tau}) = K_m(z,z)$.

**Proposition 11.** If $\ell < q(m)$, then the $Q_{\nu,m}$-seminorm vanishes on $\mathcal{M}_\ell$; thus $\mathcal{M}_\ell$ is contained in $Q_{\nu,m}$ in a trivial way.

The same is true also for the Bloch space $\mathcal{B}_m$.

**Proof.** Choose an orthonormal basis $\{\psi_j\}_{j=1}^{4m}$ for $\mathcal{P}_m$. Then, by (22),

$$
\Delta_m |f|^2(z) = K_m(\partial,\partial)|f \circ \phi_z|^2(0) = \sum_j \psi_j(\partial)\overline{\psi_j(\partial)}|f \circ \phi_z|^2(0) = \sum_j |\psi_j(\partial)(f \circ \phi_z)(0)|^2 = \sum_j |(f \circ \phi_z,\psi_j^*)|^2.
$$

(33)

Since, by Lemma 10, $\{\psi_j^*\}$ is also a basis for $\mathcal{P}_m$, this equals $\|P_m(f \circ \phi_z)\|^2$, where $P_m$ denotes the projection $g = \sum_n g_n \mapsto g_m$ onto $\mathcal{P}_m$.

Thus $f \in \mathcal{M}_\ell \implies f \circ \phi_z \in \mathcal{M}_\ell \implies P_m(f \circ \phi_z) = 0 \implies \Delta_m |f|^2 = 0 \implies f \in \mathcal{B}_m$ and $f \in Q_{\nu,m}$.

**Remark 12.** In Section 1 we used the notation

$$
\mathcal{N}_m := \{ f \text{ holomorphic on } \Omega : \Delta_m |f|^2 \equiv 0 \}
$$

for the subspace of $\mathcal{B}_m$ on which the $m$-Bloch seminorm vanishes. It follows from the last proof that, in fact,

$$
\mathcal{N}_m = \mathcal{M}_{q(m)}.
$$
We will see in a moment (cf. Corollary 15) that the Bloch spaces $B_m$ also depend only on the “height” $q(m)$ of $m$.

**Proposition 13.** If $\nu > p - 1$, then $B_m \subset Q_{\nu,m}$ continuously.

**Proof.** Since the measure $d\mu_\nu := h^\nu \, d\mu$ is finite for $\nu > p - 1$, we have, for any $\phi \in G$,

$$\int_\Omega (\Delta_m |f|^2) \circ \phi \, h^\nu \, d\mu \leq \mu_\nu(\Omega) \| \Delta_m |f|^2 \|_\infty = \mu_\nu(\Omega) \| f \|_{B_m}^2.$$ 

Taking supremum over all $\phi \in G$ yields the assertion. \qed

**Theorem 14.** If $q(m) \leq q(n)$, then $Q_{\nu,m} \subset B_n$ continuously.

**Proof.** By the $K$-invariance of $\Delta_m$ and $h$, the integral

$$\int_\Omega \Delta_m(fg) \, h^\nu \, d\mu$$

is a positive-definite $K$-invariant bilinear form in $f, g \in \mathcal{P}$. As noted in Section 2, it is a consequence of Schur’s lemma from representation theory that any such bilinear functional must be of the form

$$\sum_k c_{mk} \langle f_k, g_k \rangle_F,$$

for some coefficients $c_{mk} \geq 0$. Suppose we can show that

(34) \quad c_{mn} > 0.

Since $\Delta_n |f|^2(0) = \| P_n f \|_F^2 = \| f_n \|_F^2$, by (33), it will follow that

$$\Delta_n |f|^2(0) \leq \frac{1}{c_{mn}} \int_\Omega \Delta_m |f|^2 \, h^\nu \, d\mu.$$

Replacing $f$ by $f \circ \phi_x$, this becomes

$$\Delta_n |f|^2(x) \leq \frac{1}{c_{mn}} \int_\Omega \Delta_m |f \circ \phi_x|^2 \, h^\nu \, d\mu.$$

Taking suprema over all $x \in \Omega$ gives the assertion.

It remains to prove (34). But by the properties of the composition series,

$$c_{mn} = 0 \iff \int_\Omega \Delta_m |f_n|^2 \, h^\nu \, d\mu = 0 \quad \forall f_n \in \mathcal{P}_n$$
$\iff \Delta_m |f_n|^2(z) = 0 \quad \forall z \forall f_n$

$\iff \|P_m(f_n \circ \phi_z)\|^2 = 0 \quad \forall z \forall f_n$ by (33)

$\iff P_m(f_n \circ \phi_z) = 0 \quad \forall z \forall f_n$

$\iff P_m M_q(n) = 0$ by (27)

$\iff q(m) > q(n)$.

**Corollary 15.** If $\nu > p - 1$, then $Q_{\nu, m} = B_m$, with equivalent norms.
If $q(m) \leq q(n)$, then $B_m \subset B_n$ continuously.
If $q(m) = q(n)$, then $B_m = B_n$, with equivalent norms.
If $q(m) = q(n)$ and $\nu > p - 1$, then $Q_{\nu, m} = Q_{\nu, n}$, with equivalent norms.

The last corollary exhausts the case $\nu > p - 1$ completely. Let us now turn to $\nu \leq p - 1$.

In the sequel, similarly as we did with $h(z, z)$, we will often abbreviate $K_m(z, z)$ just to $K_m(z)$ (or even to $K_m$).

**Lemma 16.** For any signature $m$, there exist constants $\alpha > 0$ and $c > 0$

such that

$\Delta_m K_m \geq c h^\alpha$ on $\Omega$.

