The essential norm of a composition operator mapping into $Q_K$ type spaces

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Abstract. An asymptotic formula for the essential norm of the composition operator $C_\varphi(f) := f \circ \varphi$, induced by an analytic self-map $\varphi$ of the unit disc, mapping from the $\alpha$-Bloch space $B^\alpha$ or the Dirichlet type space $D^\alpha_p$ into $Q_K(p,q)$ is established in terms of an integral condition.

1. Introduction

Every analytic self-map $\varphi$ of the unit disc $\mathbb{D}$ induces the composition operator $C_\varphi(f) := f \circ \varphi$ acting on the space of all analytic functions in $\mathbb{D}$. Composition operators in spaces of analytic functions have been actively

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studied since the mid 1980’s. The research relates the operator theoretic properties of \( C_\varphi \) to the function theoretic properties of the symbol \( \varphi \). These properties of the symbol can be either of geometric or analytic nature. Ryff (1966) and Nordgren (1968) were probably the first ones to study composition operators acting on function spaces while Shapiro’s (1987) work was undoubtedly the breakthrough which led to an abundant research activity on the field. The monographs by Shapiro (1993), and Cowen and MacCluer (1995) give an excellent overview of the subject up to the early or mid 1990’s. After the appearance of these monographs a large number of interesting papers by many authors have been published. The list of references of the present paper covers only a small amount of these papers and therefore the reader is invited to consult one of the standard databases of mathematical literature for more information on the relevant references.

The main purpose of this paper is to establish an asymptotic formula for the essential norm of the composition operator \( C_\varphi \) mapping from the \( \alpha \)-Bloch space \( B^\alpha \) or the Dirichlet type space \( D_\alpha^p \) into \( Q_{K}(p, q) \) in terms of an integral condition. Characterizations of bounded and compact composition operators are also given. The proofs rely strongly on the standard tools in the field such as a change of variable formula and different kind of characterizations of Carleson measures.

The remainder of this paper is organized as follows. In Section 2, the main results are presented together with necessary definitions. In Section 3 the necessary background material including a change of variable formula and Carleson measures is introduced. Sections 4-7 contain the proofs of the main results in chronological order.

### 2. Main results

Let \( \mathcal{H}(\mathbb{D}) \) denote the algebra of all analytic functions in the unit disc \( \mathbb{D} := \{z : |z| < 1\} \), and let \( B(\mathbb{D}) \) be the subset of \( \mathcal{H}(\mathbb{D}) \) consisting of those \( \varphi \) for which \( \varphi(\mathbb{D}) \subset \mathbb{D} \). Every \( \varphi \in B(\mathbb{D}) \) induces the composition operator \( C_\varphi \) acting on \( \mathcal{H}(\mathbb{D}) \), defined by \( C_\varphi(f) := f \circ \varphi \). By Littlewood’s Subordination Principle any such composition operator maps every Hardy and Bergman space boundedly into itself. For the theory of composition operators in analytic function spaces, see [7, 33], while [10, 11, 15] are excellent references for the theory of Hardy and Bergman spaces.

Let the Green’s function of \( \mathbb{D} \) be defined as \( g(z,a) := - \log |\varphi_a(z)| \), where \( \varphi_a(z) := (a-z)/(1-\overline{a}z) \) is the automorphism of \( \mathbb{D} \) which interchanges the points zero and \( a \in \mathbb{D} \). Let \( 0 < p < \infty \) and \( -2 < q < \infty \), and let \( K \) be a right-continuous and nondecreasing function from \( (0, \infty) \) into itself. The
spaces $Q_K(p, q)$ and $Q_{K, 0}(p, q)$ consist of those $f \in H(\mathbb{D})$ for which
\[
\|f\|_{Q_K(p, q)}^p := \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q K(g(z,a)) \, dA(z) < \infty,
\]
and
\[
\lim_{|a| \to 1^-} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q K(g(z,a)) \, dA(z) = 0,
\]
respectively. Here $dA$ denotes the element of the Lebesgue area measure on $\mathbb{D}$. The spaces $Q_K(p, q)$ and $Q_{K, 0}(p, q)$ were introduced in [42]. For $1 \leq p < \infty$, the space $Q_K(p, q)$ is a Banach space with respect to the norm $\|f\|_{Q_K(p, q)} + |f(0)|$. When $0 < p < 1$, the space $Q_K(p, q)$ is a complete metric space with the (invariant) metric defined by $d(f,g) := \|f - g\|_{Q_K(p, q)}^p + |f(0) - g(0)|^p$. The metric is also $p$-homogeneous, that is, $d(\lambda f, 0) = |\lambda|^p d(f, 0)$ for $\lambda \in \mathbb{C}$, and therefore the space $Q_K(p, q)$ is a quasi-Banach space for $0 < p < 1$. From now on it is always assumed that
\[
\int_0^1 (1 - r^2)^q K \left( \log \frac{1}{r} \right) r \, dr < \infty
\]
since otherwise the space $Q_K(p, q)$ reduces to the space of constant functions; see [42]. It is also reasonable to assume that $K$ vanishes at zero since otherwise the weight function $K$ does not play any role in the definition.

