The Dirichlet Problem for elliptic equations in unbounded domains of the plane

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Abstract. In this paper we prove a uniqueness and existence theorem for the Dirichlet problem in $W^{2,p}$ for second order linear elliptic equations in unbounded domains of the plane. Here the leading coefficients are locally of class *VMO* and satisfy a suitable condition at infinity.

1. Introduction

Consider the Dirichlet problem

$$(1.1) \quad \begin{cases} u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}^{1,p}(\Omega), \\ Lu = f, \quad f \in L^p(\Omega), \end{cases}$$

where Ω is an open subset of $\mathbb{R}^n$, $n \geq 2$, with a suitable regularity property, $p \in]1, +\infty[$, and L is a second-order, linear, uniformly elliptic operator, i.e.

$$(1.2) \quad L = - \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} + a.$$

If $n \geq 3$, $p < n$ and Ω is bounded, it is well known that the sole assumption

$$(1.3) \quad a_{ij} = a_{ji} \in L^\infty(\Omega)$$

(together with suitable summability conditions for the coefficients a_i and a) is not enough to ensure the uniqueness for the problem (1.1). It has been a challenging problem to determine suitable additional hypotheses, in terms of discontinuity, on the leading coefficients a_{ij} in order to get the well-posedness of (1.1). In particular, Miranda [13] proved that it holds whenever $p = 2$ and the coefficients a_{ij} belong to $W^{1,n}(\Omega)$; later generalizations required the derivatives of the leading coefficients to belong to certain wider spaces. Recently, in two profound papers [9] and [10], it has been pointed out that the solvability of (1.1) holds, for every $p \in]1, +\infty[$, under the assumption that the a_{ij} 's are of class VMO , $a_i = a = 0$ (note that VMO contains $W^{1,n}(\Omega)$) and, later, the condition $a_i = a = 0$ has been removed in [19] and [20].

If the set Ω is unbounded, problem (1.1) has for instance been studied in [16], where the leading coefficients satisfy restrictions similar to those of [13] and $p = 2$, in [3], where the investigation has been extended to the case $p \in]1, +\infty[$, and in [4] and [5], where the a_{ij} 's verify assumptions similar to those of [9] and [10].

In the planar case ($n = 2$) and whenever Ω is bounded, Talenti [15] proved that, if $p = 2$, condition (1.3) is enough to establish certain estimates for the solutions of (1.1) which lead to an existence and uniqueness result. Further studies have shown that this result still holds even if p lies in a certain neighborhood of 2 (see for instance [12]), and this interval has recently been determined in [2]. Observe that the lower critical exponent here is precisely the one conjectured by C. Pucci [14], who also proved that if p is smaller than this exponent, then the uniqueness of the solution of (1.1) cannot be proved. In our previous paper [7] we have studied problem (1.1) when p is an arbitrary real number > 1 and the a_{ij} 's belong to $VMO(\Omega)$.

The aim of the present one is to extend the investigation of [7] to the case of unbounded open sets. In fact, under the hypotheses that the coefficients a_{ij} are in $VMO_{\text{loc}}(\bar{\Omega})$ and satisfy a suitable condition at infinity, some $W^{2,p}$ -estimates for the solutions of (1.1) and some regularity results will be obtained, and an existence and uniqueness theorem will be deduced.

2. Preliminaries

In this section we collect the notation as well as the definitions of function spaces being used throughout the paper.

Let Ω be an open subset of $\mathbb{R}^n$, and let $\Sigma(\Omega)$ be the collection of all Lebesgue measurable subsets of Ω . For each $E \in \Sigma(\Omega)$, $|E|$ and χ_E denote the Lebesgue measure and the characteristic function of E , respectively. Moreover,

$$E(x, \rho) = E \cap B(x, \rho), \quad x \in \mathbb{R}^n, \rho \in \mathbb{R}_+,$$

for any open ball $B(x, \rho)$ in $\mathbb{R}^n$ of center x and radius ρ . For convenience, we write

$$B_\rho = B(0, \rho), \quad \rho \in \mathbb{R}_+.$$

If $X(\Omega)$ is a space of functions defined on Ω , we denote as $X_{\text{loc}}(\bar{\Omega})$ the space of all functions $g : \Omega \rightarrow \mathbb{R}$ such that $\zeta g \in X(\Omega)$ for every $\zeta \in \mathfrak{D}(\bar{\Omega})$, where $\mathfrak{D}(\bar{\Omega}) = \{\xi|_\Omega \mid \xi \in C_c^\infty(\mathbb{R}^n)\}$.

