An estimate for the best constant in the L^p -Wirtinger inequality with weights

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Abstract. We prove an estimate for the best constant C in the following Wirtinger type inequality

$$\int_0^{2\pi} a|w|^p \leq C \int_0^{2\pi} b|w'|^p.$$

1. Introduction

In this paper we consider some Wirtinger-type inequalities. More precisely, our aim is to give an estimate for the best constant $C(a, b)$ in the following type inequality (for $p > 1$):

$$(1.1) \quad \int_0^{2\pi} a|w|^p \leq C(a, b) \int_0^{2\pi} b|w'|^p$$

where $w \in W^{1,p}(0, 2\pi)$ is a 2π -periodic function satisfying the constraint

$$(1.2) \quad \int_0^{2\pi} a|w|^{p-2}w = 0,$$

and a and b are two measurable and non negative functions belonging to suitable spaces.

We note that when $a = b = 1$ and $p = 2$, inequality (1.1)-(1.2) reduces to the well-known Wirtinger inequality. In this case

$$(1.3) \quad C(a, b) = C(1, 1) = 1.$$

The case $b = a$ and $p = 2$ has been analyzed by Piccinini and Spagnolo in [3], who obtained the estimate

$$(1.4) \quad C(a, a) \leq \left(\frac{4}{\pi} \arctan \sqrt{\frac{\min a}{\max a}} \right)^{-2}.$$

The case $b = a^{-1}$ and $p = 2$ has been studied in [4] (when a is a measurable function bounded from above and away from 0) and in [2] (when a is simply a function belonging to $L^1(0, 2\pi)$), where the following estimate was obtained:

$$(1.5) \quad C\left(a, \frac{1}{a}\right) = \left(\frac{1}{2\pi} \int_0^{2\pi} a \right)^2.$$

The case $a \neq b$ and $p = 2$ has been studied in [4] (when a, b are positive measurable functions on $(0, 2\pi)$ bounded from above and away from 0) and in [2] (when a and b are non negative measurable functions on $(0, 2\pi)$ such that a and $\frac{1}{b}$ belong to $L^1(0, 2\pi)$), where the following estimate was obtained:

$$(1.6) \quad C(a, b) \leq \left(\frac{\frac{1}{2\pi} \int_0^{2\pi} \sqrt{ab^{-1}}}{\frac{4}{\pi} \arctan \left(\frac{\inf ab}{\sup ab} \right)^{1/4}} \right)^2$$

In this paper we will extend the estimates (1.3)-(1.6) to the more general case $p > 1$.

We will denote by p' the conjugate exponent to p (i.e. $\frac{1}{p} + \frac{1}{p'} = 1$) and by $B(\frac{1}{p}, \frac{1}{p'})$ the Beta function

$$B\left(\frac{1}{p}, \frac{1}{p'}\right) = \Gamma\left(\frac{1}{p}\right) \Gamma\left(\frac{1}{p'}\right) = \int_0^1 t^{\frac{1}{p}-1} (1-t)^{\frac{1}{p'}-1} dt.$$

Furthermore, if $a : (0, 2\pi) \rightarrow \mathbb{R}$ is an integrable positive function we will denote by $L^p\left(\frac{1}{a^{p-1}}\right)$ the set of all measurable functions u on $[0, 2\pi]$ such that $\int_0^{2\pi} \frac{1}{a^{p-1}} |u|^p < \infty$ and by $W_{per}^{1,p}\left(\frac{1}{a^{p-1}}\right)$ the set of all functions $w \in L^1(0, 2\pi)$ such that the distributional derivative w' belongs to $L^p\left(\frac{1}{a^{p-1}}\right)$ and $w(0) = w(2\pi)$. If $b : (0, 2\pi) \rightarrow \mathbb{R}$ is any nonnegative measurable function, $W_{per}^{1,p}(b)$ is defined similarly.

We will prove the following two theorems:

Theorem 1.1. *Let $a \in L^1(0, 2\pi)$ be a nonnegative function. Then the following Wirtinger inequality holds:*

$$(1.7) \quad \int_0^{2\pi} a|w|^p \leq C \left(a, \frac{1}{a^{p-1}} \right) \int_0^{2\pi} \frac{1}{a^{p-1}} |w'|^p$$

where

$$(1.8) \quad C \left(a, \frac{1}{a^{p-1}} \right) = \left(\frac{4}{\int_0^{2\pi} a} \left(\frac{1}{p'} \right)^{\frac{1}{p}} \left(\frac{1}{p} \right)^{\frac{1}{p'}} B \left(\frac{1}{p}, \frac{1}{p'} \right) \right)^{-p}$$

for every $w \in W_{per}^{1,p}\left(\frac{1}{a^{p-1}}\right)$ such that $\int_0^{2\pi} a|w|^{p-2}w = 0$.

