Composition operators from $\mathcal{B}^\alpha$ to Q_K type spaces

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Abstract. Suppose that ϕ is an analytic self-map of the unit disk Δ . Necessary and sufficient condition are given for the composition operator $C_\phi f = f \circ \phi$ to be bounded and compact from α -Bloch spaces to Q_K type spaces which are defined by a nonnegative, nondecreasing function $K(r)$ for $0 \leq r < \infty$. Moreover, the compactness of composition operator C_ϕ from $\mathcal{B}^0$ to Q_K type spaces are studied, where $\mathcal{B}^0$ is the space of analytic functions of f with $f' \in H^\infty$ and $\|f\|_{\mathcal{B}^0} = |f(0)| + \|f'\|_\infty$.

1. Introduction

Let ϕ be an analytic self-map of the unit disk $\Delta = \{z : |z| < 1\}$ of the complex plane $\mathbb{C}$. A linear composition operator C_ϕ is defined by $C_\phi f = f \circ \phi$ for f in the set $H(\Delta)$ of analytic functions on Δ , and the study of such operators from Banach space X into Banach space Y has attracted the most attention.

The compactness of the composition operator C_ϕ from $\mathcal{B}$ to $\mathcal{B}$ was studied by K. Madigan and A. Matheson in [6]. P. Bourdon, J. Cima and A. Matheson investigated the compact composition operators C_ϕ from BMOA

to BMOA in [2]. H. Wulan in [8] studied the same problem from $\mathcal{B}$ to Q_K , a more general space containing $\mathcal{B}$ and BMOA. L. Jiang and Y. He, studied the compactness of the operators C_ϕ from $\mathcal{B}^\alpha$ to $F(p, q, s)$ in [5]. In this paper we investigate the same problem in the context of $\mathcal{B}^\alpha$ and Q_K type spaces and the latter were introduced in [9].

For $a \in \Delta$, let $\varphi_a(z) = (a - z)/(1 - \bar{a}z)$ be the Möbius transformation of Δ . Green's function with logarithmic singularity at $a \in \Delta$ is denoted by $g(z, a) = \log(1/|\varphi_a(z)|)$. Let $dA(z)$ be the usual element of Euclidean area in Δ .

We assume that $K : [0, \infty) \rightarrow [0, \infty)$ is a right-continuous and nondecreasing function. For $0 < p < \infty$, $-2 < q < \infty$, we say that a function $f \in H(\Delta)$ belongs to the space $Q_K(p, q)$ if

$$\|f\|_{K,p,q}^p = \sup_{a \in \Delta} \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \infty,$$

and $f \in H(\Delta)$ belongs to the space $Q_{K,0}(p, q)$ if

$$\lim_{|a| \rightarrow 1} \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) = 0.$$

If $p \geq 1$, it is clear that $Q_K(p, q)$ is a Banach space with the norm

$$\|f\| = |f(0)| + \|f\|_{K,p,q}.$$

We assume throughout the paper that

$$(1.1) \quad \int_0^1 (1 - r^2)^q K\left(\log \frac{1}{r}\right) r dr < \infty.$$

Otherwise, $Q_K(p, q)$ contains constant functions only (cf. [9]). It is easy to see that $Q_K = Q_K(2, 0)$. If $K(t) = t^s$, then $Q_K(p, q) = F(p, q, s)$, which were introduced and investigated by R. Zhao in [12]. The space $Q_K(p, q)(Q_{K,0}(p, q))$ subset of the (little) $\frac{q+2}{p}$ -Bloch space, consists of those functions f , analytic in Δ and

$$(1.2) \quad \|f\|_{\frac{q+2}{p}} = \sup_{z \in \Delta} (1 - |z|^2)^{\frac{q+2}{p}} |f'(z)| < \infty,$$

respectively

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^{\frac{q+2}{p}} |f'(z)| = 0.$$

The space $\mathcal{B}^\alpha$ is also a Banach space under the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \|f\|_\alpha$.

Recall that a linear operator $T : X \rightarrow Y$ is said to be bounded if there exist a constant $C > 0$ such that $\|T(f)\|_Y \leq C\|f\|_X$. Moreover,

$T : X \rightarrow Y$ is said to be compact if it takes bounded sets in X to sets in Y which have compact closure. For Banach spaces X and Y contained in $H(\Delta)$, $T : X \rightarrow Y$ is compact if and only if for each bounded sequence $\{x_n\}$ in X , the sequence $\{Tx_n\}$ in Y contains a subsequence converging to a function $f \in Y$.

Now we introduce some notation. Two quantities A and B are said to be equivalent if there exist two finite positive constants C_1 and C_2 such that $C_1 A_2 \leq A_1 \leq C_2 A_2$, written as $A_1 \approx A_2$. In the paper, C denotes a positive constant which may be different at each occurrence. For $0 < r < 1$, denote $\Omega_r = \{z \in \Delta, |\phi(z)| > r\}$ and $G_r = \{z \in \Delta, |z| > r\}$. And $\mathbb{B}_X$ denote a unit ball of the given Banach space $(X, \|\cdot\|_X)$.

From now on, we assume that the function $K : [0, \infty) \rightarrow [0, \infty)$ is bounded nondecreasing with $K(1) > 0$, and that the integral (1.1) is convergent and $K(t) \approx K(2t)$ for $0 < t < \infty$.

2. Composition operators from $\mathcal{B}^\alpha$ to $Q_K(p, q)$

Firstly, we need a necessary and sufficient condition on ϕ such that C_ϕ is bounded from $\mathcal{B}^\alpha$ to $Q_K(p, q)$. In order to state the theorem with shorter notations, denote

$$\Psi(\phi, \alpha, p, q, a) = \iint_{\Delta} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{\alpha p}} (1 - |z|^2)^q K(g(z, a)) dA(z),$$

and

$$d\mu_{\phi, \alpha, p, q, a} = \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{\alpha p}} (1 - |z|^2)^q K(g(z, a)) dA(z).$$

Theorem 2.1. *If $p \geq 1$, the following statements are equivalent.*

- (1) $C_\phi : \mathcal{B}^\alpha \rightarrow Q_K(p, q)$ is bounded;
- (2) $C_\phi : \mathcal{B}_0^\alpha \rightarrow Q_K(p, q)$ is bounded;
- (3) $\sup_{a \in \Delta} \Psi(\phi, \alpha, p, q, a) < \infty$.