For $p \in [1, +\infty[$, $M^p(\Omega)$ is the set of all functions $g \in L_{\text{loc}}^p(\bar{\Omega})$ such that

$$(2.1) \quad \|g\|_{M^p(\Omega)} = \sup_{x \in \Omega} \|g\|_{L^p(\Omega(x, 1))} < +\infty,$$

endowed with the norm defined by (2.1) and $\tilde{M}^p(\Omega)$ (resp. $M_0^p(\Omega)$) is the closure of $L^\infty(\Omega)$ (resp. $C_c^\infty(\Omega)$) in $M^p(\Omega)$. It is known that a function g (in $M^p(\Omega)$) belongs to $\tilde{M}^p(\Omega)$ if and only if the function

$$(2.2) \quad \tau_g(t) = \sup_{\substack{E \in \Sigma(\Omega) \\ \sup_{x \in \Omega} |E(x, 1)| \leq t}} \|\chi_E g\|_{M^p(\Omega)}, \quad t \in \mathbb{R}_+,$$

vanishes when t goes to zero. Then a *modulus of continuity* of g in $\tilde{M}^p(\Omega)$ is a map $\tilde{\sigma}_p[g] : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\tilde{\sigma}_p[g](t) \geq \tau_g(t), \quad t \in \mathbb{R}_+, \quad \lim_{t \rightarrow 0^+} \tilde{\sigma}_p[g](t) = 0.$$

If Ω has the property

$$(2.3) \quad |\Omega(x, \rho)| \geq A \rho^n, \quad x \in \Omega, \quad \rho \in]0, 1],$$

where A is some positive constant independent of x and ρ , one can consider the space $BMO(\Omega, t)$, $t \in \mathbb{R}_+$, consisting of all functions g in $L_{\text{loc}}^1(\bar{\Omega})$ such that

$$(2.4) \quad [g]_{BMO(\Omega, t)} = \sup_{\substack{x \in \Omega \\ \rho \in]0, t]}} \oint_{\Omega(x, \rho)} |g - \oint_{\Omega(x, \rho)} g| < +\infty,$$

where

$$\oint_{\Omega(x, \rho)} g = |\Omega(x, \rho)|^{-1} \int_{\Omega(x, \rho)} g.$$

A function $g \in BMO(\Omega) = BMO(\Omega, t_A)$, where

$$t_A = \sup \left\{ t \in \mathbb{R}_+ : \sup_{\substack{x \in \Omega \\ \rho \in]0, t]}} \frac{\rho^n}{|\Omega(x, \rho)|} \leq \frac{1}{A} \right\},$$

is said to belong to the class $VMO(\Omega)$ if

$$\lim_{t \rightarrow 0^+} [g]_{BMO(\Omega, t)} = 0.$$

Then, a function $\eta[g] : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is called a *modulus of continuity* of g in $VMO(\Omega)$ if

$$\eta[g](t) \geq [g]_{BMO(\Omega, t)}, \quad t \in \mathbb{R}_+, \quad \lim_{t \rightarrow 0^+} \eta[g](t) = 0.$$

For a more detailed account of properties of the above function spaces we refer to [16], [17] and [18].

3. Some a priori estimates

Let Ω be an unbounded open set in $\mathbb{R}^2$ satisfying the $C^{1,1}$ -regularity property, namely

$(\mathcal{P}_\Omega)$ there exist a locally finite open covering $(U_i)_{i \in \mathbb{N}}$ of $\partial\Omega$ and $C^{1,1}$ -diffeomorphisms $\Phi_i : U_i \rightarrow B_1$, $i \in \mathbb{N}$, such that

- $\{x \in \Omega : \text{dist}(x, \partial\Omega) < \delta\} \subset \bigcup_{i \in \mathbb{N}} \Phi_i^{-1}(B_{1/2})$ for some $\delta > 0$;
- $\Phi_i(U_i \cap \Omega) = \{x \in B_1 : x_2 > 0\}$ for each $i \in \mathbb{N}$;
- there is $m_0 \in \mathbb{N}$ such that the intersection of any $m_0 + 1$ distinct U_i 's is empty;
- the components of Φ_i and Φ_i^{-1} have bounded $C^{1,1}$ -norms independently of i .