Theorem 1.2. *Let $a, b : (0, 2\pi) \rightarrow [0, \infty[$ be measurable functions such that $a, \frac{1}{b}$ and $\sqrt[p]{ab^{-1}}$ belong to $L^1(0, 2\pi)$. Assume that $0 < I = \inf \sqrt[p]{a^{p-1}b} \leq S = \sup \sqrt[p]{a^{p-1}b} < \infty$ and set, for all $x, y \geq 0$, $A(x, y) = \int_x^y \frac{1}{1+t^{\frac{p}{p-1}}} dt$. Then for every $w \in W_{per}^{1,p}(b)$ such that $\int_0^{2\pi} a|w|^{p-2}w = 0$*

$$(1.9) \quad \int_0^{2\pi} a|w|^p \leq \left(\frac{\frac{1}{2\pi} \int_0^{2\pi} \sqrt[p]{a(t)b(t)^{-1}}}{\frac{2}{\pi} \frac{1}{(p-1)^{\frac{p-1}{p}}} [c_{p'} - \Psi(S, I)]} \right)^p \int_0^{2\pi} b|w'|^p,$$

where

$$\Psi(S, I) = A \left(\frac{I}{S} \left(\frac{(S-I)S^{\frac{1}{p-1}}}{I(S^{\frac{1}{p-1}} - I^{\frac{1}{p-1}})} \right)^{\frac{p-1}{p}}, \left(\frac{(S-I)S^{\frac{1}{p-1}}}{I(S^{\frac{1}{p-1}} - I^{\frac{1}{p-1}})} \right)^{\frac{p-1}{p}} \right)$$

and, for $1 < q < \infty$

$$c_q = \int_0^{+\infty} \frac{1}{1+x^q} dx = \frac{1}{q} B \left(\frac{1}{q}, \frac{1}{q'} \right).$$

Note that when a is equal to 1, Theorem 1.1 yields

$$C(1, 1) = \left(\frac{2}{\pi} \left(\frac{1}{p'} \right)^{\frac{1}{p}} \left(\frac{1}{p} \right)^{\frac{1}{p'}} B \left(\frac{1}{p}, \frac{1}{p'} \right) \right)^{-p}$$

which is the optimal constant in the Wirtinger inequality, as proved by Croce and Dacorogna in [1].

In order to prove Theorem 1.2, we need Lemmas 3.1 and 3.2. The first one gives an estimate for $C(a, a)$ (when $a = b$) that reduces to the estimate (1.5) obtained by Piccinini and Spagnolo when $p = 2$; the second one gives an estimate for $C(a, b)$ for arbitrary weight functions a, b belonging to L^∞ .

2. Proof of Theorem 1.1

First we proceed with a lemma and then with the proof of the theorem.

Lemma 2.1. *Let a be a measurable function which is bounded from above and away from zero. If $C(a, \frac{1}{a^{p-1}})$ is defined by (1.8), then*

$$(2.1) \quad \int_0^{2\pi} a|w|^p \leq C \left(a, \frac{1}{a^{p-1}} \right) \int_0^{2\pi} \frac{1}{a^{p-1}} |w'|^p$$

for every $w \in W_{per}^{1,p}(0, 2\pi)$ such that

$$(2.2) \quad \int_0^{2\pi} a|w|^{p-2} w = 0.$$

Proof. Let us still denote by a the 2π -periodic extension of the function a originally defined only in $[0, 2\pi]$ and set

$$y(x) = \int_0^x a(t)dt \quad \text{and} \quad W(y) = w(x(y)).$$

Then $W \in W^{1,p}(\mathbb{R})$ is β -periodic where $\beta = \int_0^{2\pi} a dt$.

It easily seen that (2.1) and (2.2) may be rewritten as

$$(2.3) \quad \int_0^\beta |W|^p dy \leq C \int_0^\beta |W'|^p dy$$

and

$$(2.4) \quad \int_0^\beta |W|^{p-2} W dy = 0.$$

Therefore, applying Theorem 1.1 of [1] we conclude the proof. $\square$

Proof of the Theorem 1.1. The proof is similar in spirit to the one of [2] but it differs in technical aspects. It will be divided into 3 steps.

Step 1. Let us first assume that $w \in W_{per}^{1,p}(0, 2\pi) \cap W^{1,\infty}(0, 2\pi)$ and

$$(2.5) \quad a(t) \geq 1.$$

For $h > 1$ we set

$$(2.6) \quad a_h(t) = \begin{cases} a(t) & \text{if } 1 \leq a(t) \leq h \\ h & \text{if } a(t) > h \end{cases}$$

and define

$$(2.7) \quad w_h(x) = w(x) + \epsilon_h$$

where ϵ_h is a constant, hence $w_h(0) = w_h(2\pi)$.