**Proof.** For $m = (0, \ldots, 0)$ this is trivial, so assume $|m| > 0$. Then

$\Delta_m K_m(x) = K_m(\partial_x, \partial_x) \sum_{i_1, \ldots, i_d} c_{\alpha\beta} \prod_{k} \frac{\partial^{\alpha_k} K_m(x, x)}{\partial^{\alpha_k} (\phi_x(z, \phi_x(z))) |_{z=0}}$

is an expression of the form

$\sum_{\alpha, \beta, \alpha_1, \ldots, \beta_1, \ldots} c_{\alpha\beta} \prod_{k} \frac{\partial^{\alpha_k} K_m(x, x)}{\partial^{\alpha_k} (\phi_x(z, \phi_x(z))) |_{z=0}}$

with some constants $c_{\alpha\beta}$ (independent of $x$), and with the summation extending over multiindices $\alpha, \beta, \alpha_1, \ldots, \beta_1, \ldots$ satisfying $|\alpha_j|, |\beta_j| > 0 \forall j$.

(Here $(\phi_x)_j(z)$ stands for the $j$-th coordinate of $\phi_x(z, j = 1, \ldots, d)$.) From (15) one can see that

$\phi_x'(z) = -B(x, x)^{1/2}B(z, x)^{-1},$ $B$ being the Bergman operator (12). Using the formula

$(X^{-1})' = -X^{-1}X'X^{-1}$

for the derivative of any invertible-operator-valued function $X(z)^{-1}$, it follows by iteration that for any multiindex $\gamma, |\gamma| > 1$, and any $j = 1, \ldots, d$,

$\partial^\gamma (\phi_x)_j(0) = B(x, x)^{1/2}p_{\gamma j}(x)$.
for some polynomial $p_j$ of the coordinates of $x$.

Since both $\Delta_m$ and $K_m$ are $K$-invariant, so is the function $\Delta_m K_m$; thus it is enough to evaluate it only for $x = t_1 e_1 + \cdots + t_r e_r$ for some Jordan frame $e_1, \ldots, e_r$ and $t_1, \ldots, t_r \in [0, 1]$. By (17), the quantity (37) — and, hence, also (35) — will then be an expression of the form

$$a \text{ polynomial in } t_1, \ldots, t_r \text{ and } \sqrt{1 - t_1^2}, \ldots, \sqrt{1 - t_r^2}.$$ 

Making the substitution $t_j = 1 - \tau_j^2$, $\tau_j \in [0, 1]$, $j = 1, \ldots, r$, this becomes

$$a \text{ polynomial in } \tau_1, \ldots, \tau_r \text{ and } \sqrt{2 - \tau_1^2}, \ldots, \sqrt{2 - \tau_r^2} = G(\tau_1, \ldots, \tau_r).$$

However, this is clearly a holomorphic function of $\tau_1, \ldots, \tau_r$ on the polydisc $\{ |\tau_j| < \sqrt{2} \, \forall j \}$. Let $V_k$, $k = 0, 1, 2, \ldots$, be the set of all points in this polydisc where $G(\tau)$ has a zero of order at least $k$ (i.e. vanishes together with all its partial derivatives of orders $< k$). Then there exists a $k$ for which $V_k \cap \mathbb{D}^r = \emptyset$: otherwise the decreasing chain of compact subsets $\{V_k \cap \mathbb{D}^r\}_{k \geq 0}$ would have a nonempty intersection, i.e. there would exist a point in $\mathbb{D}^r$ where $G$ vanishes together with its partial derivatives of all orders; as $G$ is holomorphic this would mean that $G$ vanishes identically, contradicting the fact that $G = \Delta_m K_m > 0$ for $\tau \in (0, 1)^r$. Now $V_k \cap \mathbb{D}^r = \emptyset$ means that $\lim_{\tau \to \sigma} \frac{G(\tau)}{\|\tau - \sigma\|^k} = +\infty \forall \sigma \in \mathbb{D}^r$. Consider $\sigma$ of the form $\sigma_1 = \cdots = \sigma_m = 0$ and $\sigma_{m+1}, \ldots, \sigma_r \in (0, 1)$. Then as $\tau \to \sigma$, we eventually have $|\tau_j| \leq 1$ for $j = m + 1, \ldots, r$ while $|\tau_j| \leq \|\tau - \sigma\|$ for $j = 1, \ldots, m$; thus $\|\tau - \sigma\|^m \geq |\tau_1 \cdots \tau_r|$. Since $|\tau_1 \cdots \tau_r| \asymp h^{1/2}$ for $\tau_1, \ldots, \tau_r \in [0, 1]$, it follows that $h^{-m/2} \leq \|\tau - \sigma\|^k$ and

$$\frac{G}{h^{k/2m}} \gtrsim \frac{G}{\|\tau - \sigma\|^k} \to +\infty$$

as $\tau \to \sigma$. As $m \geq 1$, this implies that

$$\frac{G}{h^{k/2}} \to +\infty \quad \text{as } [0, 1]^r \ni \tau \to \sigma.$$ 

It follows that the (continuous) function $G/h^{k/2} = \Delta_m K_m/h^{k/2}$ on $\Omega$ is positive on $\Omega$ and tends to $+\infty$ at $\partial \Omega$. Thus it must be bounded from below by some $c > 0$. Taking $\alpha = k/2$, the claim follows. $\square$
Theorem 17. If $\nu < 0$, then $Q_{\nu, m} = M_{q(m)}^{-1}$.