Notice that any open set Ω with the property $(\mathcal{P}_\Omega)$ also satisfies the condition (2.3) above.

From this point on, we assume that $p \in]1, +\infty[$, L is an elliptic operator in Ω of the form

$$(3.1) \quad L = - \sum_{i,j=1}^2 a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^2 a_i \frac{\partial}{\partial x_i} + a$$

and

$$(3.2) \quad L_0 = - \sum_{i,j=1}^2 a_{ij} \frac{\partial^2}{\partial x_i \partial x_j},$$

such that

$$(h_1) \quad \begin{cases} a_{ij} = a_{ji} \in L^\infty(\Omega), \quad i, j = 1, 2, \\ \exists \nu \in \mathbb{R}_+ : \sum_{i,j=1}^2 a_{ij} \xi_i \xi_j \geq \nu |\xi|^2 \quad \text{a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}^2, \end{cases}$$

$$(h_2) \quad \begin{cases} a_i \in \tilde{M}^r(\Omega), \quad i = 1, 2, \quad \text{where} \\ r = 2 \text{ if } p < 2, \quad r > 2 \text{ if } p = 2, \quad r = p \text{ if } p > 2, \\ a \in \tilde{M}^p(\Omega), \end{cases}$$

$$(h_3) \quad a_{ij} \in VMO(\Omega), \quad i, j = 1, 2.$$

We stress that, under the assumptions (h_1) and (h_2) , the operator

$$L : W^{2,p}(\Omega) \rightarrow L^p(\Omega)$$

is bounded (see e.g. Theorem 3.1 in [11]).

The next lemma provides an a priori bound.

Lemma 3.1. *Assume (h_1) , (h_2) and (h_3) . If u is a solution of the Dirichlet problem*

$$(3.3) \quad u \in W_{\text{loc}}^{2,p}(\bar{\Omega}) \cap \mathring{W}^{1,p}(\Omega), \quad Lu \in L^p(\Omega),$$

then $u \in W^{2,p}(\Omega)$. Moreover

$$(3.4) \quad \|u\|_{W^{2,p}(\Omega)} \leq c \left(\|Lu\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} \right),$$

where the constant c depends on $\Omega, p, r, \nu, \|a_{ij}\|_{L^\infty(\Omega)}, \eta[p(a_{ij})], \|a_i\|_{M^r(\Omega)}, \|a\|_{M^p(\Omega)}, \tilde{\sigma}_r[a_i], \tilde{\sigma}_p[a]$, and $p(a_{ij})$ is an extension of a_{ij} to $\mathbb{R}^2$ of class $L^\infty(\mathbb{R}^2) \cap VMO(\mathbb{R}^2)$.

Proof. Let us fix $u \in W_{\text{loc}}^{2,p}(\bar{\Omega}) \cap \mathring{W}^{1,p}(\Omega)$ such that $Lu \in L^p(\Omega)$. The regularity hypothesis on Ω implies that there exists $d \in]0, 1[$ such that, for any $x \in \Omega$, $B(x, d) \subset \Omega$ or $B(x, d) \subset U_i$ for some $i \in I$. So, if ζ is a function of class $C_\infty^\infty(\mathbb{R}^2)$ such that

$$0 \leq \zeta \leq 1, \quad \zeta = 1 \text{ in } B\left(0, \frac{d}{2}\right), \quad \text{supp } \zeta \subset B(0, d),$$

for any fixed $x \in \Omega$, the function

$$\psi = \psi_x : y \in \Omega \rightarrow \zeta(y - x)$$

is obviously in $C_0^\infty(\mathbb{R}^2)$,

$$0 \leq \psi \leq 1, \quad \psi = 1 \text{ in } B\left(x, \frac{d}{2}\right), \quad \text{supp } \psi \subset B(x, d),$$

and

$$\psi u \in W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega), \quad \text{supp } (\psi u) \subset B(x, d) \cap \bar{\Omega}.$$

Then from Lemma 3.1 in [7] it follows that

$$(3.5) \quad \|\psi u\|_{W^{2,p}(\Omega(x,d))} \leq c_1 \left(\|L(\psi u)\|_{L^p(\Omega(x,d))} + \|\psi u\|_{L^p(\Omega(x,d))} \right),$$

where the constant c_1 depends on $\Omega, p, \nu, \|a_{ij}\|_{L^\infty(\Omega)}, \eta[p(a_{ij})], \|a_i\|_{M^r(\Omega)}, \|a\|_{M^p(\Omega)}, \tilde{\sigma}_r[a_i]$ and $\tilde{\sigma}_p[a]$ ($p(a_{ij})$ denotes an extension of a_{ij} to $\mathbb{R}^2$ of class $L^\infty(\mathbb{R}^2) \cap VMO(\mathbb{R}^2)$; see Theorem 5.1 in [18] for the existence of such extension).