A simple continuity argument yields that for all h there exists ϵ_h such that

$$(2.8) \quad \int_0^{2\pi} a_h |w + \epsilon_h|^{p-2} (w + \epsilon_h) = \int_0^{2\pi} a_h |w_h|^{p-2} w_h = 0$$

hence

$$-1 - \max w \leq \epsilon_h \leq 1 - \min w,$$

i.e., the sequence $\{\epsilon_h\}$ is bounded, we claim that $\epsilon_h \rightarrow 0$. In fact if this were not true there would exist a subsequence, still denoted by ϵ_h , such that $\epsilon_h \rightarrow \epsilon$ with $\epsilon \neq 0$.

Now, we show that the value ϵ is equal to zero, thus proving the claim. The function

$$\phi(\epsilon) = \int_0^{2\pi} a |w + \epsilon|^{p-2} (w + \epsilon)$$

is strictly increasing and from (1.2)

$$(2.9) \quad \phi(0) = 0.$$

On the other hand

$$(2.10) \quad \int_0^{2\pi} a_h |w + \epsilon_h|^{p-2} (w + \epsilon_h) \rightarrow \int_0^{2\pi} a |w + \epsilon|^{p-2} (w + \epsilon) = 0.$$

Combining (2.9) and (2.10) we have

$$(2.11) \quad \epsilon_h \rightarrow 0 \quad \text{as} \quad h \rightarrow \infty.$$

By contradiction, we have proved the claim.

From Lemma 2.1 we have the following inequality for w_h and a_h

$$(2.12) \quad \int_0^{2\pi} a_h |w_h|^p \leq C \left(a_h, \frac{1}{a_h^{p-1}} \right) \int_0^{2\pi} \frac{1}{a_h^{p-1}} |w_h'|^p.$$

where

$$C \left(a_h, \frac{1}{a_h^{p-1}} \right) = \left(\frac{4}{\int_0^{2\pi} a_h} \left(\frac{1}{p'} \right)^{\frac{1}{p}} \left(\frac{1}{p} \right)^{\frac{1}{p'}} B \left(\frac{1}{p}, \frac{1}{p'} \right) \right)^{-p}.$$

Since $\epsilon_h \rightarrow 0$, $w_h \rightarrow w$ uniformly in $(0, 2\pi)$; moreover $a_h \rightarrow a$ in $L^1(0, 2\pi)$ and we have

$$(2.13) \quad \int_0^{2\pi} a_h |w_h|^p \rightarrow \int_0^{2\pi} a |w|^p.$$

On the other hand (2.6) and (2.7) imply

$$\int_0^{2\pi} \frac{1}{a_h^{p-1}} |w_h'|^p = \int_{a \leq h} \frac{1}{a^{p-1}} |w'|^p + \frac{1}{h^{p-1}} \int_{a > h} |w'|^p.$$

Notice that since w' is bounded, the last integral is infinitesimal as $h \rightarrow \infty$, hence

$$(2.14) \quad \int_0^{2\pi} \frac{1}{a_h^{p-1}} |w_h'|^p \rightarrow \int_0^{2\pi} \frac{1}{a^{p-1}} |w'|^p$$

and (1.7) is established.

Step 2. Now, still assuming that $w \in W_{per}^{1,p}(0, 2\pi) \cap W^{1,\infty}(0, 2\pi)$, we make no special assumptions on a .

Notice, however, that we may always assume, without loss of generality, that $\int_0^{2\pi} a(x) dx > 0$, otherwise (1.7) would be trivial.

For $\delta > 0$ we set

$$(2.15) \quad a_\delta = \begin{cases} a & \text{if } a \geq \delta \\ \delta & \text{if } 0 \leq a < \delta \end{cases}$$

and define

$$(2.16) \quad w_\delta(x) = w(x) + \epsilon_\delta$$

where ϵ_δ is a constant.

As in the previous step it is possible to prove that for any δ there exists ϵ_δ such that

$$(2.17) \quad \int_0^{2\pi} a_\delta |w + \epsilon_\delta|^{p-2} (w + \epsilon_\delta) = \int_0^{2\pi} a_\delta |w_\delta|^{p-2} w_\delta = 0$$

and

$$(2.18) \quad \epsilon_\delta \rightarrow 0 \quad \text{as} \quad \delta \rightarrow 0.$$

From the previous step we have the following inequality for w_δ and a_δ

$$(2.19) \quad \int_0^{2\pi} a_\delta |w_\delta|^p \leq C \left(a_\delta, \frac{1}{a_\delta^{p-1}} \right) \int_0^{2\pi} \frac{1}{a_\delta^{p-1}} |w'_\delta|^p \leq C \left(a_\delta, \frac{1}{a_\delta^{p-1}} \right) \int_0^{2\pi} \frac{1}{a_\delta^{p-1}} |w'|^p$$

where

$$C \left(a_\delta, \frac{1}{a_\delta^{p-1}} \right) = \left(\frac{4}{\int_0^{2\pi} a_\delta} \left(\frac{1}{p'} \right)^{\frac{1}{p}} \left(\frac{1}{p} \right)^{\frac{1}{p'}} B \left(\frac{1}{p}, \frac{1}{p'} \right) \right)^{-p}.$$

As in (2.13), from (2.18) we have,

$$(2.20) \quad \int_0^{2\pi} a_\delta |w_\delta|^p \rightarrow \int_0^{2\pi} a |w|^p.$$

Combining (2.19) and (2.20) we obtain (1.7).