Proof. From the Ørsted-Faraut-Koranyi theorem (27) we know that

$$M_{q(m)}^{-1} \subseteq Q_{\nu, m}$$

$$\implies \mathcal{P} \cap M_{q(m)} \subset Q_{\nu, m}$$

$$\implies \mathcal{P}_m \subset Q_{\nu, m}$$

$$\implies \sup_x \int_{\Omega} \Delta_m |f|^2 (h \circ \phi_x)^\nu \, d\mu < \infty \quad \forall f \in \mathcal{P}_m.$$ 

Since $K_m(z, z) = \sum_j |\psi_j(z)|^2$ for any basis $\{\psi_j\}$ of $\mathcal{P}_m$, we can continue by

$$\implies \sup_x \int_{\Omega} \Delta_m K_m \cdot (h \circ \phi_x)^\nu \, d\mu < \infty$$

(where we again write just $K_m$ for $K_m(z, z)$). By Lemma 16, we can in turn continue by

$$\implies \sup_x \int_{\Omega} h^\alpha (h \circ \phi_x)^\nu \, d\mu < \infty.$$ 

By Proposition 5, this is only possible if $\nu \geq 0$. \qed

Recall that the only Cartan domain of rank 1 is the unit ball $B^d$, $d \geq 1$. Thus the following theorem means that the situation for $r > 1$ differs radically from the one for $r = 1$, when $Q_{\nu}$ is nontrivial also for $p - 2 < \nu \leq p - 1$ (for the disc, even for $p - 2 \leq \nu \leq p - 1$) in view of (2) and (4).

Theorem 18. For $r > 1$ and $m = (1, 0, \ldots, 0) =: (1)$, that is,

$$f \in Q_{\nu,(1)} \iff \sup_{\phi \in \mathcal{G}} \int_{\Omega} \Delta |f \circ \phi|^2 h^\nu \, d\mu < \infty,$$

we have

$$Q_{\nu,(1)} = \begin{cases} B_{(1)}, \text{ the Timoney Bloch space,} & \text{if } \nu > p - 1, \\ \{\text{constants}\}, & \text{if } \nu \leq p - 1. \end{cases}$$

Proof. The constants are always contained in $Q_{\nu,(1)}$, by Theorem 11. As in the preceding proof, we have

$$\{\text{constants}\} \subset Q_{\nu,(1)} \implies \sup_{\phi \in \mathcal{G}} \int_{\Omega} \Delta (1) K_{(1)} (h \circ \phi)^\nu \, d\mu < \infty,$$
that is,
\[
\sup_{\phi \in C} \int_{\Omega} (\Delta \| \cdot \|)^2 (h \circ \phi)^r \, d\mu < \infty.
\]

Since the coordinate functions \( z_1, \ldots, z_d \) are a basis of \( \mathcal{P}_1 \), we have by (29)
\[
(\Delta \| \cdot \|)^2 (x) = \sum_{j=1}^d (\Delta | \cdot |^2)_j (x) = \sum_{j,k=1}^d |\partial_k (\phi_x)_j (0)|^2.
\]

However, by (36), \( \partial_k (\phi_x)_j (0) \) is precisely the \((j,k)\)-entry of the matrix \(-B(x,x)^{1/2}\). Thus
\[
(\Delta \| \cdot \|)^2 (x) = \| B(x,x)^{1/2} \|_{HS}^2
\]
is the square of the Hilbert-Schmidt norm of the operator \( B(x,x)^{1/2} \) on \( \mathbb{C}^d \). If \( x \) has the polar decomposition (16), then we know from (17) that \( B(x,x)^{1/2} \) is a diagonal operator with respect to the Peirce decomposition (13), with eigenvalues \((1 - t_j^2)^{1/2}(1 - t_j^2)^{1/2}\) on each \( \mathcal{Z}_{ij} \). Since
\[
\dim \mathcal{Z}_{ij} = \begin{cases} a & \text{for } 1 \leq i < j \leq r, \\ b & \text{for } i = 0 < j \leq r, \\ 1 & \text{for } 1 \leq i = j \leq r, \\ 0 & \text{for } i = j = 0, \end{cases}
\]
it follows that
\[
\| B(x,x)^{1/2} \|_{HS}^2 = a \sum_{1 \leq i < j \leq r} (1 - t_i^2)(1 - t_j^2) + b \sum_{1 \leq j \leq r} (1 - t_j^2) + \sum_{j=1}^r (1 - t_j^2)^2
\]
\[
: = F(t_1, \ldots, t_r).
\]

Now taking \( \phi = \text{id} \) in (38), we get by (20)
\[
\int_{\Omega} (\Delta \| \cdot \|)^2 h^r \, d\mu = \int_{\Omega} \| B(z,z)^{1/2} \|_{HS}^2 h(z)^r \, d\mu (z)
\]
\[
= \int_{[0,1]^r} F(t_1, \ldots, t_r) \prod_{j=1}^r (1 - t_j^2)^{\nu - p} \prod_{j=1}^r t_j^{2b+1} \prod_{1 \leq i < j \leq r} |t_i^2 - t_j^2|^{a} \, dt_1 \ldots \, dt_r
\]
\[
\geq \int_{t_1 = 1 - 1/2r}^{1/2r} \int_{t_2 = 1/2r}^{4/2r} \ldots \int_{t_r = (2r - 3)/2r}^{(2r - 2)/2r} F(t_1, \ldots, t_r) \cdot (1 - t_1)^{\nu - p}
\]
\[
\cdot \left( \frac{1}{r} \right)^{(r-1)p} \left( \frac{1}{(2r)^r} \cdot \left( 1 - \frac{1}{2r} \right) \right)^{2b+1} \left( \frac{1}{r} \cdot \frac{1}{2r} \right)^{\nu - p} \, dt_1 \ldots \, dt_r.
\]
Since $F(t) \geq 1 - t_2^2 \geq 1 - \frac{2}{r}$ (here the hypothesis that $r > 1$ was used) on the last domain of integration, we can continue the estimate with

$$\geq C_r \int_{1 - 1/2r}^1 (1 - t_1)^{\nu - p} \, dt_1.$$  

But the last integral is finite only for $\nu > p - 1$. Since we know that $Q_{\nu, (1)} = B_{(1)}$ for such $\nu$, by Corollary 15, this completes the proof. $\square$

Note that the proof shows that for $\nu \leq p - 1$, not only the supremum (38) is infinite, but in fact the integral occurring there is infinite for $\phi = \text{id}$ and, hence, for any $\phi \in G$ (since $(h \circ \phi)/h$ is bounded and bounded away from zero on $\Omega$ for any fixed $\phi$).