Since

$$(3.6) \quad L(\psi u) = \psi Lu - 2 \sum_{i,j=1}^2 a_{ij} u_{x_i} \psi_{x_j} + u \left(- \sum_{i,j=1}^2 a_{ij} \psi_{x_i x_j} + \sum_{i=1}^2 a_i \psi_{x_i} \right),$$

from (3.5) and (3.6) we deduce the bound

$$(3.7) \quad \|u\|_{W^{2,p}(\Omega(x, \frac{d}{2}))} \leq c_2 \left(\|Lu\|_{L^p(\Omega(x,d))} + \|u_x\|_{L^p(\Omega(x,d))} \right. \\ \left. + \|u\|_{L^p(\Omega(x,d))} + \sum_{i=1}^2 \|a_i \psi_{x_i} u\|_{L^p(\Omega(x,d))} \right),$$

with c_2 depending on the same parameters as c_1 . On the other hand, an application of Theorem 3.1 in [11] yields

$$(3.8) \quad \|a_i \psi_{x_i} u\|_{L^p(\Omega(x,d))} = \|a_i \psi_{x_i} u\|_{L^p(\Omega)} \leq c_3 \|a_i\|_{M^r(\Omega)} \|\psi_{x_i} u\|_{W^{1,p}(\Omega)},$$

where c_3 depends only on p and r . Therefore, from (3.7) and (3.8), one has that

$$(3.9) \quad \|u\|_{W^{2,p}(\Omega(x, \frac{d}{2}))} \leq c_4(\|Lu\|_{L^p(\Omega(x, d))} + \|u_x\|_{L^p(\Omega(x, d))} + \|u\|_{L^p(\Omega(x, d))}),$$

with c_4 depending on the same parameters as c_2 and on r . By (3.9) and by Lemma 4.1 in [16] we obtain that $u \in W^{2,p}(\Omega)$ and

$$(3.10) \quad \|u\|_{W^{2,p}(\Omega)} \leq c_5(\|Lu\|_{L^p(\Omega)} + \|u_x\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}),$$

where c_5 depends on the same parameters on which c_4 does. Finally, there exists $K = K(p, \Omega) \in \mathbb{R}_+$ such that

$$(3.11) \quad \|u_x\|_{L^p(\Omega)} \leq K\varepsilon \|u\|_{W^{2,p}(\Omega)} + \frac{K}{\varepsilon} \|u\|_{L^p(\Omega)},$$

for any $\varepsilon \in]0, 1]$ (see e.g. [1], Theorem 4.14). This combined with (3.10) gives (3.4). $\square$

Consider now the following hypotheses on the coefficients of L :

$$(h'_3) \quad a_{ij} \in VMO_{\text{loc}}(\bar{\Omega}), \quad i, j = 1, 2,$$

there exist functions α_{ij} , $i, j = 1, 2$, g and $\mu \in \mathbb{R}_+$ such that

$$(h_4) \quad \begin{cases} \alpha_{ij} = \alpha_{ji} \in L^\infty(\Omega) \cap VMO(\Omega), \quad i, j = 1, 2, \\ \sum_{i,j=1}^2 \alpha_{ij} \xi_i \xi_j \geq \mu |\xi|^2 \quad \text{a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}^2, \\ g \in L^\infty(\Omega), \\ \lim_{\rho \rightarrow +\infty} \sum_{i,j=1}^2 \|\alpha_{ij} - g a_{ij}\|_{L^\infty(\Omega \setminus B_\rho)} = 0. \end{cases}$$

We point out that (h'_3) - (h_4) are weaker than (h_3) , as an example in [4] shows. The next result provides another a priori estimate, but under these weaker assumptions. In the claim, for any $\rho \in \mathbb{R}_+$, ζ_ρ stands for a function of class $C^\infty_0(\mathbb{R}^2)$ such that

$$0 \leq \zeta_\rho \leq 1, \quad \zeta_\rho = 1 \quad \text{in } B_\rho, \quad \text{supp } \zeta_\rho \subset B_{2\rho}.$$