Step 3. Finally we assume that $w \in W_{per}^{1,p}(\frac{1}{a^{p-1}})$.

For $h > 0$ we set

$$(2.21) \quad T_h(w') = \begin{cases} h & \text{if } w' > h \\ w' & \text{if } -h < w' < h \\ -h & \text{if } w' < -h \end{cases}.$$

We define

$$(2.22) \quad w_h(x) = w(0) + \int_0^x \sigma_h(t) T_h(w') dt + \epsilon_h$$

where ϵ_h is a constant and

$$(2.23) \quad \sigma_h(x) \equiv 1 \quad \text{if} \quad \int_{\{T_h(w') > 0\}} T_h(w') dx = - \int_{\{T_h(w') < 0\}} T_h(w') dx$$

otherwise, if

$$\int_{\{T_h(w') > 0\}} T_h(w') dx > - \int_{\{T_h(w') < 0\}} T_h(w') dx,$$

then

$$(2.24) \quad \sigma_h(x) = \begin{cases} 1 & \text{if } x \in \{T_h(w') < 0\} \\ \frac{-\int_{\{T_h(w') < 0\}} T_h(w') dx}{\int_{\{T_h(w') > 0\}} T_h(w') dx} & \text{if } x \in \{T_h(w') > 0\}. \end{cases}$$

When $\int_{\{T_h(w') > 0\}} T_h(w') dx < -\int_{\{T_h(w') < 0\}} T_h(w') dx$, $\sigma_h(x)$ is defined similarly to (2.24).

Also in this case with similar considerations to those of the previous steps we obtain:

for any h there exists ϵ_h such that

$$(2.25) \quad \int_0^{2\pi} a|w_h|^{p-2} w_h = 0;$$

and

$$(2.26) \quad \epsilon_h \rightarrow 0 \quad \text{as} \quad h \rightarrow \infty.$$

Moreover, for any h , $w_h(0) = w_h(2\pi)$.

From the previous case, we have the Wirtinger inequality for a and w_h

$$(2.27) \quad \int_0^{2\pi} a|w_h|^p \leq C \left(a, \frac{1}{a^{p-1}} \right) \int_0^{2\pi} \frac{1}{a^{p-1}} |w'_h|^p.$$

Since $w_h \rightarrow w$ uniformly, we get

$$(2.28) \quad \int_0^{2\pi} a|w_h|^p \rightarrow \int_0^{2\pi} a|w|^p.$$

On the other hand, for any h

$$(2.29) \quad \int_0^{2\pi} \frac{1}{a^{p-1}} |w'_h|^p \leq \int_0^{2\pi} \frac{1}{a^{p-1}} |w'|^p$$

and (1.7) is proved. $\square$

3. Proof of the Theorem 1.2

In order to prove the Theorem 1.2 we need the following two lemmas. The first lemma extends an analogous one proved in Piccinini-Spagnolo (Lemma 1. of [3]). Here we will follow closely their proof by adapting to the case $p \neq 2$.

Lemma 3.1. *Let $a(t)$ be a periodic measurable function, with period 2π , such that $1 \leq a(t) \leq L$; let $w(t)$ be a periodic function, belonging to $W_{loc}^{1,p}(\mathbb{R})$, with period 2π , such that $\int_0^{2\pi} a|w|^{p-2}w = 0$. Then the following inequality holds:*

$$(3.1) \quad \int_0^{2\pi} a(t)|w(t)|^p dt \leq (\tilde{C}(L))^{-p} \int_0^{2\pi} a|w'(t)|^p dt$$

where

$$(3.2) \quad \tilde{C}(L) = \frac{2}{\pi} \frac{1}{(p-1)^{\frac{p-1}{p}}} \left[c_{p'} - \int_{\alpha(L)}^{\beta(L)} \frac{1}{1+t^{\frac{p}{p-1}}} dt \right]$$

with

$$(3.3) \quad c_{p'} = \int_0^{+\infty} \frac{1}{1+x^p} dx = \frac{1}{p'} B\left(\frac{1}{p}, \frac{1}{p'}\right);$$

$$(3.4) \quad \beta(L) = \left[\frac{L^{\frac{1}{p-1}}(L-1)}{L^{\frac{1}{p-1}} - 1} \right]^{\frac{p-1}{p}} \quad \text{and} \quad \alpha(L) = \frac{\beta(L)}{L};$$

Proof. Consider for any $a(t)$ such that $1 \leq a(t) \leq L$, the eigenvalue problem

$$(3.5) \quad \begin{cases} (a|w'|^{p-2}w')' + \lambda a|w|^{p-2}w = 0 \\ w \text{ periodic of period } 2\pi. \end{cases}$$

When $p = 2$, (3.5) becomes the problem (13) in [3].

