On the boundedness of operators in $L^p(l^q)$ and Triebel-Lizorkin Spaces

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Abstract. Given a bounded linear operator $T : L^{p_0}(\mathbb{R}^n) \to L^{p_1}(\mathbb{R}^n)$, for $1 \leq p_0, p_1 \leq \infty$, we state conditions under which $T$ defines a bounded operator between corresponding pairs of $L^p(\mathbb{R}^n;l^q)$ spaces and Triebel-Lizorkin spaces $F_{s,p,q}^a(\mathbb{R}^n)$. Applications are given to linear parabolic equations and to Schrödinger semigroups.

1. Introduction

Let $1 \leq p \leq \infty$. We define the linear spaces $L^p(\mathbb{R}^n;l^q) = L^p(l^q)$ as the set of all sequences $\{u_k\}_{k \in \mathbb{N}_0}$ of complex-valued measurable functions in $\mathbb{R}^n$ for which

$$\|\{u_k\}\|_{L^p(l^q)} = \begin{cases} \left(\sum_{k=0}^{\infty} |u_k|^q \right)^{1/q} & 1 \leq q < \infty \\ \sup_{k \geq 0} \|u_k\|_{L^p} & q = \infty \end{cases}$$

is finite. Each of the spaces $L^p(\mathbb{R}^n;l^q)$ is a Banach space with the norm $\|\cdot\|_{L^p(l^q)}$ and $L^p$ is (isometrically) embedded in $L^p(l^q)$ via the map

$$u \mapsto \{\delta_{0k}u\}_{k \in \mathbb{N}_0},$$

where $\delta_{0k}$ is the Kronecker symbol: $\delta_{0k} = 1$ if $k = 0$ and $\delta_{0k} = 0$ if $k \neq 0$. 
We will define the action of a linear operator $T$ on $L^p(l^q)$ as

$$T\left\{u_k\right\}_{k\in\mathbb{N}_0} = \left\{Tu_k\right\}_{k\in\mathbb{N}_0};$$

as an operator from $L^{p_0}(l^q)$ to $L^{p_1}(l^q)$ it is defined for those $\left\{u_k\right\}_{k\in\mathbb{N}_0} \in L^{p_0}(l^q)$ such that $\left\{Tu_k\right\}_{k\in\mathbb{N}_0} \in L^{p_1}(l^q)$. Clearly, if $T : L^{p_0}(l^q) \to L^{p_1}(l^q)$ is bounded then $T : L^{p_0} \to L^{p_1}$ is well defined and bounded. In this paper we address the following problem (Problem 1): given a bounded linear operator $T : L^{p_0}(R^n) \to L^{p_1}(R^n)$ for $1 \leq p_0, p_1 < \infty$, under which conditions does $T$ defines a bounded operator between a corresponding pair of spaces

$$T : L^{p_0}(l^q) \to L^{p_1}(l^q), \quad 1 \leq q \leq \infty.$$  

We give an answer for some special class of operators that we call majorized operators.

A class of spaces which is defined using the $L^p(l^q)$ spaces are the Triebel-Lizorkin (TL) spaces order $\eta$ and base $(p, q)$:

$$F_{p,q}^s(R^n) = \left\{ u \in S' \mid \left\{2^{sk} \psi_k(D) u\right\}_{k\in\mathbb{N}_0} \in L^p(l^q) \right\}. $$

Here the set $\left\{\psi_k\right\}_{k\in\mathbb{N}_0}$ is a suitable dyadic partition of unity of $R^n$, $D = i\nabla$ and the operator $\varphi(D)$ is defined like the in the functional calculus of operators in $L^2$ as $\varphi(D) = F^{-1}\varphi F$, $F$ being the Fourier transform. $S'$ is the space of tempered distributions.