Proof. We will first show that (3) implies (1). For all $f \in \mathbb{B}_{\mathcal{B}^\alpha}$, we have

$$|f(\phi(0))| \leq |f(0)| + \int_0^{\phi(0)} |f'(z)| dt \leq |f(0)| + \int_0^{\phi(0)} \frac{1}{(1 - |z|^2)^\alpha} dt \leq C.$$

Nextly we will show that $\|C_\phi f\|_{K, p, q} < \infty$ for all $f \in \mathbb{B}_{\mathcal{B}^\alpha}$. If $f \in \mathbb{B}_{\mathcal{B}^\alpha}$, we have

$$\|C_\phi f\|_{K, p, q}^p = \sup_{a \in \Delta} \iint_{\Delta} |(f(\phi(z)))'|^p (1 - |z|^2)^q K(g(z, a)) dA(z)$$

$$\begin{aligned}
&= \sup_{a \in \Delta} \iint_{\Delta} \frac{|f'(\phi(z))|^p (1 - |\phi(z)|^2)^{\alpha p}}{(1 - |\phi(z)|^2)^{\alpha p}} |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\
&\leq \sup_{a \in \Delta} \Psi(\phi, \alpha, p, q, a) < \infty.
\end{aligned}$$

It is easy to see that (1) implies (2) since $\mathcal{B}_0^\alpha \subset \mathcal{B}^\alpha$. We still have to show that (2) implies (3). We need the following Lemma (cf. [10]). $\square$

Lemma 2.2. *Let $\alpha \in (0, \infty)$. Then there exist two analytic functions $f_1, f_2 \in \mathcal{B}^\alpha$ such that*

$$|f_1'(z)| + |f_2'(z)| \approx \frac{1}{(1 - |z|^2)^\alpha}, \quad z \in \Delta.$$

For given $f \in \mathcal{B}^\alpha$, we set $f_s(z) = f(sz)$, $0 < s < 1$, then $f_s \in \mathcal{B}_0^\alpha$ and $\|f_s\|_{\mathcal{B}^\alpha} \leq \|f\|_{\mathcal{B}^\alpha}$. Assume (2) holds. We have $\|C_\phi f\|_{K,p,q} \leq \|C_\phi\| \|f\|_{\mathcal{B}^\alpha}$ for all $f \in \mathcal{B}_0^\alpha$. Let f_1, f_2 be the functions from Lemma 2.2. We have

$$\begin{aligned}
&\sup_{a \in \Delta} \iint_{\Delta} \frac{|s\phi'(z)|^p}{(1 - |s\phi(z)|^2)^{\alpha p}} (1 - |z|^2)^q K(g(z, a)) dA(z) \\
&\leq 2^p \|C_\phi\|^p (\|f_1\|_{\mathcal{B}^\alpha}^p + \|f_2\|_{\mathcal{B}^\alpha}^p) \\
&\leq 2^p \|C_\phi\|^p (\|f_1\|_{\mathcal{B}^\alpha}^p + \|f_2\|_{\mathcal{B}^\alpha}^p).
\end{aligned}$$

This estimate and Fatou's lemma give (3).

Theorem 2.3. *Let ϕ be an analytic self-map of Δ and $p \geq 1$. Then the following statements are equivalent.*

- (1) $C_\phi : \mathcal{B}^\alpha \rightarrow Q_K(p, q)$ is compact;
- (2) $C_\phi : \mathcal{B}_0^\alpha \rightarrow Q_K(p, q)$ is compact;
- (3) $\phi \in Q_K(p, q)$ and

$$(2.1) \quad \lim_{r \rightarrow 1} \sup_{a \in \Delta} \iint_{\Omega_r} d\mu_{\phi, \alpha, p, q, a}(z) = 0.$$

The following lemma is needed to prove the above theorem. We omit the proof here.

Lemma 2.4. *Let $K(r)$ be a nonnegative, nondecreasing function for $0 \leq r < 1$, and let ϕ be an analytic self-map of Δ . Then $C_\phi : \mathcal{B}^\alpha \rightarrow Q_K(p, q)$ is compact if and only if for every bounded sequence $\{f_n\}$ in $\mathcal{B}^\alpha$, which converges to 0 uniformly on compact subsets of Δ as $n \rightarrow \infty$, $\lim_{n \rightarrow \infty} \|C_\phi f_n\|_{K,p,q} = 0$.*

Proof of Theorem 2.3. Firstly we will show that (3) implies (1). Assume that $\phi \in Q_K(p, q)$ and (2.1) holds. Let $\{f_n\}$ be a sequence in $\mathcal{B}^\alpha$ which

converges to 0 uniformly on compact subsets of Δ . We will show that $\{C_\phi f_n\}$ converges to 0 in $Q_K(p, q)$ norm. By (2.1) for given $\epsilon > 0$, there is a $\rho \in (0, 1)$, such that

$$\sup_{a \in \Delta} \iint_{\Omega_\rho} d\mu_{\phi, \alpha, p, q, a} < \epsilon.$$

Therefore,

$$\begin{aligned} & \iint_{\Delta} |(f(\phi(z)))'|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ &= \left\{ \iint_{\Omega_\rho} + \iint_{\Delta - \Omega_\rho} \right\} |(f(\phi(z)))'|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ &\leq \|f\|_{\mathcal{B}^\alpha}^p \epsilon + \sup\{|f'_n(\phi(z))|^p, |\phi(z)| \leq \rho\} \|\phi\|_{K, p, q}^p. \end{aligned}$$

It follows that $\lim_{n \rightarrow \infty} \|C_\phi f_n\|_{K, p, q}^p = 0$ since $\sup\{|f'_n(\phi(z))|^p, |\phi(z)| \leq \rho\} \rightarrow 0$ as $n \rightarrow \infty$. By Lemma 2.4 $C_\phi : \mathcal{B}^\alpha \rightarrow Q_K(p, q)$ is compact. $\square$

It is obvious that (1) implies (2) since $\mathcal{B}_0^\alpha \subset \mathcal{B}^\alpha$. We have to prove that (2) implies (3). To complete the proof we give the following proposition which will be proved below.