It is easy to prove that the values of λ for which this problem has not constant solutions form a sequence λ_n with $0 < \lambda_1 < \lambda_2 < \dots$, and that for any function $w(t)$, periodic of period 2π , such that $\int_0^{2\pi} a|w|^{p-2}w = 0$, the following estimate holds

$$\int_0^{2\pi} a(t)|w(t)|^p dt \leq \frac{1}{\lambda_1} \int_0^{2\pi} a(t)|w'(t)|^p dt.$$

Therefore, in order to prove (3.1) it is sufficient to show that, if $\lambda \neq 0$ and $w \neq 0$ satisfy (3.5), then necessarily

$$\lambda \geq \left[\frac{2}{\pi} \frac{1}{(p-1)^{\frac{p-1}{p}}} \left[c_{p'} - \int_{\alpha(L)}^{\beta(L)} \frac{1}{1+t^{\frac{p}{p-1}}} dt \right] \right]^p.$$

It is easily seen that a solution of (3.5) in each period has at least two zeros, and that between any pair of zeros of the function there is one and only one zero of its derivative. Let t_0, t_2, t_4 be three consecutive zeros of w and let t_1 and t_3 be two zeros of w' in such a way that $t_0 < t_1 < t_2 < t_3 < t_4$. Without loss of generality we may suppose that $w(t_1) > 0$ and $w(t_3) < 0$. It is obvious that

$$(3.6) \quad t_4 - t_0 \leq 2\pi.$$

We define, for $t_0 < t \leq t_1$, the function

$$(3.7) \quad f(t) = \frac{a|w'|^{p-2}w'}{|w|^{p-2}w}.$$

According to (3.5) this function satisfies the following first order differential equation:

$$(3.8) \quad f'(t) = -\lambda a(t) - \frac{|f|^{\frac{p}{p-1}}}{a^{\frac{1}{p-1}}}(p-1).$$

We remark that, since $f' < 0$, f is strictly decreasing. Furthermore $\lim_{t \rightarrow t_0^+} f(t) = +\infty$, $f(t_1) = 0$. Hence, there is one and only one point, say

τ , in the interval (t_0, t_1) such that $f(\tau) = \left[\frac{\lambda L^{\frac{1}{p-1}}(L-1)}{(p-1)(L^{\frac{1}{p-1}}-1)} \right]^{\frac{p-1}{p}}$. Since f is decreasing, the following inequalities hold:

$$(3.9) \quad \begin{cases} -\lambda a(t) - \frac{|f|^{\frac{p}{p-1}}}{a^{\frac{1}{p-1}}}(p-1) \geq -\lambda - |f|^{\frac{p}{p-1}}(p-1) & \text{for } t_0 < t \leq \tau \\ -\lambda a(t) - \frac{|f|^{\frac{p}{p-1}}}{a^{\frac{1}{p-1}}}(p-1) \geq -\lambda L - \frac{|f|^{\frac{p}{p-1}}}{L^{\frac{1}{p-1}}}(p-1) & \text{for } \tau \leq t \leq t_1. \end{cases}$$

Thus, calling $f_0(t)$ the function such that

$$(3.10) \quad \begin{cases} f'_0(t) &= \begin{cases} -\lambda - |f_0|^{\frac{p}{p-1}}(p-1) & \text{for } t_0 < t \leq \tau \\ -\lambda L - \frac{|f_0|^{\frac{p}{p-1}}}{L^{\frac{1}{p-1}}}(p-1) & \text{for } \tau \leq t < t_1. \end{cases} \\ f_0(\tau) &= \left[\frac{\lambda L^{\frac{1}{p-1}}(L-1)}{(p-1)(L^{\frac{1}{p-1}}-1)} \right]^{\frac{p-1}{p}} \end{cases}$$

we get

$$(3.11) \quad \begin{cases} f_0(t) \geq f(t) & \text{for } t \leq \tau \\ f_0(t) \leq f(t) & \text{for } t \geq \tau. \end{cases}$$

Let us set

$$(3.12) \quad F(z) = \int_0^z \frac{1}{1 + x^{\frac{p}{p-1}}} dx$$

and

$$(3.13) \quad F(\infty) = \int_0^{+\infty} \frac{1}{1 + x^{\frac{p}{p-1}}} dx = c_{p'}.$$

The solution of (3.10) is given by

$$(3.14) \quad f_0(t) = \begin{cases} P(\lambda)F^{-1} \left[\frac{\lambda}{P(\lambda)}(\tau - t) + F \left(\frac{f_0(\tau)}{P(\lambda)} \right) \right] & \text{for } t \leq \tau \\ P(\lambda, L)F^{-1} \left[\frac{\lambda L}{P(\lambda, L)}(\tau - t) + F \left(\frac{f_0(\tau)}{P(\lambda, L)} \right) \right] & \text{for } t \geq \tau, \end{cases}$$

where $P(\lambda) = \left(\frac{\lambda}{p-1} \right)^{\frac{p-1}{p}}$ and $P(\lambda, L) = \left(\frac{\lambda L^{\frac{p}{p-1}}}{p-1} \right)^{\frac{p-1}{p}}$.