For $0 < \alpha < \infty$, the $\alpha$-Bloch space $B^\alpha$ consists of those $f \in H(\mathbb{D})$ for which
\[
\|f\|_{B^\alpha} := \sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|^2)^\alpha < \infty.
\]
The closure of polynomials in $B^\alpha$ is the little $\alpha$-Bloch space $B_0^\alpha$ which consists of those $f \in H(\mathbb{D})$ for which $|f'(z)|(1 - |z|^2)^\alpha \to 0$, as $|z| \to 1^-$. The spaces $B^1$ and $B_0^1$ are the classical Bloch space $B$ and the little Bloch space $B_0$, respectively. For the theory of Bloch spaces, see the classical reference [1], and also [27, 50].

It is known that $Q_K(p, q)$ and $Q_{K, 0}(p, q)$ are subspaces of $B^{\frac{\alpha + 2}{r}}$ and $B_0^{\frac{\alpha + 2}{r}}$, respectively, and further, if
\[
(2.1) \quad \int_0^1 (1 - r^2)^{-2} K \left( \log \frac{1}{r} \right) r \, dr < \infty,
\]
then $Q_K(p, q) = B^{\frac{\alpha + 2}{r}}$ and $Q_{K, 0}(p, q) = B_0^{\frac{\alpha + 2}{r}}$; see [42]. Furthermore, if $K(t) = t^s$ then $Q_K(p, q)$ coincides with the space $F(p, q, s)$, introduced in [48]. In particular, $F(p, q, 0)$ is the Dirichlet type space $D^p_0$ which consists
of those $f \in \mathcal{H}(\mathbb{D})$ for which
\[ \|f\|_{D^p_q}^p := \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q dA(z) < \infty. \]

An important special case of $F(p,q,s)$ is the Möbius invariant space $F(2,0,s) = Q_s$. For more relations and further information on $F(p,q,s)$, see [28, 48], while [45, 46] are excellent references for the theory of $Q_s$ spaces.

The following characterization of bounded composition operators mapping from $B^\alpha$ or $B^\alpha_0$ into $Q_K(p,q)$ was recently found in [16]. Earlier on, various particular cases of Theorem A had been studied by several authors; see, for example, [14, 35, 41, 43, 45].

**Theorem A.** Let $\varphi \in B(\mathbb{D})$, $0 < \alpha, p < \infty$, $-2 < q < \infty$ and let $K$ be nondecreasing on $[0, \infty)$. Then the following statements are equivalent:

1. $C_{\varphi} : B^\alpha \rightarrow Q_K(p,q)$ is bounded;
2. $C_{\varphi} : B^\alpha_0 \rightarrow Q_K(p,q)$ is bounded;
3. $\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{|\varphi'(z)|^p}{(1 - |\varphi(z)|^2)^{\alpha p}} (1 - |z|^2)^q K(g(z,a)) dA(z) < \infty$.

The generalized Nevanlinna counting function for $\varphi \in B(\mathbb{D})$ is the function
\[ N_{\varphi,s}(w) := \sum_{z \in \varphi^{-1}(w)} \left( \log \frac{1}{|z|} \right)^s, \quad w \in \mathbb{D} \setminus \{\varphi(0)\}, \quad s > 0, \]
where $z \in \varphi^{-1}(w)$ is repeated according to the multiplicity of the zero of $\varphi - w$ at $z$. The Nevanlinna counting function is then $N_{\varphi}(z) := N_{\varphi,1}(z)$. If $p = 2$, $q = 0$ and $K(t) = t^s$, then the change of variable formula by Stanton (a special case of Lemma D in Section 3) shows that the condition (3) in Theorem A is equivalent to
\[ \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{N_{\varphi \circ \varphi^{-1},s}(w)}{(1 - |w|^2)^{2\alpha}} dA(w) < \infty. \]

Recall that a bounded operator is compact if it maps bounded sets into relatively compact (precompact) sets. One way to measure the noncompactness of an operator is its essential norm, that is, its distance to the compact operators. The famous formula by Shapiro [32] states that the essential norm of $C_{\varphi}$ acting on the Hardy space $H^2$ equals to
\[ \limsup_{|z| \to 1^-} \frac{N_{\varphi}(z)}{-\log |z|}. \]
In general, to determine the exact formula for the essential norm of a composition operator is not an easy task. In addition to the celebrated result by Shapiro for $H^2$, the essential norm has also been computed in some particular cases. Namely, this has been done on the weighted Bergman space $A^2_\alpha$ when $\alpha \in \{-1, 0, 1\}$ by Poggi-Corradini [26], on $H^\infty$ by Zheng [49], on $\mathcal{B}$ by Montes-Rodríguez [22], from $H^\infty$ to $H^2$ by Gorkin and MacCluer [13], from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$ by MacCluer and Zhao [21], and from the classical Dirichlet space $D^2_0$ to $BMOA$ by the second author [29]. Moreover, an asymptotic formula for the essential norm has been found in the following cases: from $A^p_\alpha$ to $A^q_\beta$, by ˇCuˇckovi´c and Zhao [8, 9], and P´erez-Gonz´alez, the second author and Vukoti´c [25], from $H^p$ to $H^q$, $p \geq q$, by Gorkin and MacCluer [13], from $B^\alpha$ to $B^\beta$ by Lindström, Makhmutov and Taskinen [18], and from $D^p_\alpha$, $1 < p \leq 2$, to $Q_s$ by the second author [29]. See also the related results by Bonet, Doma´nski and Lindström [3].