Proposition 2.5. *Let $K(r)$ be a nonnegative, nondecreasing function on $0 \leq r < \infty$, and let ϕ be an analytic self-map of Δ . If $C_\phi : \mathcal{B}^\alpha(\mathcal{B}_0^\alpha) \rightarrow Q_K(p, q)$ is compact, then for any $\epsilon > 0$ there exists $\delta, 0 < \delta < 1$, such that for all f in $\mathbb{B}_{\mathcal{B}^\alpha}(\mathbb{B}_{\mathcal{B}_0^\alpha})$,*

$$(2.2) \quad \sup_{a \in \Delta} \iint_{\Omega_r} |(f(\phi(z)))'|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \epsilon$$

holds whenever $\delta < r < 1$.

Consider the function $f(z) = C(\alpha) \sum_{n=0}^{\infty} 2^{k(\alpha-1)} z^{2^k}$. We know that $f \in \mathcal{B}_0^\alpha$. We can choose $C(\alpha)$ such that $\|f\|_{\mathcal{B}^\alpha} \leq 1$. Choose a sequence $\{\lambda_n\}$ in Δ which converges to 1 as $n \rightarrow \infty$, and let $f_n(z) = (1/\lambda_n) f(\lambda_n z)$ for all $n \in \mathbb{N}$. Thus, $f_n \in \mathbb{B}_{\mathcal{B}_0^\alpha}$ for all $n \in \mathbb{N}$. For $0 \leq \theta \leq 2\pi$, denote $f_{n, \theta}(z) = f_n(e^{i\theta} z)$. It is easy to see that $f_{n, \theta} \in \mathbb{B}_{\mathcal{B}_0^\alpha}$. By Parseval's identity we have

$$\begin{aligned} (2.3) \quad M_2^2(|z|, f'_{n, \theta}) &= \frac{1}{2\pi} \int_0^{2\pi} |f'_n(e^{i\theta} z)|^2 d\theta \\ &= (C(\alpha))^2 \sum_{n=0}^{\infty} 2^{k\alpha} |\lambda_n z|^{2(2^k-1)}. \end{aligned}$$

Since $f_{n,\theta}$ is defined by a gap series with the Hadamard condition, we have

$$(2.4) \quad M_2(|z|, f'_{n,\theta}) \approx M_p(|z|, f'_{n,\theta}),$$

where

$$M_p(|z|, f'_{n,\theta}) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f'_n(e^{i\theta}z)|^p d\theta \right)^{1/p}.$$

Replace f by $f_{n,\theta}$ in (2.2) and then integrate both side with respect to $d\theta$. By Fubini's theorem, (2.3) and (2.4) we obtain

$$\begin{aligned} \epsilon &> \frac{1}{2\pi} \iint_{\Omega_r} \left(\int_0^{2\pi} |f'_n(e^{i\theta}\phi(z))|^p d\theta \right) \\ &\quad \times |\lambda_n \phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ &\approx \iint_{\Omega_r} \left((C(\alpha))^2 \sum_{k=1}^{\infty} 2^{2k\alpha} |\lambda_n \phi(z)|^{2(2^k-1)} \right)^{\frac{p}{2}} \\ &\quad \times |\lambda_n \phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z). \end{aligned}$$

Note that

$$\sum_{k=1}^{\infty} 2^{2k\alpha} r^{2^{k+1}-2} \geq \frac{C}{(1-r^2)^{2\alpha}}$$

for all $\alpha > 0$ and $r \in (r_0, 1)$, where $r_0 > 0$ is fixed (cf. [9]). Then we have

$$\sup_{a \in \Delta} \iint_{\Omega_r} \frac{|\lambda_n \phi'(z)|^p}{(1 - |\lambda_n \phi(z)|^2)^{\alpha p}} (1 - |z|^2)^q K(g(z, a)) dA(z) < C\epsilon.$$

By Fatou's lemma we obtain (2.1).

Now we turn back to prove Proposition 2.5. We only give the proof for $\mathcal{B}_0^\alpha$ and the proof for $\mathcal{B}^\alpha$ is similar. For $f \in \mathbb{B}_{\mathcal{B}_0^\alpha}$, let $f_t(z) = f(tz)$, $0 < t < 1$. Then $f_t \in \mathbb{B}_{\mathcal{B}_0^\alpha}$ and $f_t \rightarrow f$ uniformly on compact subsets of Δ as $t \rightarrow 1$. Since C_ϕ is compact, $\|C_\phi f_t - C_\phi f\|_{K,p,q} \rightarrow 0$ as $t \rightarrow 1$. That is, for given $\epsilon > 0$ there exists $t \in (0, 1)$ such that

$$\sup_{a \in \Delta} \iint_{\Delta} |f'_t(\phi(z)) - f'(\phi(z))|^p |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \epsilon.$$

For $r \in (0, 1)$, the triangle inequality gives

$$\begin{aligned} &\sup_{a \in \Delta} \iint_{\Omega_r} |f'(\phi(z))|^p |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ &= \sup_{a \in \Delta} \iint_{\Omega_r} |f'_t(\phi(z)) - f'(\phi(z))|^p |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \end{aligned}$$

$$\begin{aligned}
& + \sup_{a \in \Delta} \iint_{\Omega_r} |f'_t(\phi(z))|^p |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\
& \leq \epsilon + \|f'_t\|_\infty^p \sup_{a \in \Delta} \iint_{\Omega_r} |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z).
\end{aligned}$$