The methods of proofs of the last two theorems can be adapted a little to yield the following result.

**Theorem 19.** Let

$$\rho_m = \sup \left\{ \rho \geq 0 : \frac{\Delta_m K_m}{h^\rho} \text{ is bounded on } \Omega \right\}. \tag{40}$$

Then this supremum is attained (i.e. is a maximum) and finite, $\rho_m$ is always an integer or a half-integer, and $Q_{\nu, m}$ is nontrivial (i.e. does not reduce to $\mathcal{M}_{q(m) - 1}$) if and only if

$$\nu \geq 0 \text{ and } \nu > p - 1 - \rho_m.$$  

**Proof.** We have already seen in the proofs of Theorems 17 and 18 that

$$\mathcal{M}_{q(m) - 1} \subsetneq Q_{\nu, m} \implies \sup_{\phi \in G} \int_{\Omega} \Delta_m K_m (h \circ \phi)^\nu \, d\mu < \infty.$$  

Conversely, if the last supremum is finite, then — since $\Delta_m K_m = \sum_j \Delta_m |\psi_j|^2$ for any orthonormal basis $\psi_j$ of $P_m$ — we have

$$\sup_{\phi \in G} \int_{\Omega} \Delta_m |\psi_j|^2 (h \circ \phi)^\nu \, d\mu < \infty \quad \forall j,$$

i.e. $\psi_j \in Q_{\nu, m} \forall j$, whence $P_m \subset Q_{\nu, m}$, so $Q_{\nu, m} \supseteq \mathcal{M}_{q(m) - 1}$. We thus see that

$$Q_{\nu, m} \text{ is nontrivial } \iff \sup_{\phi \in G} \int_{\Omega} \Delta_m K_m (h \circ \phi)^\nu \, d\mu < \infty. \tag{41}$$

Next, we have seen in the proof of Lemma 16 that

$$(\Delta_m K_m)(k(t_1 e_1 + \cdots + t_r e_r)) = F(t_1, \ldots, t_r),$$
where $F$ is a polynomial in $t_1, \ldots, t_r$ and $\sqrt{1-t_1^2}, \ldots, \sqrt{1-t_r^2}, 0 \leq t_j \leq 1$; and that, hence,

$$F(t_1, \ldots, t_r) = G(\tau_1, \ldots, \tau_r), \quad t_j = 1 - \tau_j^2,$$

where $G$ is a polynomial in $\tau_1, \ldots, \tau_r$ and $\sqrt{2-\tau_1^2}, \ldots, \sqrt{2-\tau_r^2}, 0 \leq \tau_j \leq 1$, and, consequently, extends to a holomorphic function in the polydisc $(\sqrt{2D})^r = \{ |\tau_j| < \sqrt{2} \forall j \}$. Let

$$G(\tau) = \sum_{\alpha \text{ multiindex}} g_\alpha \tau^\alpha$$

be the Taylor expansion of $G$ around the origin. Let $k \geq 0$ be the greatest integer such that

$$g_\alpha = 0 \text{ whenever } \max\{\alpha_1, \ldots, \alpha_r\} \leq k.$$

Then

$$G(\tau) = (\tau_1 \ldots \tau_r)^k H(\tau)$$

where

$$H(\tau) = \sum_{\alpha} c_\alpha \tau^\alpha, \quad c_\alpha := g_\alpha + (k,k,\ldots,k),$$

is still holomorphic in $(\sqrt{2D})^r$, and there exists $\alpha$ such that $\alpha_j = 0$ for some $j$ and $c_\alpha \neq 0$. Since $F$ is symmetric in $t_1, \ldots, t_r$, and thus $G$ and $H$ are symmetric in $\tau_1, \ldots, \tau_r$, we may assume that $j = 1$. Thus $c_{\alpha_2,\ldots,\alpha_r} \neq 0$ for some $\alpha_2, \ldots, \alpha_r$; consequently,

$$H(0, \tau_2, \ldots, \tau_r) = \sum_{\alpha_2,\ldots,\alpha_r = 0}^{\infty} c_{\alpha_2,\ldots,\alpha_r} \tau_2^{\alpha_2} \cdots \tau_r^{\alpha_r}$$

does not vanish identically, and therefore assumes nonzero values in any neighbourhood of the origin in $\mathbb{R}^{r-1}$. It follows that $H(\tau)/(\tau_1 \ldots \tau_r)^r$ is unbounded in any neighbourhood of the origin for any $\epsilon > 0$. Consequently, $G(\tau)/(\tau_1 \ldots \tau_r)^k$ is bounded on $(0, 1)^r$, but $G(\tau)/(\tau_1 \ldots \tau_r)^{k+\epsilon}$ is not bounded there for any $\epsilon > 0$. Since

$$h(k(t_1 e_1 + \cdots + t_r e_r)) = \prod_{j=1}^{r} \tau_j^2 (2 - \tau_j^2) \preceq (\tau_1 \ldots \tau_r)^2,$$