Therefore $f_0(t)$ tends to infinity for t converging to $\tau - \frac{P(\lambda)}{\lambda} \left(c_{p'} - F \left(\frac{f_0(\tau)}{P(\lambda)} \right) \right)$ and vanishes for t equal to $\tau + \frac{P(\lambda, L)}{\lambda L} F \left(\frac{f_0(\tau)}{P(\lambda, L)} \right)$. It follows that

$$t_1 - t_0 \geq \frac{P(\lambda, L)}{\lambda L} F \left(\frac{f_0(\tau)}{P(\lambda, L)} \right) + \frac{P(\lambda)}{\lambda} \left(c_{p'} - F \left(\frac{f_0(\tau)}{P(\lambda)} \right) \right).$$

In a similar way we can prove that

$$t_{i+1} - t_i \geq \frac{P(\lambda, L)}{\lambda L} F \left(\frac{f_0(\tau)}{P(\lambda, L)} \right) + \frac{P(\lambda)}{\lambda} \left(c_{p'} - F \left(\frac{f_0(\tau)}{P(\lambda)} \right) \right) \text{ for } i = 1, 2, 3;$$

hence by adding the relations above we get that

$$t_4 - t_0 \geq 4 \left[\frac{P(\lambda, L)}{\lambda L} F \left(\frac{f_0(\tau)}{P(\lambda, L)} \right) + \frac{P(\lambda)}{\lambda} \left(c_{p'} - F \left(\frac{f_0(\tau)}{P(\lambda)} \right) \right) \right]. \text{ So recalling (3.6), we can state}$$

$$2\pi \geq 4 \left[\frac{P(\lambda, L)}{\lambda L} F \left(\frac{f_0(\tau)}{P(\lambda, L)} \right) + \frac{P(\lambda)}{\lambda} \left(c_{p'} - F \left(\frac{f_0(\tau)}{P(\lambda)} \right) \right) \right],$$

that is

$$\lambda \geq \left[\frac{2}{\pi} \frac{1}{(p-1)^{\frac{p-1}{p}}} \left[F\left(\frac{f_0(\tau)}{P(\lambda, L)}\right) - F\left(\frac{f_0(\tau)}{P(\lambda)}\right) + c_{p'} \right] \right]^p.$$

Since $\frac{f_0(\tau)}{P(\lambda, L)} = \alpha(L)$ and $\frac{f_0(\tau)}{P(\lambda)} = \beta(L)$ the proof is complete. $\square$

Lemma 3.2. Suppose a and $b \in L^\infty(\mathbb{R})$ and $\inf a > 0$, $\inf b > 0$. Assume that $S = \sup \sqrt[p]{a^{p-1}b}$, $I = \inf \sqrt[p]{a^{p-1}b} > 0$ and set $A(x, y) = \int_x^y \frac{1}{1+t^{\frac{p}{p-1}}} dt$. Then

$$(3.15) \quad C(a, b) \leq \left(\frac{\frac{1}{2\pi} \int_0^{2\pi} \sqrt[p]{a(t)b(t)^{-1}}^{-1}}{\frac{2}{\pi} \frac{1}{(p-1)^{\frac{p-1}{p}}} [c_{p'} - \Psi(S, I)]} \right)^p,$$

where

$$(3.16) \quad \Psi(S, I) = A \left(\frac{I}{S} \left(\frac{(S-I)S^{\frac{1}{p-1}}}{I(S^{\frac{1}{p-1}} - I^{\frac{1}{p-1}})} \right)^{\frac{p-1}{p}}, \left(\frac{(S-I)S^{\frac{1}{p-1}}}{I(S^{\frac{1}{p-1}} - I^{\frac{1}{p-1}})} \right)^{\frac{p-1}{p}} \right)$$

and

$$(3.17) \quad c_{p'} = \int_0^{+\infty} \frac{1}{1+x^{p'}} dx = \frac{1}{p'} B\left(\frac{1}{p}, \frac{1}{p'}\right).$$

Proof. Under the change of variable

$$y(x) = \frac{1}{k} \int_0^x \sqrt[p]{\frac{a(t)}{b(t)}} dt,$$

where k is defined by

$$(3.18) \quad k = \frac{1}{2\pi} \int_0^{2\pi} \sqrt[p]{\frac{a(t)}{b(t)}} dt$$

setting $a(x) = \alpha(y(x))$, $b(x) = \beta(y(x))$, $w(x) = W(y(x))$, we obtain

$$dx = k \sqrt[p]{\frac{b(x(y))}{a(x(y))}} dy, \quad w'(x) = W'(y(x)) \frac{1}{k} \sqrt[p]{\frac{a(x)}{b(x)}},$$

and therefore:

$$\begin{aligned}\int_0^{2\pi} a|w|^p dx &= k \int_0^{2\pi} \sqrt[p]{\alpha^{p-1}\beta} |W|^p dy \\ \int_0^{2\pi} b|w'|^p dx &= \frac{1}{k^{p-1}} \int_0^{2\pi} \sqrt[p]{\alpha^{p-1}\beta} |W'|^p dy \\ \int_0^{2\pi} a|w|^{p-2} w dx &= k \int_0^{2\pi} \sqrt[p]{\alpha^{p-1}\beta} |W|^{p-2} W dy = 0.\end{aligned}$$