Now we prove that for given $\|f'_t\|_\infty^p > 0$ there exists $\delta \in (0, 1)$ such that

$$(2.5) \quad \|f'_t\|_\infty^p \sup_{a \in \Delta} \iint_{\Omega_r} |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \epsilon.$$

for $\delta < r < 1$. Indeed, choose $f_n(z) = n^{\alpha-1} z^n \in \mathcal{B}_0^\alpha$, and we have $C_\phi f_n = n^{\alpha-1} \phi^n \in Q_K(p, q)$. Since C_ϕ is compact and $\{f_n\}$ converge to 0 uniformly on compact subsets of Δ , we have $\lim_{n \rightarrow \infty} \|n^{\alpha-1} \phi^n\|_{K, p, q} = 0$. Thus, for given $\epsilon > 0$ and $\|f'_t\|_\infty^p > 0$ there exists $N \in \mathbb{N}$ such that

$$\|f'_t\|_\infty^p \sup_{a \in \Delta} \iint_{\Delta} n^{p\alpha} |\phi^{n-1}(z)|^p |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \epsilon,$$

whenever $n \geq N$. Hence, for $0 < r < 1$,

$$\begin{aligned}
& N^{p\alpha} \sup_{a \in \Delta} \iint_{\Delta} |\phi^{N-1}(z)|^p |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\
& \geq N^{p\alpha} \sup_{a \in \Delta} \iint_{\Omega_r} |\phi^{N-1}(z)|^p |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\
& \geq N^{p\alpha} r^{p(N-1)} \sup_{a \in \Delta} \iint_{\Omega_r} |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z).
\end{aligned}$$

We choose r such that $N^{p\alpha} r^{p(N-1)} = 1$. In fact we have obtained (2.5). Thus we have already proved that for any $\epsilon > 0$ and $f \in \mathbb{B}_{\mathcal{B}_0^\alpha}$, there exists $\delta = \delta(\epsilon, f)$ such that

$$\sup_{a \in \Delta} \iint_{\Omega_r} |(f \circ \phi)'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \epsilon$$

holds whenever $\delta < r < 1$. We have to show that $\delta = \delta(\epsilon, f)$ is independent of $f \in \mathbb{B}_{\mathcal{B}_0^\alpha}$. Since C_ϕ is compact, $C_\phi(\mathbb{B}_{\mathcal{B}_0^\alpha})$ is relatively compact in $Q_K(p, q)$. It means that there are $f_1, f_2, \dots, f_m \in \mathbb{B}_{\mathcal{B}_0^\alpha}$ for any $\epsilon > 0$ and $f \in \mathbb{B}_{\mathcal{B}_0^\alpha}$, we can choose $f_j (1 \leq j \leq m)$ satisfying

$$\sup_{a \in \Delta} \iint_{\Delta} |f'(\phi(z)) - f'_j(\phi(z))|^p |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \epsilon.$$

If $\delta = \max_{1 \leq j \leq m} \delta_j(\epsilon, f_j) < r < 1$, we have for all $j = 1, 2, \dots, m$,

$$\sup_{a \in \Delta} \iint_{\Omega_r} |f'_j(\phi(z))|^p |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \epsilon.$$

By the triangle inequality we obtain (2.2). The proof of Theorem 2.3 is complete by finishing the proof of Proposition 2.5.

3. Composition operators from $\mathcal{B}^\alpha(\mathcal{B}_0^\alpha)$ to $Q_{K,0}(p, q)$

In this section, we consider compact composition operators $C_\phi : \mathcal{B}^\alpha(\mathcal{B}_0^\alpha) \rightarrow Q_{K,0}(p, q)$. Obviously, $Q_{K,0}(p, q)$ is a subspace of $Q_K(p, q)$. The necessary and sufficient condition for the compactness of the composition operator C_ϕ from $\mathcal{B}^\alpha$ to $Q_{K,0}(p, q)$ differs from the one for the composition operator C_ϕ from $\mathcal{B}_0^\alpha$ to $Q_{K,0}(p, q)$.

Theorem 3.1. *Let ϕ be an analytic self-map of Δ and $p \geq 1$. Then the following statements are equivalent.*

- (1) $C_\phi : \mathcal{B}^\alpha \rightarrow Q_{K,0}(p, q)$ is bounded;
- (2) $C_\phi : \mathcal{B}^\alpha \rightarrow Q_{K,0}(p, q)$ is compact;
- (3) $\phi \in Q_{K,0}(p, q)$ and $\lim_{|a| \rightarrow 1} \Psi(\phi, \alpha, p, q, a) = 0$.

We need the following lemma to prove Theorem 3.1.

Lemma 3.2. *If $\Psi(\phi, \alpha, p, q, a)$ is finite at some point $a \in \Delta$, then $\Psi(\phi, \alpha, p, q, a)$ is a continuous function of a on Δ .*

Proof. Remember that $K(t) \approx K(2t)$ and $g(z, a)$ can be replaced by $1 - \varphi_a(z)$ (cf. [4]). Then we will show that

$$T(a) = \iint_{\Delta} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{\alpha p}} (1 - |z|^2)^q K(1 - \varphi_a(z)) dA(z)$$

is a continuous function of a on Δ if $T(a)$ is finite at some point $a \in \Delta$. The rest are omitted since it is similar to the proof of Lemma 2.3 in [7].