The general philosophy seems to suggest that one candidate for the asymptotic (and exact) formula for the essential norm could be found by observing the conditions which characterize bounded and/or compact operators. Theorem 1 compared with Theorem A shows that this also happens when $C_\varphi$ maps $\mathcal{B}^\alpha$ boundedly into $Q_K(p, q)$.

**Theorem 1.** Let $\varphi \in B(\mathbb{D})$, $0 < \alpha, p < \infty$, $-2 < q < \infty$ and let $K$ be nondecreasing on $[0, \infty)$. If $C_\varphi : \mathcal{B}^\alpha \rightarrow Q_K(p, q)$ is bounded, then

$$
\|C_\varphi\|^p_{p} \simeq \limsup_{r \to 1^-} \sup_{a \in \mathbb{D}} \int_{\{r(z) > r\}} \frac{|\varphi'(z)|^p}{(1 - |\varphi(z)|^2)^{p\alpha}} \frac{(1 - |\varphi(z)|^2)^q K(g(z, a))}{(1 - |\varphi(z)|^2)^q} dA(z).
$$

Here and from now on the symbol $\simeq$ means that the quantities in other sides of the symbol are comparable, that is, their quotient is bounded and bounded away from zero. Moreover, the notation $a \lesssim b$ means that $a \leq Cb$ for some positive constant $C$, independent of $a$ and $b$, and $a \gtrsim b$ is understood in an analogous manner.

Theorem 1 yields the following characterization of compact composition operators from $\mathcal{B}^\alpha$ into $Q_K(p, q)$, which was recently proved in [16]. Earlier on, various particular cases of this result had been studied by several authors. See, for instance, [14, 35, 40, 43, 45].

**Corollary 2.** Let $\varphi \in B(\mathbb{D})$, $0 < \alpha, p < \infty$, $-2 < q < \infty$ and let $K$ be nondecreasing on $[0, \infty)$. Then $C_\varphi : \mathcal{B}^\alpha \rightarrow Q_K(p, q)$ is compact if and only if

$$
\limsup_{r \to 1^-} \sup_{a \in \mathbb{D}} \int_{\{|\varphi(z)| > r\}} \frac{|\varphi'(z)|^p}{(1 - |\varphi(z)|^2)^{p\alpha}} \frac{(1 - |\varphi(z)|^2)^q K(g(z, a))}{(1 - |\varphi(z)|^2)^q} dA(z) = 0.
$$
The following result characterizes bounded composition operators mapping from $D^s_\alpha$ into $Q_K(p,q)$ when $p \geq s$. This generalizes the corresponding results in [17, 19, 25, 34, 35, 39].

**Theorem 3.** Let $0 < s \leq p < \infty$, $-1 < \alpha < \infty$, $-2 < q < \infty$, and $\varphi \in B(\mathbb{D})$. Then $C_\varphi : D^s_\alpha \to Q_K(p,q)$ is bounded if and only if

\[
\text{(2.3) } \sup_{a, b \in \mathbb{D}} \int_{\mathbb{D}} |\varphi'(\varphi(z))|^{\frac{p}{2}}(1 - |z|^2)^q K(g(z, b)) \, dA(z) < \infty.
\]

If $K$ satisfies (2.1), then $Q_K(p, \beta p - 2) = B^\beta$, and Theorem 3 implies that $C_\varphi : D^p_\alpha \to B^\beta$ is bounded if and only if

\[
\text{(2.3) } \sup_{a, b \in \mathbb{D}} \int_{\mathbb{D}} |\varphi'(\varphi(z))|^{\alpha + 2} |\varphi'(z)|^p (1 - |z|^2)^{\beta q} K(g(z, b)) \, dA(z) < \infty.
\]

However, this reduces to known result, since (2.3) is satisfied if and only if

\[
\text{(2.4) } \sup_{z \in \mathbb{D}} \frac{|\varphi'(z)|}{(1 - |\varphi(z)|^2)^\frac{\alpha + 2}{p}} (1 - |z|^2)^\beta < \infty.
\]

To see this, note first that (2.4) clearly implies (2.3) since (2.1) is satisfied. Moreover, the opposite implication can be verified by a straightforward calculation which uses the fact that the function $|\varphi''(\varphi(z))|^{\alpha + 2} |\varphi'(z)|^p$ is subharmonic in $\mathbb{D}$. For characterizations of bounded and compact composition operators from $Q_K(p, q)$ into $B^\alpha$; see [16].

Theorem 4 characterizes compact composition operators mapping from $D^s_\alpha$ into $Q_K(p, q)$ when $p \geq s$. This result is a natural analogue of Theorem 3 and therefore it generalizes the corresponding compactness results in [17, 19, 25, 34, 35, 37, 39].

**Theorem 4.** Let $0 < s \leq p < \infty$, $-1 < \alpha < \infty$, $-2 < q < \infty$, and $\varphi \in B(\mathbb{D})$. Then $C_\varphi : D^s_\alpha \to Q_K(p,q)$ is compact if and only if

\[
\text{(2.5) } \lim_{|a| \to 1^-} \sup_{b \in \mathbb{D}} \int_{\mathbb{D}} |\varphi'(\varphi(z))|^{\frac{p}{2}}(1 - |z|^2)^q K(g(z, b)) \, dA(z) = 0.
\]

If $1 < s \leq p < \infty$, then Theorem 5 shows that the quantity in (2.5) gives an asymptotic formula for the essential norm of a bounded operator $C_\varphi$ from $D^s_\alpha$ into $Q_K(p,q)$.