By the above equalities, (1.1)-(1.2) take the form:

$$(3.19) \quad \int_0^{2\pi} \sqrt[p]{\alpha^{p-1}\beta} |W|^p dy \leq \frac{C(a,b)}{k^p} \int_0^{2\pi} \sqrt[p]{\alpha^{p-1}\beta} |W'|^p dy$$

with the constraint

$$(3.20) \quad \int_0^{2\pi} \sqrt[p]{\alpha^{p-1}\beta} |W|^{p-2} W dy = 0.$$

If $\sqrt[p]{\alpha^{p-1}\beta} = \sqrt[p]{a^{p-1}b} \in L^1$ and $1 \leq \sqrt[p]{\alpha^{p-1}\beta} \leq L$ an application of Lemma 3.1 enables us to write

$$\begin{aligned}\frac{C(a,b)}{k^p} &= \frac{C(a,b)}{\left(\frac{1}{2\pi} \int_0^{2\pi} \sqrt[p]{\frac{a(t)}{b(t)}} dt\right)^p} \\ &= C(\sqrt[p]{\alpha^{p-1}\beta}, \sqrt[p]{\alpha^{p-1}\beta}) \\ &\leq \left(\frac{2}{\pi} \frac{1}{(p-1)^{\frac{p-1}{p}}} \left[c_{p'} - \tilde{C}(\sup \sqrt[p]{a^{p-1}b})\right]\right)^{-p},\end{aligned}$$

where $\tilde{C}$ is defined as in (3.2).

If $I = \inf \sqrt[p]{a^{p-1}b} \neq 1$ we set

$$\bar{a} = \frac{a}{I} \quad \text{and} \quad \bar{b} = \frac{b}{I}$$

and from (1.1) we consider

$$\int_0^{2\pi} \frac{a}{\bar{I}} |w|^p \leq C(a,b) \int_0^{2\pi} \frac{b}{\bar{I}} |w'|^p.$$

Since $\bar{I} = \inf \sqrt[p]{\bar{a}^{p-1}\bar{b}} = 1$ from the previous case we obtain

$$\frac{C(a,b)}{\left(\frac{1}{2\pi} \int_0^{2\pi} \sqrt[p]{\frac{a(t)}{b(t)}} dt\right)^p} = C(\sqrt[p]{\bar{a}^{p-1}\bar{b}}, \sqrt[p]{\bar{a}^{p-1}\bar{b}})$$

$$\leq \left(\frac{2}{\pi} \frac{1}{(p-1)^{\frac{p-1}{p}}} \left[c_{p'} - \tilde{C}(\sup \sqrt[p]{a^{p-1}b}) \right] \right)^{-p}$$

where $\tilde{C}(\sup \sqrt[p]{a^{p-1}b}) = \tilde{C}\left(\frac{S}{I}\right)$ and (3.15) is proved. $\square$

Proof of Theorem 1.2. The proof of this Theorem is analogous to the one of Theorem 3.1 in [2].

We divide the proof into 3 steps.

Step 1. Let us first assume $w \in W_{per}^{1,p}(0, 2\pi) \cap W^{1,\infty}(0, 2\pi)$ and

$$(3.21) \quad a(t) \geq 1 \quad b(t) \leq \gamma$$

for some constant $\gamma > 0$.

Let a_h and w_h be defined by (2.6) and (2.7) respectively, and for $h > 1$

$$(3.22) \quad b_h = \begin{cases} b & \text{if } b \geq \frac{1}{h} \\ \frac{1}{h} & \text{if } b < \frac{1}{h} \end{cases}$$

then

$$(3.23) \quad \frac{1}{2\pi} \int_0^{2\pi} \sqrt[p]{\frac{a_h}{b_h}} \leq \frac{1}{2\pi} \int_0^{2\pi} \sqrt[p]{\frac{a}{b}}.$$

From Lemma 3.2 we deduce the Wirtinger inequality for a_h, b_h and w_h

$$(3.24) \quad \int_0^{2\pi} a_h |w_h|^p \leq \left(\frac{\frac{1}{2\pi} \int_0^{2\pi} \sqrt[p]{a_h(t)b_h(t)^{-1}}}{\frac{2}{\pi} \frac{1}{(p-1)^{\frac{p-1}{p}}} [c_{p'} - \Psi(S_h, I_h)]} \right)^p \int_0^{2\pi} b_h |w_h'|^p,$$

where $S_h = \sup \sqrt[p]{a_h^{p-1}b_h}$ and $I_h = \inf \sqrt[p]{a_h^{p-1}b_h}$. In view of (2.7) and (3.22) we have

$$\int_0^{2\pi} b_h |w_h'|^p = \int_0^{2\pi} b_h |w'|^p = \int_{b \geq \frac{1}{h}} b |w'|^p + \frac{1}{h} \int_{b < \frac{1}{h}} |w'|^p.$$