Proof of Theorem 3.1. (3) implies (2). Since $\lim_{|a| \rightarrow 1} \Psi(\phi, \alpha, p, q, a) = 0$, there is a $r > 0$ such that $\Psi(\phi, \alpha, p, q, a) < \epsilon$ for given $\epsilon > 0$ whenever $a \in G_r = \{z \in \Delta, |z| > r\}$. So for any compact set $G \subset \Delta$, we have

$$(3.1) \quad T(a) = \iint_{\Delta - G} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{\alpha p}} (1 - |z|^2)^q K(g(z, a)) dA(z) < \epsilon$$

for all $a \in G_r$. By Lemma 3.2, we see that $T(a)$ is continuous. For given a_i , there is a compact set $G_i \in \Delta$ such that (3.1) holds. And there is a

neighborhood $U(a_i)$ of a_i such that (3.1) holds by the continuity of $T(a)$. By the compactness of $\Delta - G_r$, we can choose finite $U(a_i)$ ($i = 1, 2, \dots, m$) such that $\Delta - G_r = \bigcup_{i=1}^m U(a_i)$. Replace G by $\bigcup_{i=1}^m G_i$ in (3.1). Then (3.1) holds for all $a \in \Delta$.

Let $\{f_n\}$ be a sequence in $\mathbb{B}_{\mathcal{B}^\alpha}$ which converges to 0 uniformly on compact subsets of Δ as $n \rightarrow \infty$. We see that $\{f'_n\}$ converges to 0 uniformly on compact subsets of Δ . There exists a number $N \in \mathbb{N}$ such that $\sup_{z \in G} \{|f'_n(\phi(z))|(1 - |\phi(z)|^2)^\alpha\}^p < \epsilon$ for $n > N$. Then for $n > N$ and $a \in \Delta$ we have

$$\begin{aligned} & \iint_{\Delta} |(f_n(\phi(z)))'|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ &= \iint_{\Delta} \frac{|f'_n(\phi(z))|^p (1 - |\phi(z)|^2)^{\alpha p}}{(1 - |\phi(z)|^2)^{\alpha p}} |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ &< \left(\iint_G \epsilon + \iint_{\Delta - G} \right) \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{\alpha p}} (1 - |z|^2)^q K(g(z, a)) dA(z) \\ &< (k + 1)\epsilon, \end{aligned}$$

where $k = \sup_{a \in \Delta} \Psi(\phi, \alpha, p, q, a)$. We have seen that (3) implies (2). It is obvious that (2) implies (1). It remains to prove that (1) implies (3). Since $z \in \mathcal{B}^\alpha$, we have $\phi \in Q_{K,0}(p, q)$. Let $f_1, f_2 \in \mathcal{B}^\alpha$ be two functions as in Lemma 2.2. Then

$$\begin{aligned} \Psi(\phi, \alpha, p, q, a) &\approx \iint_{\Delta} |(f_1(\phi(z)) + f_2(\phi(z)))'|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ &\leq \iint_{\Delta} 2^p (|(f_1(\phi(z)))'|^p + |(f_2(\phi(z)))'|^p) (1 - |z|^2)^q \\ &\quad \times K(g(z, a)) dA(z). \end{aligned}$$

Since $f_i \circ \phi \in Q_{K,0}(p, q)$, $i = 1, 2$, we have $\lim_{|a| \rightarrow 1} \Psi(\phi, \alpha, p, q, a) = 0$.

We obtain the following results by choosing p, q and K .

Corollary 3.3. *Let ϕ be an analytic self-map of Δ . Then the following statements are equivalent:*

- (1) $C_\phi : \mathcal{B}^\alpha \rightarrow Q_{K,0}$ is bounded;
- (2) $C_\phi : \mathcal{B}^\alpha \rightarrow Q_{K,0}$ is compact;
- (3) $\phi \in Q_{K,0}(p, q)$ and

$$\lim_{|a| \rightarrow 1} \iint_{\Delta} \frac{|\phi'(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} K(g(z, a)) dA(z) = 0.$$

Corollary 3.4. *Let ϕ be an analytic self-map of Δ . Let $p \geq 1, 0 < s < \infty$ and $q + s > -1$. Then the following results are equivalent:*

- (1) $C_\phi : \mathcal{B}^\alpha \rightarrow F_0(p, q, s)$ is bounded;
- (2) $C_\phi : \mathcal{B}^\alpha \rightarrow F_0(p, q, s)$ is compact;
- (3) $\phi \in Q_{K,0}(p, q)$ and

$$\lim_{|a| \rightarrow 1} \iint_{\Delta} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{\alpha p}} (1 - |z|^2)^q g^s(z, a) dA(z) = 0.$$

Proposition 3.5. *Let ϕ be an analytic self-map of Δ and $p \geq 1$. The following statements are equivalent:*

- (1) $C_\phi : \mathcal{B}_0^\alpha \rightarrow Q_{K,0}(p, q)$ is bounded;
- (2) $\phi \in Q_{K,0}(p, q)$ and $\sup_{a \in \Delta} \Psi(\phi, \alpha, p, q, a) < \infty$.

Proof We will first show that (1) implies (2). If (1) holds, then $C_\phi : \mathcal{B}_0^\alpha \rightarrow Q_{K,0}(p, q)$ is bounded, and we obtain $\sup_{a \in \Delta} \Psi(\phi, \alpha, p, q, a) < \infty$ by Theorem 2.1. Let $f(z) = z$, the boundedness of C_ϕ implies $\phi \in Q_{K,0}(p, q)$.