it follows that $(\Delta_\mu K_\mu)/h^\rho$ is bounded for $\rho = k/2$, but not for any $\rho > k/2$. Thus the first part of the theorem follows, with $\rho_\mu = k/2$;
moreover, we see that the function
\[
\frac{F(t_1, \ldots, t_r)}{\prod_{j=1}^{r}(1 - t_j^2)^{k/2}} =: E(t_1, \ldots, t_r)
\]
is continuous on \([0, 1]^r\) and positive at some point \(t\) with \(t_1 = 1\). By continuity, there must exist \(t \in [0, 1]^r\) such that
\[
E(t) \geq 2\delta, \quad t_1 = 1, \quad t_2, \ldots, t_r \in [2\delta, 1 - 2\delta], \quad |t_j - t_k| \geq 2\delta \ \forall j \neq k,
\]
for some \(\delta > 0\). Let \(U\) be a cubical neighbourhood of this point in \([0, 1]^r\) so small that
\[
E(t) \geq \delta, \quad t_1 \in [1 - \delta, 1], \quad t_2, \ldots, t_r \in [\delta, 1 - \delta], \quad |t_j - t_k| \geq \delta \ \forall j \neq k,
\]
for all \(t \in U\). We may assume that \(\delta < 1\), so that \(\delta < 1 - \delta\). Proceeding as in the proof of Theorem 18, we then have
\[
\int_{\Omega} \Delta_m K_m h^\nu \, d\mu \\
= \int_{[0, 1]^r} F(t) \prod_{j=1}^{r}(1 - t_j^2)^{\nu-p} \prod_{j=1}^{r} t_j^{2b+1} \prod_{1 \leq i < j \leq r} |t_i^2 - t_j^2|^a \, dt_1 \ldots dt_r \\
= \int_{[0, 1]^r} E(t) \prod_{j=1}^{r}(1 - t_j^2)^{\nu-p+\rho_m} \prod_{j=1}^{r} t_j^{2b+1} \prod_{1 \leq i < j \leq r} |t_i^2 - t_j^2|^a \, dt_1 \ldots dt_r \\
\geq \int_{U} \ldots \\
\geq \int_{U} \delta \cdot (1 - t_1)^{\nu-p+\rho_m} \delta^{(\nu-p+\rho_m)(r-1)} \cdot (2\delta^2)^{\frac{r(r+1)}{2}} dt_1 \ldots dt_r \\
\geq C \delta \int_{1-\epsilon}^{1} (1 - t_1)^{\nu-p+\rho_m} \, dt_1.
\]
The last integral is finite only for \(\nu - p + \rho_m > -1\), i.e. \(\nu > p - 1 - \rho_m\). Thus, by (41), \(Q_{\nu, m}\) is trivial if \(\nu \leq p - 1 - \rho_m\).

From Theorem 17, we also already know that \(Q_{\nu, m}\) is trivial for \(\nu < 0\). Thus it remains to prove that the supremum in (41) is finite if \(\nu \geq 0\) and \(\nu > p - 1 - \rho_m\).

However, from the boundedness of \((\Delta_m K_m)/h^{\rho_m}\) it follows that \(\Delta_m K_m \leq C h^{\rho_m}\) for some \(0 < C < \infty\), so for any \(\phi \in G\)
\[
\int_{\Omega} (\Delta_m K_m) (h \circ \phi)^\nu \, d\mu \leq C \int_{\Omega} h^{\rho_m} (h \circ \phi)^\nu \, d\mu.
\]
But by Proposition 5, the supremum of the right-hand side over all $\phi \in G$ is finite if $\nu \geq 0$ and $\nu + \rho_m > p - 1$. The proof is complete. \qed

As in the preceding theorem, we even see that for $\nu \leq p - 1 - \rho_m$, not only the supremum in (41) is infinite, but in fact the integral there is infinite for any $\phi \in G$.

We finish this section by a result which characterizes the Bloch spaces $\mathcal{B}_m$ as maximal spaces of holomorphic functions on each given quotient $\mathcal{M}_q/\mathcal{M}_t$ of the composition series. It generalizes the analogous characterizations for the ordinary Bloch space $\mathcal{B}_1$ of Timoney and for the top quotient Bloch space $\mathcal{B}_{(s')}$ on tube-type domains.

**Theorem 20.** Let $X$ be any semi-Banach space of holomorphic functions on $\Omega$ such that

1. $X$ is Möbius invariant, i.e. $f \in X$ and $\phi \in G$ imply $f \circ \phi \in X$ and $\|f \circ \phi\|_X = \|f\|_X$;
2. if $\sigma$ is any finite Borel measure on the stabilizer subgroup $K$ then the operator of convolution with $\sigma$,

$$C_\sigma f(z) := \int_K f(k^{-1}z) d\sigma(k),$$

is bounded on $X$.

Let further $m$ be any signature such that

$$f \in X \text{ and } \|f\|_X > 0 \quad \text{for some } f \in \mathcal{P}_m.$$ (42)

Then $X \subset \mathcal{B}_m$ continuously.