Theorem 3.2. *Assume (h_1) , (h_2) , (h'_3) and (h_4) . Then there exist positive real numbers ρ_0, c such that*

$$(3.12) \quad \|u\|_{W^{2,p}(\Omega)} \leq c(\|Lu\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}), \quad \forall u \in W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega),$$

where c depends only on $\Omega, p, r, \nu, \mu, \|a_{ij}\|_{L^\infty(\Omega)}, \|\alpha_{ij}\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \eta[p(\zeta_{2\rho_0} a_{ij})], \eta[p(\alpha_{ij})], \tilde{\sigma}_r[a_i], \tilde{\sigma}_p[a]$.

Proof. The proof is similar to the one given in [4] (Theorem 3.1), taking into account to apply our Lemma 3.1 in place of Theorem 5.1 in [6]. $\square$

4. Some regularity results

The aim of this section is to prove two regularity results, the first one of local type and the other of global type. Note that for the local regularity case, we assume only the conditions (h_1) and (h'_3) on the leading coefficients, while for the coefficients of lower order the condition (h_2) is replaced by the weaker assumption

$$(h'_2) \quad \begin{cases} a_i \in L^r_{\text{loc}}(\bar{\Omega}), \quad i = 1, 2, \\ \text{where } r > 2 \quad \text{if } p \leq 2, \quad r = p \quad \text{if } p > 2, \\ a \in L^p_{\text{loc}}(\bar{\Omega}). \end{cases}$$

The following result is proved.

Lemma 4.1. *Assume $(h_1), (h'_2)$ and (h'_3) . If u is a solution of the problem*

$$(\mathcal{R}_1) \quad \begin{cases} u \in W^{2,q}_{\text{loc}}(\bar{\Omega}) \cap \mathring{W}^{1,q}_{\text{loc}}(\bar{\Omega}), \\ Lu \in L^p_{\text{loc}}(\bar{\Omega}), \end{cases}$$

where $q \in]1, p]$, then u belongs to $W^{2,p}_{\text{loc}}(\bar{\Omega})$.

Proof. We may assume that $q < p$. Let k be the positive integer such that

$$(4.1) \quad k - 1 < 2\left(\frac{1}{q} - \frac{1}{p}\right) \leq k,$$

so that

$$(4.2) \quad \frac{1}{q} - \frac{h}{2} > \frac{1}{p}, \quad h = 0, \dots, k-1, \quad \frac{1}{q} - \frac{k}{2} \leq \frac{1}{p}.$$

Put

$$(4.3) \quad \frac{1}{q_h} = \frac{1}{q} - \frac{h}{2}, \quad h = 0, \dots, k-1, \quad \frac{1}{q_k} = \frac{1}{p}.$$

In order to prove the claim it is enough to show that

$$(4.4) \quad L(\zeta u) \in L^{q_1}(\Omega), \quad \zeta \in \mathcal{D}(\bar{\Omega}),$$

since by (4.4) and by Lemma 3.3 in [7], it will follow that $u \in W_{\text{loc}}^{2,q_1}(\bar{\Omega})$, and so the statement can be deduced iterating the above argument k times.

Fixed $\zeta \in \mathcal{D}(\bar{\Omega})$, since

$$\begin{aligned}
 (4.5) \quad L(\zeta u) &= L_0(\zeta u) + \sum_{i=1}^2 a_i(\zeta u)_{x_i} + a\zeta u \\
 &= \zeta Lu + uL_0\zeta - 2 \sum_{i,j=1}^2 a_{ij}u_{x_i}\zeta_{x_j} + u \sum_{i=1}^2 a_i\zeta_{x_i},
 \end{aligned}$$

it is enough to check that

$$(4.6) \quad u\zeta_{x_ix_j}, \quad u_{x_i}\zeta_{x_j}, \quad a_i u\zeta_{x_i} \in L^{q_1}(\Omega), \quad i, j = 1, 2.$$

By Sobolev embedding, one obtains that $u\zeta_{x_ix_j}, u_{x_i}\zeta_{x_j}$ belong to L^{q_1} ; on the other hand, from Theorem 3.1 in [11] we can conclude that $a_i u\zeta_{x_i}$ also lies in L^{q_1} , and so (4.6) holds. $\square$

We prove now the following regularity result of global type.