**Theorem 5.** Let $1 < s \leq p < \infty$, $-1 < \alpha < \infty$, $-2 < q < \infty$, and $\varphi \in B(\mathbb{D})$. If $C_\varphi : D^s_\alpha \to Q_K(p,q)$ is bounded, then

\[
\|C_\varphi\|_e \simeq \lim_{|a| \to 1^-} \sup_{b \in \mathbb{D}} \int_{\mathbb{D}} |\varphi'(\varphi(z))|^{\frac{p}{2}}(1 - |z|^2)^q K(g(z, b)) \, dA(z).
\]
It is worth noticing that the conditions (2.2) and (2.5) with $\alpha = -1$ characterize bounded and compact operators $C_\varphi$ from the space $\{f \in \mathcal{H}(\mathbb{D}) : f' \in H^s\}$ into $Q_K(p,q)$, and moreover, (2.6) with $\alpha = -1$ gives an asymptotic formula for the essential norm in the case $1 < s \leq p < \infty$. This can be seen by following the proofs with appropriate modifications. For example, Theorem B should be replaced by Carleson’s original result [5, 6] and its generalization due to Duren [10], while the test functions $f_\alpha$ defined in the proof of Theorem 3 with $\alpha = -1$ will do the trick also in this case. The details are omitted.

3. Auxiliary results and background material

A positive Borel measure $\mu$ on $\mathbb{D}$ is a bounded $t$-Carleson measure, if

$$\sup_I \frac{\mu(S(I))}{|I|^t} < \infty, \quad 0 < t < \infty,$$

where $|I|$ denotes the arc length of a subarc $I$ of the boundary of $\mathbb{D}$, $S(I) = \{z \in \mathbb{D} : z/|z| \in I, 1 - |I| \leq |z|\}$ is the Carleson box based on $I$, and the supremum is taken over all subarcs $I$ such that $|I| \leq 1$. Moreover, if

$$\lim_{|I| \to 0} \frac{\mu(S(I))}{|I|^t} = 0, \quad 0 < t < \infty,$$

then $\mu$ is a compact (vanishing) $t$-Carleson measure. If $t = 1$, then a bounded (resp. compact) 1-Carleson measure is just a standard bounded (resp. compact) Carleson measure. These measures (for $t = 1$) were introduced by Carleson [5, 6]; see [24] for a list of related references.

The first auxiliary result needed is Luecking’s [20] characterization of Carleson measures in terms of functions in the Dirichlet type spaces. In comparison with the original result, $f$ has been replaced by $f'$ since this appears to be convenient for the purposes of the present study.

**Theorem B.** Let $\mu$ be a positive measure on $\mathbb{D}$, and let $0 < s \leq p < \infty$ and $-1 < \alpha < \infty$. Then $\mu$ is a bounded $\mathcal{D}_s^p(\alpha+2)$-Carleson measure if and only if there is a positive constant $C$, depending only on $p$ and $s$, such that

$$\int_{\mathbb{D}} |f'(z)|^p \, d\mu(z) \leq C \|f\|_{\mathcal{D}_s^p}^p$$

for all analytic functions $f$ in $\mathbb{D}$, in particular for all $f \in \mathcal{D}_s^p$. Moreover, if $\mu$ is a bounded $\mathcal{D}_s^p(\alpha+2)$-Carleson measure, then $C = C_1C_2$, where $C_1 > 0$.
The essential norm of a composition operator mapping depends only on $p$, $s$ and $\alpha$, and

$$C_2 = \sup_I \frac{\mu(S(I))}{|I|^s \omega(\alpha+2)}.$$ 

It is well-known that the bounded $t$-Carleson measures can be characterized by a global integral condition, namely,

$$(3.1) \quad \sup_I \frac{\mu(S(I))}{|I|^t} \simeq \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\varphi'_a(z)|^t \, d\mu(z), \quad 0 < t < \infty;$$

see [2]. The following lemma is a partial boundary version of this result. See [30] for a proof.

**Lemma C.** Let $0 < r < 1$, $1 \leq t < \infty$ and let $\mu$ be a positive Borel measure on $\mathbb{D}$. Then

$$\sup_I \frac{\mu(S(I) \setminus \Delta(0,r))}{|I|^t} \lesssim \sup_{|a| \geq r} \int_{\mathbb{D}} |\varphi'_a(z)|^t \, d\mu(z),$$

where $\Delta(0,r) := \{z : |z| < r\}$.

The following change of variables formula is a special case of the Area Formula [7, Theorem 2.32]; see also [12, 32, 36]. It plays a key role in the proofs of some of the main results.

**Lemma D.** Let $g$ and $u$ be positive measurable functions on $\mathbb{D}$, and let $\varphi \in B(\mathbb{D})$. Then

$$\int_{\mathbb{D}} (g \circ \varphi)(z)|\varphi'(z)|^2 u(z) \, dA(z) = \int_{\mathbb{D}} g(w)U(\varphi, w) \, dA(w),$$

where $U(\varphi, w) := \sum_{z \in \varphi^{-1}(\{w\})} u(z)$ for $w \in \mathbb{D} \setminus \{\varphi(0)\}$.