**Proof.** From the hypothesis 2. and the representation

$$\mathcal{P}_m f(z) = \int_K f(k^{-1}z) \chi_m(k) dk,$$ (43)

where $\chi_m$ is the character of $K$ associated with $m$, it follows that the canonical projection $\mathcal{P}_m$ onto $\mathcal{P}_m$ is bounded on $X$. From (42), we further have $\|\mathcal{P}_m x\|_X > 0$ and, by the property (27) of the composition series, $\mathcal{M}_q(m) \cap \mathcal{P} \subset X$; in particular, $\mathcal{P}_m \subset X$. Finally, as $\mathcal{P}_m$ is finite-dimensional,

$$\alpha_m \|f\|_X \leq \|f\|_F \leq \beta_m \|f\|_X \quad \forall f \in \mathcal{P}_m$$

for some constants $\alpha_m$ and $\beta_m$. Let $f \in X$ and $\phi \in G$. Then

$$\|f\|_X = \|f \circ \phi\|_X \geq \frac{\|\mathcal{P}_m (f \circ \phi)\|_X}{\|\mathcal{P}_m\|_{X \to X}} \geq \frac{\|\mathcal{P}_m (f \circ \phi)\|_F}{\beta_m \|\mathcal{P}_m\|_{X \to X}}.$$
Taking supremum over all \( \phi \in G \) and recalling that \( \| P_m(f \circ \phi) \|_F^2 = \Delta_m |f|^2(\phi(0)) \), we obtain
\[
\| f \|_X \geq \frac{\| f \|_{B_m}}{\beta_m \| P_m \|_{X \rightarrow X}}.
\]
This means that \( X \subset B_m \) continuously. \( \square \)

Note that \( P_m \subset B_m \) for any \( m \). In fact, even
\[
(44) \quad H^\infty(\Omega) \subset B_m \quad \text{continuously}
\]
for any \( m \); this can be seen as follows. From the representation (43), we obtain for any \( z \in \Omega \) and \( f \) holomorphic on \( \Omega \)
\[
|P_m f(z)| = \left| \int_K f(k^{-1}z) \chi_m(k) \, dk \right|
\leq \| f \|_\infty \left( \int_K |\chi_m(k)|^2 \, dk \right)^{1/2}
= \| f \|_\infty
\]
(where \( \| \cdot \|_\infty \) stands for the supremum norm on \( \Omega \)). Thus \( \| P_m f \|_\infty \leq \| f \|_\infty \). Also, since \( P_m \) is finite-dimensional, the Fock norm is equivalent to the \( L^2(\Omega) \)-norm on \( P_m \). Thus for any \( \phi \in G \),
\[
\| P_m(f \circ \phi) \|_F^2 = \int_\Omega |P_m(f \circ \phi)(z)|^2 \, dz
\leq \text{vol}(\Omega) \| P_m(f \circ \phi) \|_\infty^2
\leq \text{vol}(\Omega) \| f \circ \phi \|_\infty^2
= \text{vol}(\Omega) \| f \|_\infty^2.
\]
Taking supremum over all \( \phi \in G \) and recalling that \( \| P_m(f \circ \phi) \|_F^2 = \Delta_m |f|^2(\phi(0)) \), it follows that
\[
\| f \|_{B_m} \leq \text{vol}(\Omega)^{1/2} \| f \|_\infty,
\]
proving (44).

Thus \( B_m \) is maximal among the spaces of holomorphic functions that contain \( P_m \) and whose seminorm does not vanish identically on \( P_m \). Since, by the property (27) of the composition series, \( P_m \subset \{ f \in X : \| f \|_X = 0 \} \) implies \( P_n \subset \{ f \in X : \| f \|_X = 0 \} \) whenever \( q(m) = q(n) \), it follows that \( B_m \subset B_n \) and, hence, by symmetry, \( B_m = B_n \) with equivalent norms, whenever \( q(m) = q(n) \). This gives another proof of Theorem 14 as well as of the second and the third parts of Corollary 15.
5. Concluding remarks

Theorem 19 reduces the question of nontriviality of $Q_{\nu,m}$ to the determination of the number $\rho_m$, i.e. to a question concerning the boundary behaviour of the function $\Delta_m K_m$. Unfortunately, in general we do not have a complete answer to the latter question either.

For $m = (0, \ldots, 0) =: (0)$, one has trivially $\Delta(0) K(0) = 1$ and $\rho(0) = 0$, in agreement with Proposition 7; we will thus assume that $|m| > 0$ from now on.

On the unit disc $D$, the operator $\Delta_m$, being a polynomial in the invariant Laplacian $\tilde{\Delta} = (1 - |z|^2)^2 \Delta$, always contains the factor $(1 - |z|^2)^2$, and thus $\rho_m \geq 2 \forall m$. Since $p - 1 = 1$ in this case, the spaces $Q_{\nu,m}(D)$ are thus nontrivial if and only if $\nu \geq 0$. For $\nu > 1$, $Q_{\nu,m} = B(D)$, the Bloch space, by Corollary 15. For $0 \leq \nu \leq 1$, the spaces $Q_{\nu,m}$ are the familiar spaces from (2) if $m = (1)$, but we do not know anything about them for any other nonzero $m$.

Conjecture 21. For any $m \neq (0)$ and any $0 \leq \nu \leq 1$, the spaces $Q_{\nu,m}$ are independent of $m$, i.e. $Q_{\nu,m} = Q_{\nu,(1)}$ with equivalent norms.

For the unit ball $B^d$, $d > 1$, $\Delta_m$ are still polynomials in the invariant Laplacian $\tilde{\Delta}$, but now $\tilde{\Delta} = (1 - \|z\|^2)(\Delta - \mathcal{R}\mathcal{R})$, where $\mathcal{R}$ stands for the radial derivative, contains only the factor $(1 - \|z\|^2)$ instead of $(1 - |z|^2)^2$; thus $\rho_m \geq 1$. Computations seem to indicate that $\rho_m = 1$ for all $m$, so that $Q_{\nu,m}$ is nontrivial — i.e. does not reduce to the constants — if and only if $\nu > p - 2 = d - 1$. For $m = (1)$, this recovers the familiar spaces from (4); for other nonzero $m$, again nothing is known.

