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On the boundedness of operators in $L^p(l^q)$ and Triebel-Lizorkin Spaces

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Abstract. Given a bounded linear operator $T : L^{p_0}(\mathbb{R}^n) \to L^{p_1}(\mathbb{R}^n)$, for $1 \leq p_0, p_1 \leq \infty$, we state conditions under which T defines a bounded operator between corresponding pairs of $L^p(\mathbb{R}^n; l^q)$ spaces and Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$. Applications are given to linear parabolic equations and to Schrödinger semigroups.

1. Introduction

Let $1 \leq p \leq \infty$. We define the linear spaces $L^p(\mathbb{R}^n; l^q) = L^p(l^q)$ as the set of all sequences $\{u_k\}_{k\in\mathbb{N}_0}$ of complex-valued measurable functions in $\mathbb{R}^n$ for which

$$\|\{u_k\}\|_{L^p(l^q)} = \begin{cases} \|(\sum_{k=0}^{\infty} |u_k|^q)^{1/q}\|_{L^p} & 1 \leq q < \infty \\ \|\sup_{k\geq 0}(|u_k|)\|_{L^p} & q = \infty \end{cases}$$

is finite. Each of the spaces $L^p(\mathbb{R}^n; l^q)$ is a Banach space with the norm $\|.\|_{L^p(l^q)}$ and L^p is (isometrically) embedded in $L^p(l^q)$ via the map

$$u \mapsto \{\delta_{0k} u\}_{k\in\mathbb{N}_0},$$

where δ_{0k} is the Kronecker symbol: $\delta_{0k} = 1$ if $k = 0$ and $\delta_{0k} = 0$ if $k \neq 0$.

We will define the action of a linear operator T on $L^p(l^q)$ as

$$T\left(\{u_k\}_{k\in\mathbb{N}_0}\right) = \{Tu_k\}_{k\in\mathbb{N}_0};$$

as an operator from $L^{p_0}(l^q)$ to $L^{p_1}(l^q)$ it is defined for those $\{u_k\}_{k\in\mathbb{N}_0} \in L^{p_0}(l^q)$ such that $\{Tu_k\}_{k\in\mathbb{N}_0} \in L^{p_1}(l^q)$. Clearly, if $T : L^{p_0}(l^q) \to L^{p_1}(l^q)$ is bounded then $T : L^{p_0} \to L^{p_1}$ is well defined and bounded. In this paper we address the following problem (Problem 1)**:** given a bounded linear operator $T : L^{p_0}(\mathbb{R}^n) \to L^{p_1}(\mathbb{R}^n)$ for $1 \le p_0, p_1 < \infty$, under which conditions does T defines a bounded operator between a corresponding pair of spaces

$$T : L^{p_0}(l^q) \to L^{p_1}(l^q), \qquad 1 \le q \le \infty.$$

We give an answer for some special class of operators that we call majorized operators.

A class of spaces which is defined using the $L^p(l^q)$ spaces are the Triebel-Lizorkin (TL) space order s and base (p,q):

$$F^s_{p,q}(\mathbb{R}^n) = \left\{u \in \mathcal{S}' \mid \left\{2^{sk}\psi_k(D)\,u\right\}_{k\in\mathbb{N}_0} \in L^p(l^q)\right\}.$$

Here the set $\{\psi_k\}_{k\in\mathbb{N}_0}$ is a suitable dyadic partition of unity of $\mathbb{R}^n$, $D = i\nabla$ and the operator $\varphi(D)$ is defined like the in the functional calculus of operators in L^2 as $\varphi(D) \equiv \mathcal{F}^{-1}\varphi\mathcal{F}$, $\mathcal{F}$ being the Fourier transform. $\mathcal{S}'$ is the space of tempered distributions.

The topology in $F^s_{p,q}(\mathbb{R}^n)$ is given by the norm

$$\|u\|_{F^s_{p,q}} \equiv \left\| \left(\sum_{k=0}^{\infty} 2^{skq} \left|\psi_k(D)\,u\right|^q \right)^{1/q} \right\|_{L^p}$$

for $1 \le q < \infty$ and by the norm

$$\|u\|_{F^s_{p,\infty}} \equiv \left\| \sup_{k\ge 0} \left(2^{sk}\left|\psi_k(D)\,u\right|\right) \right\|_{L^p}$$

for $q = \infty$. With these norms, $F^s_{p,q}$ are Banach spaces. We refer to ([7]) for details on the TL spaces, let us just note that

$$\mathcal{S} \subset F^s_{p,q}(\mathbb{R}^n) \subset \mathcal{S}', \quad 1 \le p < \infty,\ 1 \le q \le \infty,\ s \in \mathbb{R}$$