This section is completed by a characterization of compact composition operators, needed in the proof of Theorem 4. Lemma E follows by a more general result for linear operators [37]. The original result is stated for Banach spaces only, but the proof applies also for complete metric spaces.

**Lemma E.** Let $0 < p_1, p_2 < \infty$, $-2 < q_1, q_2 < \infty$ and let $K_1, K_2$ be nonnegative and nondecreasing on $[0, \infty)$. If $\varphi \in B(\mathbb{D})$, then $C_\varphi : Q_{K_1}(p_1, q_1) \to Q_{K_2}(p_2, q_2)$ is compact if and only if for any bounded sequence $\{f_n\}$ in $Q_{K_1}(p_1, q_1)$ with $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, $\|C_\varphi(f_n)\|_{Q_{K_2}(p_2, q_2)} \to 0$ as $n \to \infty$. 
4. Proof of Theorem 1

The upper and lower estimates for the essential norm are proved separately. In order to simplify the formulas, denote

$$
\Lambda := \limsup_{r \to 1^-} \sup_{a \in \partial} \int_{\{|\varphi(z)| > r\}} \frac{d\mu_a(z)}{(1 - |\varphi(z)|^2)^{\pi}}.
$$

where $d\mu_a(z) := |\varphi'(z)|^p(1 - |z|^2)^q K(g(z, a)) dA(z)$.

4.1. Proof of $\Lambda \lesssim \|C_\varphi\|_p^p$. This asymptotic inequality is proved by modifying the proof of [18, Theorem 6]. Let $\{b_m\} \subset (\frac{1}{2}, 1)$ such that $b_m \to 1$ as $m \to \infty$, and define

$$
\kappa_{n,m,\theta} := \frac{z^{2n}}{b_m} \sum_{k=1}^{\infty} \frac{2^k(b_m e^{i\theta})^k}{1 + 2^{n-k}} z^{2k} = \frac{1}{b_m} \sum_{k=1}^{\infty} \frac{2^k}{2^k + 2^n(b_m e^{i\theta})^k} z^{2k + 2^n}
$$

for $n, m \in \mathbb{N}$ and $\theta \in [0, 2\pi)$. Since

$$
0 \leq \lim_{k \to \infty} \left| \frac{2^k(b_m e^{i\theta})^k}{1 + 2^{n-k}} \right|^2 = \lim_{k \to \infty} \frac{(b_m)^{2k}}{1 + 2^{n-k}} \leq \lim_{k \to \infty} (b_m)^{2k} = 0,
$$

the function $f_{n,m,\theta}$ belongs to $B_0^\alpha$ for all $n, m \in \mathbb{N}$ and $\theta \in [0, 2\pi)$ by [47, Theorem 1]. Moreover,

$$
\sup_{k \in \mathbb{N}} \left| \frac{2^k(b_m e^{i\theta})^k}{2^k + 2^n} \right| (2^k + 2^n)^{-\alpha} = \sup_{k \in \mathbb{N}} \frac{(b_m)^{2k}2^k a}{(2^k + 2^n)^\alpha} \leq 1,
$$

and the proof of [47, Theorem 1] shows that there exists a positive constant $M$ such that $\|f_{n,m,\theta}\|_{B_0^\alpha} \leq M$ for all $n, m \in \mathbb{N}$ and $\theta \in [0, 2\pi)$. Define $g_{n,m,\theta} := f_{n,m,\theta}/M$. Then the sequence $\{g_{n,m,\theta}\}_{n=1}^\infty$ is contained in the closed unit ball of $B_0^\alpha$. Moreover, $g_{n,m,\theta}$ tends to zero uniformly on compact subsets of $\partial$ for every $m$ and $\theta$ as $n \to \infty$, and therefore $g_{n,m,\theta}$ tends to zero weakly as $n \to \infty$; see, [18], [21] for analogous results. It follows that for any compact operator $J : B^\alpha \to Q_K(p, q)$,

$$
\|C_\varphi - J\| \geq \limsup_{n \to \infty} \sup_{m,\theta} \|(C_\varphi - J)(g_{n,m,\theta})\|
$$

$$
\geq \limsup_{n \to \infty} \sup_{m,\theta} \|C_\varphi(g_{n,m,\theta})\|_{Q_k(p, q)} - \limsup_{n \to \infty} \sup_{m,\theta} \|J(g_{n,m,\theta})\|_{Q_k(p, q)}
$$

$$
= \limsup_{n \to \infty} \sup_{m,\theta} \|C_\varphi(g_{n,m,\theta})\|_{Q_k(p, q)}.
$$

which yields

\[ \|C_\varphi\|_p^p \geq M^{-p} \limsup_{n \to \infty} \sup_{m, \theta} \sup_{a \in \mathbb{D}} \int_D |f'_{n,m,\theta}(\varphi(z))|^p \, d\mu_a(z). \]