We must show that (2) implies (1). Suppose that (2) holds and let $f \in \mathcal{B}_0^\alpha$. Then $C_\phi : \mathcal{B}_0^\alpha \rightarrow Q_{K,0}(p, q)$ is bounded by Theorem 2.1. We will show that $C_\phi f \in Q_{K,0}(p, q)$. For given $\epsilon > 0$, we can choose $r \in (0, 1)$ such that

$$|f'(\phi(z))|^p (1 - |\phi(z)|^2)^{\alpha p} < \epsilon$$

for $z \in \Omega_r$. Thus

$$\begin{aligned} (3.2) \quad & \sup_{a \in \Delta} \iint_{\Omega_r} |(f \circ \phi)'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ & \leq \epsilon \sup_{a \in \Delta} \Psi(\phi, \alpha, p, q, a). \end{aligned}$$

Since $\phi(z) \in Q_{K,0}(p, q)$, for r as above, we have

$$\begin{aligned} (3.3) \quad & \lim_{|a| \rightarrow 1} \iint_{|\phi(z)| \leq r} |(f \circ \phi)'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ & \leq \|f\|_{\mathcal{B}^\alpha}^p \lim_{|a| \rightarrow 1} \iint_{|\phi(z)| \leq r} \frac{|\phi'(z)|^p}{1 - |\phi(z)|^2} (1 - |z|^2)^q K(g(z, a)) dA(z) \\ & \leq \frac{\|f\|_{\mathcal{B}^\alpha}^p}{(1 - r^2)^{\alpha p}} \lim_{|a| \rightarrow 1} \iint_{\Delta} |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) = 0. \end{aligned}$$

By (3.2) and (3.3), we obtain that $C_\phi f \in Q_{K,0}(p, q)$.

Theorem 3.6. *Let ϕ be an analytic self-map of Δ and $p \geq 1$. Then the composition operator $C_\phi : \mathcal{B}_0^\alpha \rightarrow Q_{K,0}(p, q)$ is compact if and only if*

$\phi \in Q_{K,0}(p, q)$ and

$$(3.4) \quad \lim_{r \rightarrow 1} \sup_{a \in \Delta} \iint_{G_r} d\mu_{\phi, \alpha, p, q, a}(z) = 0.$$

Proof. Firstly, we assume that (3.4) holds. For given $\epsilon > 0$, there is a $\delta \in (0, 1)$ such that

$$(3.5) \quad \sup_{a \in \Delta} \iint_{G_r} d\mu_{\phi, \alpha, p, q, a}(z) < \epsilon$$

whenever $\delta < r < 1$. Let $\{f_n\}$ be a sequence in $\mathbb{B}_{\mathcal{B}_0^\alpha}$ which converges to 0 uniformly on compact subsets of Δ . We will show that $\lim_{n \rightarrow \infty} \|C_\phi f_n\|_{K, p, q} = 0$. For any $r \in (0, 1)$, $\{f'_n \circ \phi\}$ converges to 0 uniformly on $\Delta - G_r$. Since $\phi \in Q_{K,0}(p, q)$, we have

$$(3.6) \quad \lim_{n \rightarrow \infty} \sup_{a \in \Delta} \iint_{\Delta - G_r} |(f_n \circ \phi)'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) = 0.$$

On the other hand, by (3.5) for all $n \in \mathbb{N}$ we have

$$(3.7) \quad \begin{aligned} & \sup_{a \in \Delta} \iint_{G_r} |(f_n \circ \phi)'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ & \leq \sup_{a \in \Delta} \iint_{G_r} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{\alpha p}} (1 - |z|^2)^q K(g(z, a)) dA(z) \\ & < \epsilon. \end{aligned}$$

Combining (3.6) and (3.7) we obtain that $\lim_{n \rightarrow \infty} \|C_\phi f_n\|_{K, p, q} = 0$.

Assume C_ϕ is compact. Choose $f(z) = z$, then $\phi \in Q_{K,0}(p, q)$. Obviously $C_\phi(\mathbb{B}_{\mathcal{B}_0^\alpha})$ is a relatively compact subsets of $Q_{K,0}(p, q)$ spaces. For given $\epsilon > 0$ and $f \in \mathbb{B}_{\mathcal{B}_0^\alpha}$, there is a finite set of functions $f_k \in Q_{K,0}(p, q)$, $k = 1, 2, \dots, m$ such that

$$\sup_{a \in \Delta} \iint_{\Delta} |(C_\phi f)'(z) - f'_k(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \epsilon.$$

Hence, for any $0 < r < 1$, we have

$$\sup_{a \in \Delta} \iint_{G_r} |(C_\phi f)'(z) - f'_k(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \epsilon.$$

Since $f_k \in Q_{K,0}(p, q)$, there is a $\delta_k \in (0, 1)$ such that

$$\sup_{a \in \Delta} \iint_{G_r} |f'_k(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \epsilon$$

whenever $\delta_k < r < 1$. Choose $r > \delta = \max_{1 \leq k \leq m}(\delta_k)$, for all $f \in Q_{K,0}(p, q)$, we have

$$(3.8) \quad \sup_{a \in \Delta} \iint_{G_r} |(f \circ \phi)'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \epsilon$$

by the triangle inequality. We complete the proof by applying Lemma 2.2 and Fatou's lemma to (3.7).

Corollary 3.7. *Let ϕ be an analytic self-map of Δ . Then composition operator $C_\phi : \mathcal{B}_0^\alpha \rightarrow Q_{K,0}$ is compact if and only if $\phi \in Q_{K,0}$ and*

$$\lim_{r \rightarrow 1} \sup_{a \in \Delta} \int_{G_r} \frac{|\phi'(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} K(g(z, a)) dA(z) = 0.$$

Corollary 3.8. *Let ϕ be an analytic self-map of Δ . Let $p \geq 1, 0 < s < \infty$ and $q + s > -1$. Then the composition operator $C_\phi : \mathcal{B}_0^\alpha \rightarrow F_0(p, q, s)$ is compact if and only if $\phi \in F_0(p, q, s)$ and*

$$\lim_{r \rightarrow 1} \sup_{a \in \Delta} \iint_{G_r} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{\alpha p}} (1 - |z|^2)^q g^s(z, a) dA(z) = 0.$$

4. Composition operators from $\mathcal{B}^0$ to $Q_K(p, q)(Q_{K,0}(p, q))$

In this section we will investigate the compactness of the composition operator C_ϕ from $\mathcal{B}^0$ to $Q_K(p, q)$, where $\mathcal{B}^0$ is the space of analytic functions f with $f' \in H^\infty$, and $\|f\|_{\mathcal{B}^0} = |f(0)| + \|f'\|_\infty$.