$\mathcal{S}$ being dense in $F^s_{p,q}$ for $q \ne \infty$, $F^s_{p,q} \subset L^p$ for $1 \le p < \infty$, $1 \le q \le \infty$ and $s > 0$ and $F^s_{p,2} = H^{s,p}$ for $1 < p < \infty$, $F^s_{p,p} = B^s_{p,p}$ (Besov spaces) for $1 \le p < \infty$ (all real s). Also, remark that we have omitted the value

$p=\infty$ from the definitions above; we shall not deal with the spaces $F^s_{\infty,q}$ whose definition require some modification of the above.

As a consequence of our study of Problem 1 we are able give one solution to the following problem (Problem 2): given a bounded linear operator $T: L^{p_0}(\mathbb{R}^n) \to L^{p_1}(\mathbb{R}^n)$ for $1 \le p_0, p_1 < \infty$, under which conditions does T defines a bounded operator between a corresponding pair of Triebel-Lizorkin spaces

$$T: F^s_{p_0,q}(\mathbb{R}^n) \to F^s_{p_1,q}(\mathbb{R}^n),$$

$1 \le q \le \infty$, $s \in \mathbb{R}$. We shall see that, if an operator is translation invariant, then boundedness in Triebel-Lizorkin spaces follows from boundedness in the corresponding $L^p(l^q)$ spaces. Hence a majorized translation invariant operator is bounded on Triebel-Lizorkin spaces if its majorized operator is bounded on the corresponding L^p spaces, and this independently of q.

Here is the plan of the paper. We first prove a boundedness result on $L^p(l^q)$ spaces for majorized operators (Problem 1). In the following section we see that if a majorized operator is also translation invariant then boundedness in Triebel-Lizorkin spaces follows from boundedness in the corresponding $L^p(l^q)$ spaces so we obtain a solution to Problem 2. We close with applications to some classical differential operators and to parabolic evolution equations.

2. Operators majorized by positive operators

Let P be a linear operator defined on L^{p_0}, with values on L^{p_1}, for some $1 \le p_0, p_1 \le \infty$. We shall say that P is a positive operator if $Pu \ge 0$ whenever $u \ge 0$. If P is positive and $T: L^{p_0} \to L^{p_1}$ is a linear operator we shall say that T is majorized by P, and write $T = \mathcal{O}(P)$, if for some positive constant M we have

$$|Tu| \le MP|u| \ \ a.e.$$

It is well known that a linear positive operator from L^{p_0} taking values in L^{p_1} is bounded ([4]), as $\|Tu\|_{L^{p_1}} \le M\|P|u|\|_{L^{p_1}}$ we see that $T = \mathcal{O}(P)$ implies that T is bounded with

$$\|T\|_{\mathcal{L}(L^{p_0};L^{p_1})} \le M\|P\|_{\mathcal{L}(L^{p_0};L^{p_1})}.$$

The simplest examples of majorized operators, besides positive operators themselves, are integral operators

$$Tu(x) = \int K(x,y)\,u(y)\,dy,$$

for which $|K(x,y)|$ is the kernel of a bounded operator $L^{p_0} \to L^{p_1}$.

Theorem 1. *Suppose that T is a linear operator with $T = \mathcal{O}(P)$ for some linear positive operator P. Suppose that for some $1 \le p_0, p_1 < \infty$, the operator P is a map from L^{p_0} into L^{p_1} with*

$$\|Pu\|_{L^{p_1}} \le C \|u\|_{L^{p_0}}.$$

Then, for all $1 \le q \le \infty$, $T : L^{p_0}(l^q) \to L^{p_1}(l^q)$ is bounded with

$$\|Tu\|_{L^{p_1}(l^q)} \le CM \|u\|_{L^{p_0}(l^q)}. \tag{1}$$

Proof. It is enough to prove that the hypothesis imply that

$$T : L^{p_0}(l^q) \to L^{p_1}(l^q)$$

is bounded when $q = 1$ and when $q = \infty$ in both cases with an operator bound $\le$ to CM. Complex interpolation then shows that the same is true for all $1 \le q \le \infty$, see ([8]).