This implies that for a given \( \varepsilon > 0 \) there exist positive constants \( \delta \) for every \( |\theta| < \delta \) such that

\[ M^p \|C_\varphi\|_p^p + \varepsilon \geq \int_D |f'_{n,m,\theta}(\varphi(z))|^p \, d\mu_a(z) \]

for all \( m, \theta \) and \( a \) when \( n \leq N_\varepsilon \). An integration of this inequality with respect to \( \theta \) with an application of Fubini’s theorem gives

\[
2\pi (M^p \|C_\varphi\|_p^p + \varepsilon) \\
\geq \int_0^{2\pi} \int_D |f'_{n,m,\theta}(\varphi(z))|^p \, d\theta \, d\mu_a(z) \\
= \int_0^{2\pi} \int_0^\infty \sum_{k=1}^\infty 2^{k\alpha} |e^{i2^k\theta}(b_m \varphi(z))|^{2^k-1} \, d\theta |\varphi(z)|^{2^n-1} \, d\mu_a(z).
\]

By Zygmund’s result on lacunary series [51, p. 215] and [24, Lemma 3.1] there exist positive constants \( C_1 \) and \( C_2 \) such that

\[
\int_0^{2\pi} \left| \sum_{k=1}^\infty 2^{k\alpha} e^{i2^k\theta}(b_m \varphi(z))^{2^k-1} \right|^p \, d\theta \geq C_1 \left( \sum_{k=1}^\infty 2^{k\alpha} |b_m \varphi(z)|^{2^k-1} \right)^{\frac{p}{2}} \\
\geq \frac{C_2}{(1 - |b_m \varphi(z)|^{2^k-1})^{\frac{p\alpha}{2}}}
\]

for \( |\varphi(z)| \geq e^{-\frac{\alpha}{2}} \), and it follows that

\[ \int_{|\varphi(z)| \geq e^{-\frac{\alpha}{2}}} |\varphi(z)|^{2^n-1} |b_m|^p |\varphi(z)|^{2^n-1} \, d\mu_a(z) \]

for every \( m, a \) and \( n \geq N_\varepsilon \). Denote \( E_n := \{ z \in \mathbb{D} : |\varphi(z)| \geq 1 - 2^{-2^n} \} \), and let \( N \in \mathbb{N} \) such that \( 1 - 2^{-2^n} \geq e^{-\frac{\alpha}{2}} \) for all \( n \geq N \). Take now limit inferior as \( m \to \infty \) on both sides of (4.1) and apply Fatou’s lemma to obtain

\[
2\pi (M^p \|C_\varphi\|_p^p + \varepsilon) \geq C_2 \int_{|\varphi(z)| \geq e^{-\frac{\alpha}{2}}} |\varphi(z)|^{2^n-1} \frac{d\mu_a(z)}{(1 - |\varphi(z)|^{2^n-1})^{\frac{p\alpha}{2}}} \\
\geq C_2 (1 - 2^{-2^n})^{-\frac{p\alpha}{2}} \int_{E_n} \frac{d\mu_a(z)}{(1 - |\varphi(z)|^{2^n-1})^{\frac{p\alpha}{2}}}
\]
for all $a \in \mathbb{D}$ and $n \geq N'_\varepsilon := \max\{N_\varepsilon, N\}$. This yields
\[
2\pi e(C_2)^{-1}(M^p\|C_{\varphi}\|_p^p + \varepsilon) \geq \limsup_{r \to 1^-} \sup_{a \in \mathbb{D}} \int_{\{|\varphi(z)| > r\}} \frac{d\mu_a(z)}{(1 - |\varphi(z)|)^{pa}} \geq \Lambda
\]
for all $\varepsilon > 0$, and it follows that $\Lambda \lesssim \|C_{\varphi}\|_p^p$.

4.2. Proof of $\|C_{\varphi}\|_p \lesssim \Lambda$. For $k \in \mathbb{N}$, define $C_k(f) := C_{\psi_k}(f)$, where $\psi_k(z) := \frac{kz}{k + 1}$. Since the operator $C_k$ is compact on $B^\alpha$ for all $k \in \mathbb{N}$; see, for example, [44, p. 115], and $C_{\varphi} : B^\alpha \to Q_K(p, q)$ is bounded, it follows that
\[
\|C_{\varphi}\|_p \leq \|C_{\varphi} - C_{\varphi}C_k\|_p = \|C_{\varphi}(\text{Id} - C_k)\|_p,
\]
where $\text{Id}(f) := f$. Therefore
\[
\|C_{\varphi}\|_p \leq 2^p \sup_{\|f\|_{B^\alpha} \leq 1} \sup_{a \in \mathbb{D}} \int\{\{\varphi(z)\} \leq r\} |(f - f \circ \psi_k)'(\varphi(z))|^p d\mu_a(z)
\]
\[
+ 2^p \sup_{\|f\|_{B^\alpha} \leq 1} \sup_{a \in \mathbb{D}} \int\{\{|\varphi(z)| > r\} |(f - f \circ \psi_k)'(\varphi(z))|^p d\mu_a(z)
\]
\[
+ 2^p \sup_{\|f\|_{B^\alpha} \leq 1} |(f - f \circ \psi_k)(\varphi(0))|^p
\]
\[
= \Lambda(k, r) + B(k, r) + C(k)
\]
for all $r \in (0, 1)$ and $k \in \mathbb{N}$. Since $f - f \circ \psi_k$ and its derivative tend to zero uniformly in a compact subset of $\mathbb{D}$ as $k \to \infty$, it follows that $C(k) \to 0$ and $A(k, r) \to 0$ for every $r \in (0, 1)$. Therefore, $\|C_{\varphi}\|_p \leq \liminf_{k \to \infty} B(k, r)$. Moreover,
\[
\|f - f \circ \psi_k\|_{B^\alpha} \leq \|f\|_{B^\alpha} + \|f \circ \psi_k\|_{B^\alpha} \leq 2
\]
and it follows that
\[
\|C_{\varphi}\|_p \leq 2^p \sup_{a \in \mathbb{D}} \int\{\{|\varphi(z)| > r\} \frac{d\mu_a(z)}{(1 - |\varphi(z)|)^{pa}}
\]
for all $r \in (0, 1)$. This yields $\|C_{\varphi}\|_p \lesssim \Lambda$. \qed
5. Proof of Theorem 3