Therefore, as $h \rightarrow \infty$ we have

$$(3.25) \quad \int_0^{2\pi} b_h |w_h'|^p \rightarrow \int_0^{2\pi} b |w'|^p.$$

Furthermore, one can easily check that

$$(3.26) \quad \lim_{h \rightarrow \infty} S_h = S$$

$$(3.27) \quad \lim_{h \rightarrow \infty} I_h = I$$

Combining (2.13), (3.23), (3.24), (3.25), (3.26) and (3.27) we obtain (1.9).

Step 2. Let us assume now, $w \in W_{per}^{1,p}(0, 2\pi) \cap W^{1,\infty}(0, 2\pi)$, and

$$a \geq 0 \quad b < \infty.$$

For $\delta > 0$, let a_δ and w_δ be defined by (2.15) and (2.16) respectively, and

$$(3.28) \quad b_\delta = \begin{cases} b & \text{if } b \leq \frac{1}{\delta} \\ \frac{1}{\delta} & \text{if } b > \frac{1}{\delta}. \end{cases}$$

From the step 1, we deduce the Wirtinger inequality for a_δ, b_δ and w_δ

$$(3.29) \quad \int_0^{2\pi} a_\delta |w_\delta|^p \leq \left(\frac{\frac{1}{2\pi} \int_0^{2\pi} \sqrt[p]{a_\delta(t) b_\delta(t)^{-1}}}{\frac{2}{\pi} \frac{1}{(p-1)^{\frac{p-1}{p}}} [c_{p'} - \Psi(S_\delta, I_\delta)]} \right)^p \int_0^{2\pi} b_\delta |w'_\delta|^p,$$

where $S_\delta = \sup \sqrt[p]{a_\delta^{p-1} b_\delta}$ and $I_\delta = \inf \sqrt[p]{a_\delta^{p-1} b_\delta}$.

From the definitions (2.15), (2.16) and (3.28) of a_δ, w_δ and b_δ we have

$$(3.30) \quad \begin{aligned} \int_0^{2\pi} \sqrt[p]{\frac{a_\delta}{b_\delta}} &= \int_{\{\delta < a\} \cap \{b < \frac{1}{\delta}\}} \sqrt[p]{\frac{a}{b}} + \int_{\{\delta < a\} \cap \{b > \frac{1}{\delta}\}} \sqrt[p]{a\delta} \\ &= \int_{\{a < \delta\} \cap \{b < \frac{1}{\delta}\}} \sqrt[p]{\frac{\delta}{b}} + \int_{\{a < \delta\} \cap \{b > \frac{1}{\delta}\}} \sqrt[p]{\delta^2} \end{aligned}$$

and

$$(3.31) \quad \int_0^{2\pi} b_\delta |w'_\delta|^p \leq \int_0^{2\pi} b |w'|^p.$$

Notice that the last three integrals on the right side of (3.32) are all infinitesimal, therefore

$$(3.32) \quad \lim_{\delta \rightarrow 0^+} \int_0^{2\pi} \sqrt[p]{\frac{a_\delta}{b_\delta}} = \lim_{\delta \rightarrow 0^+} \int_{\{\delta < a\} \cap \{b < \frac{1}{\delta}\}} \sqrt[p]{\frac{a}{b}} = \int_0^{2\pi} \sqrt[p]{\frac{a}{b}}.$$

As in the previous case, we get

$$(3.33) \quad \lim_{\delta \rightarrow 0^+} S_\delta = S$$

$$(3.34) \quad \lim_{\delta \rightarrow 0^+} I_\delta = I.$$

Combining (2.20), (3.29), (3.31), (3.32), (3.33) and (3.34) we obtain (1.9).

Step 3. Finally we assume that $w \in W_{per}^{1,p}(0, 2\pi)$.

We define w_h as in (2.22). By using previous case, we deduce the Wirtinger inequality for a, b and w_h

$$(3.35) \quad \int_0^{2\pi} a|w_h|^p \leq \left(\frac{\frac{1}{2\pi} \int_0^{2\pi} \sqrt[p]{ab^{-1}}}{\frac{2}{\pi} \frac{1}{(p-1)^{\frac{p-1}{p}}} [c_{p'} - \Psi(S, I)]} \right)^p \int_0^{2\pi} b|w_h'|^p.$$

Then the result follows since

$$(3.36) \quad \int_0^{2\pi} b|w_h'|^p \leq \int_0^{2\pi} b|w'|^p.$$

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