Take first $q = 1$ and $\{u_k\} \in L^{p_0}\left(l^1\right)$. Since the series of positive terms $\sum_{k=0}^{\infty} |u_k|$ converges almost everywhere and defines a L^{p_0} function, we have $u_k \in L^{p_0}$ for all k. Then $|Tu_k| \le MP|u_k|$ for all k, hence

$$\|T\{u_k\}\|_{L^{p_1}(l^1)} = \left\| \left(\sum_{k=0}^{\infty} |Tu_k| \right) \right\|_{L^{p_1}} \le M \left\| \left(\sum_{k=0}^{\infty} P|u_k| \right) \right\|_{L^{p_1}}.$$

The series $\sum_{k=0}^{\infty} |u_k|$ also converges in L^{p_0} so, by continuity of P,

$$\sum_{k=0}^{\infty} P|u_k| = P\left(\sum_{k=0}^{\infty} |u_k| \right)$$

and

$$\begin{aligned} \|T\{u_k\}\|_{L^{p_1}(l^1)} &\le M \left\| P\left(\sum_{k=0}^{\infty} |u_k| \right) \right\|_{L^{p_1}} \le CM \left\| \sum_{k=0}^{\infty} |u_k| \right\|_{L^{p_0}} \\ &\le CM \|\{u_k\}\|_{L^{p_0}(l^1)} \end{aligned}$$

This shows that $T : L^{p_0}\left(l^1\right) \to L^{p_1}\left(l^1\right)$ is bounded. Now take $q = \infty$ and $\{u_k\} \in L^{p_0}(l^\infty)$. As $|u_k| \le \sup_{k \ge 0} |u_k| \in L^{p_0}$ for every k, we have $u_k \in L^{p_0}$ and

$$\sup_{k \ge 0} |Tu_k| \le M \sup_{k \ge 0} P(|u_k|) \le MP\left(\sup_{k \ge 0} |u_k| \right),$$

the last inequality follows from the positivity of P. Taking the L^{p_1} norm and using the continuity assumption we get

$$\begin{aligned}\|T\{u_k\}\|_{L^{p_1}(l^\infty)} &= \left\|\sup_{k\geq 0}|Tu_k|\right\|_{L^{p_1}} \leq M\left\|P\left(\sup_{k\geq 0}|u_k|\right)\right\|_{L^{p_1}} \\ &\leq CM\|\{u_k\}\|_{L^{p_1}(l^\infty)}\end{aligned}$$

and the proof is done. □

3. Translation invariant operators

A translation invariant operator is an operator that commutes with translations by all vectors $y \in \mathbb{R}^n$. It is known that if T is a translation invariant operator, continuity of T from L^{p_0} to L^{p_1} then implies that $p_0 \leq p_1$ (or else $T \equiv 0$) and T is given on $\mathcal{S}$ as $\mathcal{F}^{-1}m\mathcal{F}$ for a uniquely determined $m \in \mathcal{S}'$. We say then that m is a Fourier multiplier of type (p_0, p_1). In a previous work [1] we have considered this boundedness problem in different scales of spaces, we have proven in this paper that every Fourier multiplier of type (p_0, p_1) is also a Fourier multiplier of the same type in the Sobolev spaces $W^{k,p}$, in the Bessel potential spaces $H^{s,p}$ and in the Besov spaces $B^s_{p,q}$, (and their homogenous versions) see ([1]) for details.

For translation invariant operator T, boundedness of T in $L^p(l^q)$ implies boundedness of T in $F^s_{p,q}$:

Theorem 2. *Suppose that, for some $1 \leq p_0, p_1 < \infty$, T is a continuous translation invariant linear operator from L^{p_0} to L^{p_1} and that $T : L^{p_0}(l^q) \to L^{p_1}(l^q)$ is bounded for some $1 \leq q < \infty$:*

$$\|T\{u_k\}\|_{L^{p_1}(l^q)} \leq M\|\{u_k\}\|_{L^{p_0}(l^q)}.$$

Then, $T : F^s_{p_0,q} \to F^s_{p_1,q}$ is bounded for all $s \in \mathbb{R}$ and we have

$$\|Tu\|_{F^s_{p_1,q}} \leq M\|u\|_{F^s_{p_0,q}}. \tag{2}$$

Proof. By translation invariance of T

$$\psi_k(D)Tu = T\psi_k(D)u$$

for every $k \in \mathbb{N}_0$ and $u \in \mathcal{S}$. Taking the $L^{p_1}(l^q)$ norm of the sequence $\left\{2^{sk}\psi_k(D)Tu\right\}$ we get