Theorem 4.1. *Let ϕ be an analytic self-map of Δ and $p \geq 1$. Then the composition operator $C_\phi : \mathcal{B}^0 \rightarrow Q_K(p, q)$ is bounded if and only if $\phi \in Q_K(p, q)$.*

Proof. Suppose that $\phi \in Q_K(p, q)$ and $f \in \mathcal{B}^0$, we show that $f \circ \phi \in Q_K(p, q)$. Since $f \in \mathcal{B}^0 \subset \mathcal{B}$, we have

$$(4.1) \quad |f(\phi(0))| \leq |f(0)| + \int_0^{\phi(0)} |f'(z)| dz \leq |f(0)| + \|f'\|_{\mathcal{B}^0}.$$

On the other hand,

$$(4.2) \quad \sup_{a \in \Delta} \iint_{\Delta} |(f \circ \phi)'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z)$$

$$\begin{aligned}
&= \sup_{a \in \Delta} \iint_{\Delta} |f'(\phi(z))|^p |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\
&\leq \|f\|_{\mathcal{B}^0}^p \sup_{a \in \Delta} \iint_{\Delta} |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\
&= \|f\|_{\mathcal{B}^0}^p \|\phi\|_{K, p, q}^p.
\end{aligned}$$

(4.1) and (4.2) give

$$\|C_\phi f\| \leq C \|f\|_{\mathcal{B}^0}.$$

If $C_\phi : \mathcal{B}^0 \rightarrow Q_K(p, q)$ is bounded, then $C_\phi f \in Q_K(p, q)$ for all $f \in \mathcal{B}^0$. Take $f(z) = z$ to see that $\phi \in Q_K(p, q)$. The proof is complete.

Theorem 4.2. *Let ϕ be an analytic self-map of Δ and $p \geq 1$. Then $C_\phi : \mathcal{B}^0 \rightarrow Q_K(p, q)$ is compact if and only if $\phi \in Q_K(p, q)$ and*

$$(4.3) \quad \lim_{r \rightarrow 1} \sup_{a \in \Delta} \iint_{\Omega_r} |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) = 0$$

Proof. Assume that $\phi \in Q_K(p, q)$ and (4.3) holds. Let $\{f_n\}$ be a sequence in $\mathcal{B}_{\mathcal{B}^0}$ which converges to 0 uniformly on compact subsets of Δ . We will show that $\{C_\phi f_n\}$ converges to 0 in $Q_K(p, q)$ norm. By (4.3) for given $\epsilon > 0$, there exists $\rho \in (0, 1)$ such that

$$\sup_{a \in \Delta} \iint_{\Omega_\rho} |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \epsilon.$$

Therefore,

$$\begin{aligned}
&\iint_{\Delta} |(f \circ \phi)'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\
&= \left\{ \iint_{\Omega_\rho} + \iint_{\Delta - \Omega_\rho} \right\} |(f \circ \phi)'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\
&\leq \epsilon \|f\|_{\mathcal{B}^0}^p + \sup\{|f'_n(\phi(z))|^p, |\phi(z)| \leq \rho\} \|\phi\|_{K, p, q}^p.
\end{aligned}$$

It follows that $\lim_{n \rightarrow \infty} \|C_\phi f_n\|_{K, p, q}^p = 0$ since $\sup\{|f'_n(\phi(z))|^p, |\phi(z)| \leq \rho\} \rightarrow 0$ as $n \rightarrow \infty$. So $C_\phi : \mathcal{B}^0 \rightarrow Q_K(p, q)$ is compact.

Conversely, suppose $C_\phi : \mathcal{B}^0 \rightarrow Q_K(p, q)$ is compact. It means that $C_\phi : \mathcal{B}^0 \rightarrow Q_K(p, q)$ is bounded, so $\phi \in Q_K(p, q)$ by Theorem 4.1.

Choose the function $f_n(z) = z^n/n$. Since $f_n(z)$ is norm bounded and converges uniformly to 0 on compact subsets of Δ , we have

$$\lim_{n \rightarrow \infty} \|C_\phi f_n\|_{K, p, q} = \lim_{n \rightarrow \infty} \|\phi^n/n\|_{K, p, q} = 0.$$

Hence, for given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that

$$n^{p-1} \sup_{a \in \Delta} \iint_{\Delta} |\phi^{(n-1)}(z)|^p |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \epsilon$$

for all $n > N$. Then we have

$$\begin{aligned} & N^{p-1} \sup_{a \in \Delta} \iint_{\Delta} |\phi^{N-1}(z)|^p |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ & \geq N^{p-1} \sup_{a \in \Delta} \iint_{\Omega_r} |\phi^{N-1}(z)|^p |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ & \geq N^{p\alpha} r^{p(N-1)} \sup_{a \in \Delta} \iint_{\Omega_r} |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z). \end{aligned}$$

We can choose $r, 0 < r < 1$ such that $N^{p-1} r^{p(N-1)} = 1$. We obtain (4.3).

Corollary 4.3. *Let ϕ be an analytic self-map of Δ . Then $C_\phi : \mathcal{B}^0 \rightarrow Q_K$ is compact if and only if $\phi \in Q_K$ and*

$$\lim_{r \rightarrow 1} \sup_{a \in \Delta} \iint_{\Omega_r} |\phi'(z)|^2 K(g(z, a)) dA(z) = 0.$$

Corollary 4.4. *Let ϕ be an analytic self-map of Δ . Let $p \geq 1, 0 < s < \infty$ and $q + s > -1$. Then $C_\phi : \mathcal{B}^0 \rightarrow F(p, q, s)$ is compact if and only if $\phi \in F(p, q, s)$ and*

$$\lim_{r \rightarrow 1} \sup_{a \in \Delta} \iint_{\Omega_r} |\phi'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) = 0.$$

We should mention that necessary and sufficient condition for the compactness of the composition operator from $\mathcal{B}^0$ to $Q_{K,0}(p, q)$ is equivalent to that of the boundedness.