The topology in $F_{p,q}^s(R^n)$ is given by the norm

$$\|u\|_{F_{p,q}^s} \equiv \left\| \left( \sum_{k=0}^{\infty} 2^{skq} |\psi_k(D) u|^q \right)^{1/q} \right\|_{L^p}$$

for $1 \leq q < \infty$ and by the norm

$$\|u\|_{F_{p,\infty}^s} \equiv \left\| \sup_{k\geq 0} (2^{sk} |\psi_k(D) u|) \right\|_{L^p}$$

for $q = \infty$. With these norms, $F_{p,q}^s$ are Banach spaces. We refer to ([7]) for details on the TL spaces, let us just note that

$$S \subset F_{p,q}^s(R^n) \subset S', \quad 1 \leq p < \infty, \quad 1 \leq q < \infty, \quad s \in \mathbb{R}$$

$S$ being dense in $F_{p,q}^s$ for $q \neq \infty$, $F_{p,q}^s \subset L^p$ for $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $s > 0$ and $F_{p,2}^s = H^{s,p}$ for $1 < p < \infty$, $F_{p,p}^s = B_{p,p}^s$ (Besov spaces) for $1 \leq p < \infty$ (all real $s$). Also, remark that we have omitted the value
As a consequence of our study of Problem 1 we are able to give one solution to the following problem (Problem 2): given a bounded linear operator $T : L^{p_0} (\mathbb{R}^n) \to L^{p_1} (\mathbb{R}^n)$ for $1 \leq p_0, p_1 < \infty$, under which conditions does $T$ define a bounded operator between a corresponding pair of Triebel-Lizorkin spaces

$$T : F^{s}_{p_0,q} (\mathbb{R}^n) \to F^{s}_{p_1,q} (\mathbb{R}^n),$$

$1 \leq q \leq \infty$, $s \in \mathbb{R}$. We shall see that, if an operator is translation invariant, then boundedness in Triebel-Lizorkin spaces follows from boundedness in the corresponding $L^p (l^q)$ spaces. Hence a majorized translation invariant operator is bounded on Triebel-Lizorkin spaces if its majorized operator is bounded on the corresponding $L^p$ spaces, and this independently of $q$.

Here is the plan of the paper. We first prove a boundedness result on $L^p (l^q)$ spaces for majorized operators (Problem 1). In the following section we see that if a majorized operator is also translation invariant then boundedness in Triebel-Lizorkin spaces follows from boundedness in the corresponding $L^p (l^q)$ spaces so we obtain a solution to Problem 2. We close with applications to some classical differential operators and to parabolic evolution equations.

### 2. Operators majorized by positive operators

Let $P$ be a linear operator defined on $L^{p_0}$, with values on $L^{p_1}$, for some $1 \leq p_0, p_1 \leq \infty$. We shall say that $P$ is a positive operator if $Pu \geq 0$ whenever $u \geq 0$. If $P$ is positive and $T : L^{p_0} \to L^{p_1}$ is a linear operator we shall say that $T$ is majorized by $P$, and write $T = O (P)$, if for some positive constant $M$ we have

$$|Tu| \leq MP |u| \text{ a.e.}$$

It is well known that a linear positive operator from $L^{p_0}$ taking values in $L^{p_1}$ is bounded ([4]), as $\|Tu\|_{L^{p_1}} \leq M \|P|u|\|_{L^{p_1}}$ we see that $T = O (P)$ implies that $T$ is bounded with

$$\|T\|_{L(L^{p_0};L^{p_1})} \leq M \|P\|_{L(L^{p_0};L^{p_1})}.$$

The simplest examples of majorized operators, besides positive operators themselves, are integral operators

$$Tu (x) = \int K (x,y) u (y) dy,$$
Theorem 1. Suppose that $T$ is a linear operator with $T = O(P)$ for some linear positive operator $P$. Suppose that for some $1 \leq p_0, p_1 < \infty$, the operator $P$ is a map from $L^{p_0}$ into $L^{p_1}$ with

$$\|Pu\|_{L^{p_1}} \leq C \|u\|_{L^{p_0}}.$$ 

Then, for all $1 \leq q \leq \infty$, $T : L^{p_0}(l^q) \to L^{p_1}(l^q)$ is bounded with

$$(1) \quad \|Tu\|_{L^{p_1}(l^q)} \leq CM \|u\|_{L^{p_0}(l^q)}.$$ 

Proof. It is enough to prove that the hypothesis imply that $T : L^{p_0}(l^1) \to L^{p_1}(l^1)$ is bounded when $q = 1$ and when $q = \infty$ in both cases with an operator bound $\leq CM$. Complex interpolation then shows that the same is true for all $1 \leq q \leq \infty$, see (8).