$$\begin{aligned}\left\|\left\{2^{sk}\psi_k(D)Tu\right\}\right\|_{L^{p_1}(l^q)} &= \left\|T\left\{2^{sk}\psi_k(D)u\right\}\right\|_{L^{p_1}(l^q)} \\ &\leq M\left\|\left\{2^{sk}\psi_k(D)u\right\}\right\|_{L^{p_0}(l^q)}\end{aligned}$$

hence the result follows from the density of $\mathcal{S}$ in $F^s_{p_0,q}$ (that's why q must be different from ∞). □

So, for a solution to Problem 2, we have:

Corollary 3. *Suppose that T is a translation invariant linear operator with $T = \mathcal{O}(P)$ for some linear positive operator P. Suppose that for some $1 \leq p_0, p_1 < \infty$, the operator P is continuous map from L^{p_0} into L^{p_1} with*

$$\|Pu\|_{L^{p_1}} \leq C\|u\|_{L^{p_0}}.$$

Then, if $1 \leq q < \infty$, $T : F^s_{p_0,q} \to F^s_{p_1,q}$ is bounded for all $s \in \mathbb{R}$ and we have

$$\|Tu\|_{F^s_{p_1,q}} \leq M\|u\|_{F^s_{p_0,q}}.$$

The proof is clear.

The Corollary 3 applies to special types of Fourier multipliers:

Corollary 4. *Suppose that $m \in \mathcal{S}'$ with $\mathcal{F}^{-1}m \in L^r(\mathbb{R}^n)$ for some $1 \leq r \leq \infty$. Then m is a Fourier multiplier from $F^s_{p_0,q}$ to $F^s_{p_1,q}$ for $0 \leq \frac{1}{p_0} - \frac{1}{p_1} = 1 - \frac{1}{r}$, $1 < p_0, p_1 < \infty$, $1 < q < \infty$ and all $s \in \mathbb{R}$.*

Proof. Since, for $u \in \mathcal{S}$,

$$\mathcal{F}^{-1}m\mathcal{F}u = \left(\mathcal{F}^{-1}m\right) * u,$$

if $\mathcal{F}^{-1}m \in L^r$ for some $1 \leq r \leq +\infty$, by Young inequality, $\mathcal{F}^{-1}m\mathcal{F}$ extends to a bounded operator $L^{p_0} \to L^{p_1}$ for $1 \leq p_0 < \infty$, $1 \leq p_1 \leq \infty$ and $\frac{1}{r} = 1 + \frac{1}{p_1} - \frac{1}{p_0}$ and the result follows from the theorem and the previous Corollary. □

Remark 5. *In the results above, we can replace the target space $F^s_{p_1,q}$ by the space $F^{s_2}_{p_2,q_2}$, with $q_2 \geq q$, $0 < s_2 \leq s$ and $s - \frac{n}{p_1} = s_2 - \frac{n}{p_2}$, due to the embedding $F^s_{p_1,q} \subset F^{s_2}_{p_2,q_2}$.*

Remark 6. *By suitably modification of the definition of the $L^p(l^q)$ spaces, we could prove a version of Corollary 3 for the homogenous versions of the spaces $F^s_{p,q}$.*

On the remaining sections of this paper we give some miscellaneous applications of Theorem 1 and Corollary 3.

4. Powers of the resolvent of the Laplacian

Consider the operators $T_{\lambda,z} = (z - \triangle)^{-\lambda}$, for $\operatorname{Re} z > 0$ and $\lambda > 0$. We have $(z - \triangle)^{-\lambda} u = G_{\lambda,z} * u$ where

$$G_{\lambda,z}(x) = \frac{(4\pi t)^{-\frac{n}{2}}}{\Gamma(\lambda)} \int_0^\infty e^{-|x|^2/4t} e^{-zt} t^{\lambda-1} dt.$$

$T_{\lambda,z}$ is a translation invariant operator and $|G_{\lambda,z}(x)| \leq G_{\lambda,\operatorname{Re} z}(x)$ which is the kernel of the positive operator $T_{\lambda,\operatorname{Re} z}$.