Conjecture 22. For any $m \neq (0)$, the space $Q_{\nu,m}(B^d)$, $d > 1$, is nontrivial if and only if $\nu > d - 1$, and then coincides with $Q_{\nu,(1)}(B^d)$, with equivalent norms.

For domains of higher rank, it is immediate from (39) that $\rho_{(1)} = 0$; and from Example 9 that

$$\Delta(s') K(s') = h^p |N(\partial)^s N^s|^2 = (s)_{(s')} h^p$$

so that $\rho_{(s')} = p$ for $\Omega$ a tube type domain with $\frac{d}{r} =: s$ an integer.

Using computer, we were also able to compute $\rho_m$ for a few signatures $m$ for the Cartan domains $\Omega = I_{22}$ and $\Omega = I_{23}$, that is, for the unit balls of all $2 \times 2$ and $2 \times 3$ complex matrices, respectively; the results are summarized in the table below, which gives the values of $\rho_m$ and the corresponding ranges of $\nu$ for which $Q_{\nu,m}$ is nontrivial. Note that in this case $r = 2$,
236  \( Q_p \)-spaces on bounded symmetric domains

\( a = 2 \), and \( b = 0 \) and \( 1 \), respectively, so that \( p = 4 \) for \( I_{22} \) and \( p = 5 \) for \( I_{23} \).

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \Omega = I_{22} )</th>
<th>( \Omega = I_{23} )</th>
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<td>(0,0)</td>
<td>( \rho_m = 0, \nu &gt; p - 1 )</td>
<td>( \rho_m = 0, \nu &gt; p - 1 )</td>
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<td>(1,0)</td>
<td>( \rho_m = 0, \nu &gt; p - 1 )</td>
<td>( \rho_m = 0, \nu &gt; p - 1 )</td>
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<tr>
<td>(1,1)</td>
<td>( \rho_m = 0, \nu &gt; p - 1 )</td>
<td>( \rho_m = 0, \nu &gt; p - 1 )</td>
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<tr>
<td>(2,0)</td>
<td>( \rho_m = 0, \nu &gt; p - 1 )</td>
<td>( \rho_m = 0, \nu &gt; p - 1 )</td>
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<tr>
<td>(2,1)</td>
<td>( \rho_m = 0, \nu &gt; p - 1 )</td>
<td>( \rho_m = 0, \nu &gt; p - 1 )</td>
</tr>
<tr>
<td>(2,2)</td>
<td>( \rho_m = 4, \nu \geq 0 )</td>
<td>( \rho_m = 2, \nu &gt; p - 3 )</td>
</tr>
<tr>
<td>(3,0)</td>
<td>( \rho_m = 0, \nu &gt; p - 1 )</td>
<td>( \rho_m = 0, \nu &gt; p - 1 )</td>
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<tr>
<td>(3,1)</td>
<td>( \rho_m = 0, \nu &gt; p - 1 )</td>
<td>( \rho_m = 0, \nu &gt; p - 1 )</td>
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<tr>
<td>(3,2)</td>
<td>( \rho_m = 4, \nu \geq 0 )</td>
<td>( \rho_m = 0, \nu &gt; p - 1 )</td>
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<tr>
<td>(3,3)</td>
<td>( \rho_m = 4, \nu \geq 0 )</td>
<td>( \rho_m = 0, \nu &gt; p - 1 )</td>
</tr>
</tbody>
</table>

It is not completely clear from the table what \( \rho_m \) might be in general, except for the case of tube type domains.

**Conjecture 23.** Let \( \Omega \) be a tube type domain with \( \frac{d}{p} \) an integer. Then \( \rho_m = 0 \) if \( q(m) < q \), and \( \rho_m = p \) if \( q(m) = q \). Consequently, for \( \nu \leq p - 1 \), \( Q_{\nu,m} \) is nontrivial if and only if \( q(m) = q \) and \( \nu \geq 0 \).

Note that even for the non-tube type domain \( I_{23} \), the table suggests that only the top quotient of the composition series, i.e. the signatures with \( q(m) = q \), are of interest.

Similarly to the disc and the ball, we also conjecture that

**Conjecture 24.** For any real \( \nu \) and any \( \Omega \), \( Q_{\nu,m} = Q_{\nu,n} \) (with equivalent norms) whenever \( q(m) = q(n) \).

By Corollary 15, the last conjecture is definitely valid for \( \nu > p - 1 \).

Another conjecture which has emerged from the computations behind the last table is the following. Recall that we have shown in the proof of Lemma 16 that \( \Delta_m K_m(k \sum_j t_j e_j) \) is always a polynomial in \( t_j \) and \( \sqrt{1 - t^2_j}, \ j = 1, \ldots, r \).

**Conjecture 25.** \( \Delta_m K_m(k \sum_j t_j e_j) \) is actually a polynomial in \( t_1, \ldots, t_r \).

Of course, all the above conjectures could probably be solved if we had some explicit formula for \( \Delta_m K_m \). We conclude this paper by a result which, though short of giving such a formula, at least relates it to another well-known problem.

For any signatures \( m \) and \( n \), the product of the invariant differential operators \( \Delta_m \) and \( \Delta_n \) is again an invariant differential operator; by
Proposition 2, there must therefore exist coefficients \( q^k_{mn} \) (only finitely many of which are nonzero, for each fixed \( m \) and \( n \)) such that

\[
\Delta_m \Delta_n = \sum_k q^k_{mn} \Delta_k.
\]

(45)

(We remark that, similarly, there exist \( \gamma^k_{mn} \) such that

\[
K_m K_n = \sum_k \gamma^k_{mn} K_k.
\]

The coefficients \( \gamma^k_{mn} \) are known as the Pieri (or branching, or Clebsch-Gordan) coefficients; however, there seems to be no established name for \( q^k_{mn} \). Obviously, \( \gamma^k_{mn} = q^k_{mn} \) if \(|k| = |m| + |n|\), by comparing the top order terms in (45). Similarly, \( q^k_{mn} \) is nonzero only if \(|k| \leq |m| + |n|\).)