Lemma 4.2. *Assume $(h_1), (h_2)$ (with $r > 2$), (h'_3) and (h_4) . If u is a solution of the problem*

$$(R_2) \quad \begin{cases} u \in W_{\text{loc}}^{2,q}(\bar{\Omega}) \cap \overset{\circ}{W}_{\text{loc}}^{1,q}(\bar{\Omega}) \cap L^{q_0}(\Omega), \\ Lu \in L^p(\Omega), \end{cases}$$

where $q \in]1, p]$ and $q_0 \in [1, p]$, then u belongs to $W^{2,p}(\Omega)$.

Proof. The proof follows the lines of that one of Lemma 4.2 in [4], but taking care to apply our Lemma 4.1 and Theorem 3.2 in place of Lemma 4.1 and Theorem 3.1 in [4], respectively. $\square$

5. Uniqueness and existence results

Now we turn to the Dirichlet problem (1.1) and we first prove the following uniqueness result.

Theorem 5.1. *Assume $(h_1), (h'_2), (h'_3)$ and that*

$$(5.1) \quad \exists a_0 \in \mathbb{R}_+ \text{ such that } a \geq a_0 \text{ a.e. in } \Omega.$$

Moreover, if $p \leq 2$, suppose also (h_2) (with $r > 2$) and (h_4) . Then the problem

$$(\mathcal{D}_1) \quad \begin{cases} u \in W_{\text{loc}}^{2,p}(\bar{\Omega}) \cap \mathring{W}^{1,p}(\Omega), \\ Lu = 0 \end{cases}$$

admits only the zero solution in Ω .

Proof. Note that Sobolev embedding yields $u \in C^0(\bar{\Omega}) \cap \mathring{W}^{1,p}(\Omega)$, hence $u|_{\partial\Omega} = 0$. If $p > 2$, then $\lim_{|x| \rightarrow +\infty} u(x) = 0$, and so the result follows applying Corollary 4.2 in [8]. If $p \in]1, 2]$, Lemma 4.2 above guarantees that $u \in W^{2,p}(\Omega)$; so by Sobolev embedding u lies in $W^{1,\bar{p}}(\Omega)$, with $\frac{1}{\bar{p}} \in \left[\frac{1}{p} - \frac{1}{2}, \frac{1}{2}\right]$. Therefore $\lim_{|x| \rightarrow +\infty} u(x) = 0$, and so, also in this case, the statement follows by Corollary 4.2 in [8]. $\square$

Finally, adding the following assumption on the coefficients of L

$$(h_5) \quad \begin{cases} (\alpha_{ij})_{x_h}, a_i \in M_0^r(\Omega), \quad i, j, h = 1, 2, \\ \text{where } r > 2 \text{ if } p \leq 2, \quad r = p \text{ if } p > 2, \\ a = a' + b, \text{ where } a' \in M_0^p(\Omega), b \in L^\infty(\Omega), b_0 = \text{ess inf}_\Omega b > 0, \\ g \in \text{Lip}(\bar{\Omega}), \quad g_0 = \text{ess inf}_\Omega g > 0, \end{cases}$$

we are now in position to state the following uniqueness and existence result.

Theorem 5.2. *Under (h_1) , (h'_3) , (h_4) , (h_5) and (5.1), the Dirichlet problem*

$$(\mathcal{D}_2) \quad \begin{cases} u \in W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega), \\ Lu = f, \quad f \in L^p(\Omega) \end{cases}$$

is uniquely solvable.

Proof. One can proceed as in the proof of Theorem 4.3 in [5], but replacing the applications of Theorem 5.2 in [4] and Theorem 5.1 in [6] with the ones of previous Theorem 5.1 and Lemma 3.1, respectively. $\square$

Remark 5.3. An example of functions a_{ij} verifying the assumptions (h_1) , (h'_3) , (h_4) and (h_5) of Theorem 5.2, but such that $(a_{ii})_{x_h}$ do not belong to $M_0^p(\Omega)$ for any $p \in [1, +\infty[$ and $i, h = 1, 2$, can be obtained considering $\Omega =]-\infty, \infty[\times]-1, 1[$, $\alpha_{ij} = 2\delta_{ij}$ and

$$a_{ij} = \alpha_{ij} + \frac{\sin(1 + e^{|x|^2})}{1 + |x|} \delta_{ij}, \quad i, j = 1, 2.$$

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