It is known ([5]) that for $p_0 \leq p_1$ such that $\frac{1}{p_0} - \frac{1}{p_1} \leq \frac{2\lambda}{n}$ we have $(z - \triangle)^{-\lambda} : L^{p_0}(\mathbb{R}^n) \to L^{p_1}(\mathbb{R}^n)$ with

$$\left\| (z - \triangle)^{-\lambda} u \right\|_{L^{p_1}} \leq C (\operatorname{Re} z)^{-\lambda} \|u\|_{L^{p_0}}$$

so by the Corollary 3 we have $(z - \triangle)^{-\lambda} : F^s_{p_0,q} \to F^s_{p_1,q}$ with

$$\left\| (z - \triangle)^{-\lambda} u \right\|_{F^s_{p_1,q}} \leq C (\operatorname{Re} z)^{-\lambda} \|u\|_{F^s_{p_0,q}}$$

for complex z with $\operatorname{Re} z > 0$, $\lambda > 0$, $0 \leq \frac{1}{p_0} - \frac{1}{p_1} \leq \frac{2\lambda}{n}$.

An interesting application occurs in the limiting case $z = 0$ where we may still consider the operator $T_\beta = (-\triangle)^{-\beta/2}$ for $0 < \beta < n$. This is a positive operator, bounded from L^{p_0} to L^{p_1} for $\frac{1}{p_1} = \frac{1}{p_0} - \frac{\beta}{n}$, a fact known as the fractional integration theorem. T_β is given on $\mathcal{S}$ as convolution with the L^1_{loc} function $|y|^{-\beta+n}$ and an application of Corollary 3 leads to the fact that $(-\triangle)^{-\beta/2} : F^s_{p_0,q} \to F^s_{p_1,q}$ for all $s > 0$ or, since $(-\triangle)^{\beta/2} F^{s+\beta}_{p_0,q} = F^s_{p_0,q}$,

$$F^{s+\beta}_{p_0,q} \subset F^s_{p_1,q}$$

for $s + \beta - \frac{n}{p_0} = s - \frac{n}{p_1}$. This is the known expression for the embedding theorem for Triebel-Lizorkin spaces and this shows how this embedding theorem follows easily from the fractional integration theorem.

5. Parabolic equations

We consider now the parabolic equation

$$\frac{\partial u}{\partial t} = Lu \tag{3}$$

where

$$L = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial}{\partial x_i} + c(x,t)$$

is a elliptic operator. Under some mild conditions on the coefficients a_{ij}, b_i, c ([3, Chap. 9]), the solution to (3), which satisfies $u(0,x) = u_0(x)$, is given by

$$u(x,t) = \int K(x,y,t)\, u_0(y)\, dy = T(t)\, u$$

with

$$|K(x,y,t)| \leq Ct^{-n/2} e^{-\mu \frac{|x-y|^2}{t}} \tag{4}$$

for some positive constants $C, \mu > 0$ ([3, Chapter 9]). An easy computation shows that $T(t)$ is majorized by $\left(\frac{\pi}{\mu}\right)^{n/2} S\left(\frac{t}{4\mu}\right)$, where $S(t)$ is the semigroup that gives the solution of the heat equation on $\mathbb{R}^n$, $\frac{\partial u}{\partial t} = \triangle u$.

It is known ([5]) that, for each $t > 0$, $S(t)$ is bounded from $L^{p_0}(\mathbb{R}^n) \to L^{p_1}(\mathbb{R}^n)$ for $1 \leq p_0 < \infty, 1 \leq p_1 \leq \infty$ and $p_0 \leq p_1$ with

$$\|S(t)u\|_{L^{p_1}} \leq Ct^{-\frac{1}{2}n\left(\frac{1}{p_0}-\frac{1}{p_1}\right)} \|u\|_{L^{p_0}}.$$

Thus, it follows from Theorem 1, under the above assumptions on the parabolic operator L, that $T(t) : L^{p_0}(l^q) \to L^{p_1}(l^q)$ is bounded for all $1 \leq p_0 \leq \infty$, $1 \leq p_1 \leq \infty$, $p_0 \leq p_1$ and $1 \leq q \leq \infty$ with

$$\|T(t)\, u\|_{L^{p_1}(l^q)} \leq Ct^{-\frac{1}{2}n\left(\frac{1}{p_0}-\frac{1}{p_1}\right)} \|u\|_{L^{p_0}(l^q)}, \quad t > 0$$

and $C = O\left(\mu^{-\frac{1}{2}n(1-\frac{1}{p_0}+\frac{1}{p_1})}\right)$. If the coefficients of the parabolic operator L are constant, $T(t)$ are translation invariant operators and so, with $1 \leq p_0 < \infty$, $1 \leq p_1 < \infty$, $p_0 \leq p_1$ and $1 \leq q < \infty$, $T : F^s_{p_0,q} \to F^s_{p_1,q}$ is bounded, for all $s \in \mathbb{R}$,

$$\|T(t)\, u\|_{F^s_{p_1,q}} \leq Ct^{-\frac{1}{2}n\left(\frac{1}{p_0}-\frac{1}{p_1}\right)} \|u\|_{F^s_{p_0,q}}, \quad t > 0.$$