Theorem 4.5. *Let ϕ be an analytic self-map of Δ and $p \geq 1$. Then the following statements are equivalent:*

- (1) $\phi \in Q_{K,0}(p, q)$;
- (2) $C_\phi : \mathcal{B}^0 \rightarrow Q_{K,0}(p, q)$ is bounded;
- (3) $C_\phi : \mathcal{B}^0 \rightarrow Q_{K,0}(p, q)$ is compact.

Proof. (3) implies (2) is obvious.

We will show that (2) implies (1). If C_ϕ is bounded, then $f \circ \phi \in Q_{K,0}(p, q)$ for all $f \in \mathcal{B}^0$. Choose $f(z) = z$, we obtain $\phi \in Q_{K,0}(p, q)$.

(1) implies (2). If $\phi \in Q_{K,0}(p, q)$, then $C_\phi : \mathcal{B}^0 \rightarrow Q_K(p, q)$ is bounded by Theorem 4.1. So it is enough to show that $C_\phi(\mathcal{B}^0) \subset Q_{K,0}(p, q)$. Choose

$\phi \in Q_{K,0}(p, q)$ for given $f \in \mathcal{B}^0$, we have

$$\begin{aligned} & \lim_{|a| \rightarrow 1} \iint_{\Delta} |(f \circ \phi)'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ &= \lim_{|a| \rightarrow 1} \iint_{\Delta} |f'(\phi(z))|^p |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ &\leq \|f\|_{\mathcal{B}^0}^p \lim_{|a| \rightarrow 1} \iint_{\Delta} |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) = 0. \end{aligned}$$

That is, $C_\phi \in Q_{K,0}(p, q)$.

We will show that (1) implies (3). Suppose that $\phi \in Q_{K,0}(p, q)$. We have $C_\phi(\mathcal{B}^0) \subset Q_{K,0}(p, q)$ since $C_\phi : \mathcal{B}^0 \rightarrow Q_{K,0}(p, q)$ is bounded. To prove that C_ϕ from $\mathcal{B}^0$ to $Q_{K,0}(p, q)$ is compact, we still have to show that C_ϕ from $\mathcal{B}^0$ to $Q_K(p, q)$ is compact. Let $\{f_n\}$ converge to 0 on compact subsets of Δ and $\{f_n\} \in \mathcal{B}^0$. If we can show that

$$(4.4) \quad \lim_{n \rightarrow \infty} \|C_\phi f_n\|_{K,p,q} = 0,$$

then we complete the proof of compactness of the composition operator C_ϕ . Since $\phi \in Q_{K,0}(p, q)$, for given $\epsilon > 0$, there is $\rho \in (0, 1)$ such that

$$\iint_{\Delta} |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \epsilon$$

for $|a| > \rho$. For $|a| > \rho$ and $\{f_n\} \in \mathcal{B}^0$, we have

$$\begin{aligned} (4.5) \quad & \iint_{\Delta} |(f_n \circ \phi)'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ &= \iint_{\Delta} |f_n'(\phi(z))|^p |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ &\leq \|f_n\|_{\mathcal{B}^0} \iint_{\Delta} |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ &\leq \epsilon. \end{aligned}$$

We can prove the operator

$$T(a) = \iint_{\Delta} |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z)$$

is a continuous of a on Δ in the same way in Lemma 3.2. For given ϵ and $a_i, |a_i| \leq \rho$, there is a compact set $G_i \in \Delta$ such that

$$(4.6) \quad \iint_{\Delta - G_i} |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \epsilon$$

holds. And there is a neighborhood $U(a_i)$ of a_i such that (4.6) holds by the continuity of $T(a)$. By the compactness of $\Delta - G_r$, there exists finite $U(a_i)(i = 1, 2, \dots, m)$ such that $\Delta - G_r = \bigcup_{i=1}^m U(a_i)$. Let $G = \bigcup_{i=1}^m G_i$ and G is a compact subsets of Δ . For all $|a| \leq \rho$, we have

$$\begin{aligned}
 (4.7) \quad & \iint_{\Delta} |(f_n \circ \phi)'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\
 &= \left(\iint_G + \iint_{\Delta - G} \right) |f'_n(\phi(z))|^p \\
 &\quad \times |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\
 &\leq \sup\{f'_n(\phi(z)), z \in G\} \|\phi\|_{K,p,q}^p + \epsilon \|f_n\|_{\mathcal{B}^0}^p.
 \end{aligned}$$

We obtain (4.4) from (4.5) and (4.7) as $n \rightarrow \infty$.

Corollary 4.6. *Let ϕ be an analytic self-map of Δ . Then the following statements are equivalent:*

- (1) $\phi \in Q_{K,0}$;
- (2) $C_\phi : \mathcal{B}^0 \rightarrow Q_{K,0}$ is bounded;
- (3) $C_\phi : \mathcal{B}^0 \rightarrow Q_{K,0}$ is compact.

Corollary 4.7. *Let ϕ be an analytic self-map of Δ . Let $p \geq 1, 0 < s < \infty$ and $q + s > -1$. Then the following statements are equivalent:*

- (1) $\phi \in F_0(p, q, s)$;
- (2) $C_\phi : \mathcal{B}^0 \rightarrow F_0(p, q, s)$ is bounded;
- (3) $C_\phi : \mathcal{B}^0 \rightarrow F_0(p, q, s)$ is compact.

Remark. After submitting the paper, the author has learned that some of the results in our Section 2 have been independently obtained in a paper “On composition operators on Q_K type spaces” by M. Kotilainen, to appear also in J. Funct. Spaces Appl. (2007).

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