**Theorem 26.** For any \( m \) and \( n \),

\[
\Delta_m K_n = \sum_k q^n_{km} \frac{d_n}{d_k} K_k.
\]

(46)

The series on the right-hand side converges absolutely and uniformly on \( \overline{\Omega} \).

Here, as before, \( d_m \) stands for the dimension of the Peter-Weyl space \( \mathcal{P}_m \).

**Proof.** Arguing as we did (for \( m = n \)) in the proof of Lemma 16 shows that for any fixed Jordan frame \( e_1, \ldots, e_r \),

\[
\Delta_m K_n (k(t_1 e_1 + \cdots + t_r e_r)) = F(t_1, \ldots, t_r)
\]

where \( F \) is a polynomial in \( t_j \) and \( \sqrt{1 - t_j^2} \), \( j = 1, \ldots, r \); in particular, \( F \) extends to a holomorphic function on the polydisc \( \mathbb{D}^r \subset \mathbb{C}^r \) and is continuous on its closure, and the Taylor expansion

\[
F(t) = \sum_{\alpha \text{ multiindex}} f_\alpha t^\alpha
\]

of \( F \) converges absolutely and uniformly on \( \overline{\mathbb{D}^r} \). It is known that the stabilizer subgroup \( K \) acts transitively on the set of all Jordan frames; since \( \pm e_1, \ldots, \pm e_r \) and \( e_{\sigma(1)}, \ldots, e_{\sigma(r)} \) are also Jordan frames, for any choice of the signs \( \pm \) and for any permutation \( \sigma \) of \( \{1, \ldots, r\} \), respectively, \( F \) must be invariant under all signed permutations of the variables \( t_1, \ldots, t_r \).
Consequently, we even have
\[
(47) \quad F(t) = \sum_{\alpha} f_{2\alpha} t^{2\alpha}
\]
and \( f_{2\sigma(\alpha)} = f_{2\alpha} \) for any permutation \( \sigma \) of \( \{1, \ldots, r\} \). Let \( F_{2m}(t) = \sum_{|\alpha|=m} f_{2\alpha} t^{2\alpha} \) be the \( 2m \)-homogeneous part of (47), \( m = 0, 1, 2, \ldots \); then \( F_{2m} \) is a homogeneous symmetric polynomial of \( t_1^2, \ldots, t_r^2 \) of degree \( m \). On the other hand, it is known that
\[
K_k(k(t_1 e_1 + \cdots + t_r e_r)) = j_k J^{(2/a)}_k(t_1^2, \ldots, t_r^2)
\]
where \( j_k \) is a positive constant and \( J^{(2/a)}_k \) is the Jack symmetric polynomial with parameter \( \frac{2}{a} \) [11, Section 10 of Chapter VI]; furthermore, the Jack polynomials \( J^{(2/a)}_k \), \( |k| = m \), form a basis of the space of homogeneous polynomials of degree \( m \). This means that there must exist constants \( c^k_{mn} \) such that
\[
F_{2m} = \sum_{|k|=m} c^k_{mn} K_k.
\]
Feeding this back into (47), we conclude that, indeed,
\[
(48) \quad \Delta_m K_n = \sum_k c^k_{mn} K_k
\]
with the series converging absolutely and uniformly on the closure of \( \Omega \). It remains to identify the constants \( c^k_{mn} \).

Let \( \{\psi_k\}_{j=1}^{d_k} \) and \( \{\psi_l\}_{i=1}^{d_l} \) be any orthonormal bases of \( P_k \) and \( P_l \), respectively. Then by (22) and (21)
\[
\Delta_l K_k(0) = K_l(\partial, \partial) K_k(z, \bar{z}) \big|_{z=0} = \sum_{j,i} |\psi_l(\partial) \psi_k(0)|^2 = \sum_{j,i} |\langle \psi_l, \psi_k^* \rangle_F|^2 = \sum_i \|P_k \psi_l\|_F^2 \quad \text{by Lemma 10} = \delta_{kl} d_l.
\]
Since, for any smooth function \( g \), \( \Delta_k g(0) \) depends only on the \( (|k|, |k|)- \)homogeneous part of the Taylor expansion of \( g \), it is legitimate to apply \( \Delta_1 \) to the series in (48) term-by-term. This yields
\[
\Delta_l(\Delta_m K_n)(0) = \sum_k c^k_{mn} \Delta_1 K_k(0) = c^d_{mn} d_l.
\]
On the other hand, by (45),

\[
\Delta_l \Delta_m K_n(0) = \sum_k q^k_{lm} \Delta_k K_n(0) = q^n_{lm} d_n.
\]

Thus \(c^l_{mn} = \frac{d_n}{d_l} q^n_{lm}\), completing the proof. \(\square\)

Note that Conjecture 25 is thus tantamount to the fact that the series in (46) terminates.

The following assertion is an immediate consequence of the symmetry \(q^n_{km} = q^n_{mk}\), combined with the property (26) of the composition series, which implies that \(\Delta_m K_n \equiv 0\) if \(q(n) < q(m)\) (cf. the proof of Proposition 11).

**Proposition 27.** \(q^n_{km} = 0\) if \(q(n) < q(m)\) or \(q(n) < q(k)\).

**References**

Q_p-spaces on bounded symmetric domains


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