5.1. Proof of the sufficiency of (2.2). First note that it suffices to consider seminorms; see, for example, [16]. By Lemma D,

\[ I_{rb}(f) := \int_{D} |(f \circ \varphi)'(z)|^p (1 - |z|^2)^q K(g(z,b)) \, dA(z) = \int_{D} |f'(w)|^p \, d\mu_b(w), \]

where \( d\mu_b(w) = \sum_{z \in \varphi^{-1}(w)} |\varphi'(z)|^{p-2} (1 - |z|^2)^q K(g(z,b)) \, dA(w). \) By Theorem B, Lemma D and the asymptotic equality (3.1),

\[ I_{rb}(f) \lesssim \sup_{a \in D} \int_{D} |\varphi'_a(w)|^{\frac{p}{r_0} + \frac{2}{s}} \, d\mu_b(w) \|f\|^p_{D^s_{\alpha}}, \]

and it follows that \( C_{\varphi} : D^s_{\alpha} \to Q_K(p,q) \) is bounded if (2.2) is satisfied. \( \square \)

5.2. Proof of the necessity of (2.2). If \( C_{\varphi} : D^s_{\alpha} \to Q_K(p,q) \) is bounded, then the condition (2.2) follows by using the test functions

\[ f_a(z) := \int_{0}^{z} \left( \frac{1 - |a|^2}{(1 - \zeta |a|^2)^2} \right) dw, \quad a \in D, \]

for which \( f_a(0) = 0 \) and \( \|f_a\|^p_{D^s_{\alpha}} = \pi \) for all \( a \in D \); see [11, 15] or the original reference [38]. \( \square \)

6. Proof of Theorem 4

6.1. Proof of the sufficiency of (2.5). Let \( \{f_n\} \) be a bounded sequence in \( D^s_{\alpha} \) such that \( f_n \) tends uniformly to zero on compact subsets of \( D \) as \( n \to \infty \). By Lemma D,

\[ \|C_{\varphi}(f_n)\|^p_{Q_K(p,q)} = \sup_{b \in D} \int_{D} |f'_n(w)|^p \, d\mu_b(w) \]

\[ = \sup_{b \in D} \left( \int_{\Delta(0,r)} + \int_{D \setminus \Delta(0,r)} \right) |f'_n(w)|^p \, d\mu_b(w) \]

\[ =: A(r) + B(r), \]

where \( d\mu_b(w) = \sum_{z \in \varphi^{-1}(w)} |\varphi'(z)|^{p-2} (1 - |z|^2)^q K(g(z,b)) \, dA(w). \) By Theorem B,

\[ B(r) \leq C(r) \|f_n\|^p_{D^s_{\alpha}} \lesssim C(r). \]
where
\[ C(r) := \sup_{b \in \overline{D}} \sup_{I} \frac{1}{|I|^\frac{p}{2} (\alpha + 2)} \int_{S(I) \setminus \Delta(0, r)} d\mu_b(w) \]

\[ \lesssim \sup_{b \in \overline{D}} \sup_{|a| \geq r} \int_{\overline{D}} |\varphi'_a(w)|^\frac{p}{2} (\alpha + 2) d\mu_b(w) \]

by Lemma C. On one hand, by (2.5) it is possible to fix \( r_0 \) such that \( B(r_0) < \varepsilon / 2 \). On the other hand, there exists \( N \in \mathbb{N} \) such that

\[ A(r_0) = \sup_{b \in \overline{D}} \int_{\Delta(0, r_0)} |f'_a(w)|^p d\mu_b(w) < \frac{\varepsilon}{2} \]

for \( n \geq N \). These together with (6.1) imply that \( \|C_\varphi(f_n)\|_{Q_K(p,q)} \to 0 \) as \( n \to \infty \), and therefore \( C_\varphi : \mathcal{D}^s_\alpha \to Q_K(p,q) \) is compact by Lemma E. \( \Box \)