Of course, this last estimate applies in particular to $S(t)$. In this case, for $p_1 = p_0$, we can refine it in the following way: let $P(-i\nabla)$ be a (constant coefficient) p.d.o. of order $2m$ and $g(\xi) = P(\xi)e^{-|\xi|^2 t}$, $t > 0$. The kernel for $P(-i\nabla)\, S(t)$ is given by $K = \mathcal{F}^{-1} g \in L^r(\mathbb{R}^n)$ for $r \geq 1$ and it is

known ([2]) that

$$\|P(-i\nabla)S(t)u\|_{L^p} \leq C\max\left(t^{-m},1\right)\|u\|_{L^p}$$

for $u \in L^p(\mathbb{R}^n)$ and $1 \leq p \leq \infty$; if P is homogeneous of degree $2m$ this simplifies to

$$\|P(-i\nabla)S(t)u\|_{L^p} \leq Ct^{-m}\|u\|_{L^p}.$$

In particular, for $P(-i\nabla) = (1-\triangle)^m$ we obtain

$$\|(1-\triangle)^m S(t)u\|_{F^s_{p,q}} \leq C\max\left(t^{-m},1\right)\|u\|_{F^s_{p,q}}$$

for $s > 0$ and $1 \leq p < \infty$. Since $(1-\triangle)^m : F^{s+2m}_{p,q} \to F^s_{p,q}$ is an isomorphism, using interpolation, we can obtain a TL space version of the infinite smoothing property of the heat semigroup:

$$\|S(t)u\|_{F^{s_1}_{p,q}} \leq C\max\left(t^{-\frac{1}{2}(s_1-s_0)},1\right)\|u\|_{F^{s_0}_{p,q}}$$

for $s_1 \geq s_0 > 0$, $1 \leq p < \infty$ and $t > 0$.

6. Schrödinger semigroups

A final example, which is closely related with the one of the previous section, is the case of the so called Schrödinger operators. This is the name given to a partial differential operator on $\mathbb{R}^n$ of the form $H = -\triangle + V$, where V is a (real-valued) function on $\mathbb{R}^n$ which is not supposed to be smooth or, in fact, to be defined at all points. In nonrelativistic quantum mechanics the operator H is the Hamiltonian (energy) operator, V being the potential function. We will consider $V = V_+ - V_-$, where $V_+(V_-)$ is the positive(negative) part of V and suppose that $V_- \in K_n$, $V_+ \in K_n^{loc}$, $K_n(K_n^{loc})$ being the Kato(local Kato) class of potentials; see ([5]) for the meaning of these terms as well as for the results that we will use bellow.

Now, it is known that, with the above hypothesis on V, the Schrödinger semigroup e^{-tH} is a bounded operator from L^{p_0} to L^{p_1}, for all $1 \leq p_0 \leq p_1 \leq \infty$ and all $t > 0$, and that the integral kernel of e^{-tH}, $K(x,y)$, satisfies an estimate similar to (4):

$$|K(x,y)| \leq C_1 t^{-n/2} e^{\frac{k}{2}t} \exp\left[-\frac{(y-x)^2}{2t}\right], \quad t > 0,$$

where C_1 is a positive constant and $k > -\inf\sigma(H)$. In terms of the results that we have proven above, this means that $e^{-tH} : L^{p_0}(l^q) \to L^{p_1}(l^q)$ is

bounded for all $1 \le p_0 \le p_1 < \infty$ and $1 \le q \le \infty$ with

$$\left\|e^{-tH}u\right\|_{L^{p_1}(l^q)} \le Ct^{-\frac{1}{2}n\left(\frac{1}{p_0}-\frac{1}{p_1}\right)} e^{\frac{k}{2}t} \|u\|_{L^{p_0}(l^q)}, \quad t > 0.$$

The difference from the previous example is that V can be more singular than the coefficient c in the operator L.

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