6.2. Proof of the necessity of (2.5). Assume that \( C_\varphi : \mathcal{D}^s_\alpha \to Q_K(p,q) \) is compact. Consider the test functions \( f_\alpha(z) := \int_0^1 (\varphi'_a(w))^{\frac{s}{2}} dw, a \in \overline{D} \), defined in the proof of Theorem 3. Since \( f_\alpha \) tends uniformly to zero on compact subsets of \( \overline{D} \) as \( |a| \to 1^- \), and

\[ \|C_\varphi(f_\alpha)\|_{Q_K(p,q)}^p = \sup_{b \in \overline{D}} \int_{\overline{D}} |\varphi'_a(\varphi(z))|^\frac{p}{2} (\alpha + 2) |\varphi'(z)|^p (1-|z|^2)^q K(g(z,b)) dA(z), \]

the condition (2.5) follows by Lemma E. \( \Box \)

7. Proof of Theorem 5.

Denote
\[ \Lambda := \limsup_{|a| \to 1^-} \sup_{b \in \overline{D}} \int_{\overline{D}} |\varphi'_a(w)|^\frac{p}{2} (\alpha + 2) d\mu_b(w), \]

where \( d\mu_b(w) := \sum_{z \in \varphi^{-1}(\{w\})} |\varphi'_a(z)|^{p-2} (1-|z|^2)^q K(g(z,b)) dA(w) \).

7.1. Proof of \( \Lambda \lesssim \|C_\varphi\|_{p}^p \). Recall that the test functions \( f_\alpha \), defined in the proof of Theorem 3, have inter alia the property that \( f_\alpha \to 0 \) uniformly on compact subsets of \( \overline{D} \) as \( |a| \to 1^- \). Moreover, since the space \( \mathcal{D}^s_\alpha \) is reflexive when \( 1 < s < \infty \), \( f_\alpha \to 0 \) weakly in \( \mathcal{D}^s_\alpha \) as \( |a| \to 1^- \) by [4, Proposition 2]. Therefore, if \( J : \mathcal{D}^s_\alpha \to Q_K(p,q) \) is compact, then

\[ \|C_\varphi - J\|_{p}^p \geq \limsup_{|a| \to 1^-} \|C_\varphi(f_\alpha) - J(f_\alpha)\|_{Q_K(p,q)}^p \]

\[ \geq \limsup_{|a| \to 1^-} \|C_\varphi(f_\alpha)\|_{Q_K(p,q)}^p - \limsup_{|a| \to 1^-} \|J(f_\alpha)\|_{Q_K(p,q)}^p \]
The essential norm of a composition operator mapping

\[
= \limsup_{|a| \to 1^-} \| C_\varphi(f_a) \|_{q, p}^p,
\]
and it follows that \( \| C_\varphi \|_e^p \geq \Lambda. \)

7.2. Proof of \( \| C_\varphi \|_e^p \lesssim \Lambda. \) For an analytic function \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) in \( D, \) define

\[
T_n f(z) := \sum_{k=0}^{n} a_k z^k, \quad R_n f(z) := \sum_{k=n+1}^{\infty} a_k z^k.
\]

Since \( T_n \) is compact on \( D^*_\alpha, \)

\[
\| C_\varphi \|_e = \| C_\varphi (T_n + R_n) \|_e \leq \| C_\varphi T_n \|_e + \| C_\varphi R_n \|_e = \| C_\varphi T_n \|_e \leq \| C_\varphi R_n \|,
\]
and it follows that \( \| C_\varphi \|_e \leq \liminf_{n \to \infty} \| C_\varphi R_n \|. \) Therefore, by Lemma D,

\[
\| C_\varphi \|_e^2 \leq \liminf_{n \to \infty} \| C_\varphi R_n \|^p
\]
\[
\leq \liminf_{n \to \infty} \sup_{\| f \|_p \leq 1} \left( \| (C_\varphi R_n) (f) \|_{q, p} + \| R_n f (\varphi (0)) \|_{q, p} \right)^p
\]
\[
= \liminf_{n \to \infty} \sup_{\| f \|_p \leq 1} \sup_{b \in \mathbb{D}} \int_{\mathbb{D}} |(R_n f)'(\varphi (z))|^p |\varphi'(z)|^p (1 - |z|^2)^q K(g(z, b)) \, dA(z)
\]
\[
= \liminf_{n \to \infty} \sup_{\| f \|_p \leq 1} \sup_{b \in \mathbb{D}} \int_{\mathbb{D}} |(R_n f)'(w)|^p \, d\mu_b(w).
\]

Since

\[
\liminf_{n \to \infty} \sup_{\| f \|_p \leq 1} \int_{\Delta(0, r)} |(R_n f)'(w)|^p \, d\mu_b(w) = 0
\]
for any \( r \in (0, 1), \) Theorem B yields

\[
\| C_\varphi \|_e^p \lesssim \liminf_{n \to \infty} \sup_{\| f \|_p \leq 1} \| R_n f \|_{D^*_\alpha}^p \sup_{b \in \mathbb{D}} C(b) \leq \sup_{b \in \mathbb{D}} C(b),
\]
where

\[
C(b) := \sup_{t \in \mathbb{R}} \frac{1}{|t|^{\frac{\alpha+2}{2}}} \int_{S(1) \setminus \Delta(0, r)} \mu_b(w).
\]

By Lemma C,

\[
\| C_\varphi \|_e^p \lesssim \sup_{b \in \mathbb{D}} \int_{|z| \geq r} |\varphi'(z)|^p |z|^{\alpha+2} \, d\mu_b(w),
\]
and therefore the assertion follows by applying Lemma D and letting \( r \to 1^- \). \( \square \)
References


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