Take first $q = 1$ and $\{u_k\} \in L^{p_0}(l^1)$. Since the series of positive terms $\sum_{k=0}^{\infty} |u_k|$ converges almost everywhere and defines a $L^{p_0}$ function, we have $u_k \in L^{p_0}$ for all $k$. Then $|Tu_k| \leq MP |u_k|$ for all $k$, hence

$$\|T\{u_k\}\|_{L^{p_1}(l^1)} = \left\| \left( \sum_{k=0}^{\infty} |Tu_k| \right) \right\|_{L^{p_1}} \leq M \left\| \left( \sum_{k=0}^{\infty} P |u_k| \right) \right\|_{L^{p_1}}.$$ 

The series $\sum_{k=0}^{\infty} |u_k|$ also converges in $L^{p_0}$ so, by continuity of $P$,

$$\sum_{k=0}^{\infty} P |u_k| = P \left( \sum_{k=0}^{\infty} |u_k| \right)$$

and

$$\|T\{u_k\}\|_{L^{p_1}(l^1)} \leq M \left\| \left( \sum_{k=0}^{\infty} |u_k| \right) \right\|_{L^{p_1}} \leq CM \left\| \sum_{k=0}^{\infty} |u_k| \right\|_{L^{p_0}} \leq CM \|\{u_k\}\|_{L^{p_0}(l^1)}.$$ 

This shows that $T : L^{p_0}(l^1) \to L^{p_1}(l^1)$ is bounded. Now take $q = \infty$ and $\{u_k\} \in L^{p_0}(l^\infty)$. As $|u_k| \leq \sup_{k \geq 0} |u_k| \in L^{p_0}$ for every $k$, we have $u_k \in L^{p_0}$ and

$$\sup_{k \geq 0} |Tu_k| \leq M \sup_{k \geq 0} P (|u_k|) \leq MP \left( \sup_{k \geq 0} |u_k| \right),$$

for which $|K(x, y)|$ is the kernel of a bounded operator $L^{p_0} \to L^{p^1}$. 

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the last inequality follows from the positivity of $P$. Taking the $L^{p_1}$ norm and using the continuity assumption we get

$$
\|T \{u_k\}\|_{L^{p_1}(l^\infty)} = \left\| \sup_{k \geq 0} |Tu_k| \right\|_{L^{p_1}} \leq M \left\| P \left( \sup_{k \geq 0} |u_k| \right) \right\|_{L^{p_1}} \\
\leq CM \left\| \{u_k\}\right\|_{L^{p_1}(l^\infty)}
$$

and the proof is done. \qed

3. Translation invariant operators

A translation invariant operator is an operator that commutes with translations by all vectors $y \in \mathbb{R}^n$. It is known that if $T$ is a translation invariant operator, continuity of $T$ from $L^{p_0}$ to $L^{p_1}$ then implies that $p_0 \leq p_1$ (or else $T \equiv 0$) and $T$ is given on $\mathcal{S}$ as $\mathcal{F}^{-1}m\mathcal{F}$ for a uniquely determined $m \in \mathcal{S}'$. We say then that $m$ is a Fourier multiplier of type $(p_0, p_1)$. In a previous work [1] we have considered this boundedness problem in different scales of spaces, we have proven in this paper that every Fourier multiplier of type $(p_0, p_1)$ is also a Fourier multiplier of the same type in the Sobolev spaces $W^{k,p}$, in the Bessel potential spaces $H^{s,p}$ and in the Besov spaces $B^{s,p,q}$, (and their homogenous versions) see ([1]) for details.

For translation invariant operator $T$, boundedness of $T$ in $L^p(l^q)$ implies boundedness of $T$ in $F^{s,p,q}$:

**Theorem 2.** Suppose that, for some $1 \leq p_0, p_1 < \infty$, $T$ is a continuous translation invariant linear operator from $L^{p_0}$ to $L^{p_1}$ and that $T : L^{p_0}(l^q) \rightarrow L^{p_1}(l^q)$ is bounded for some $1 \leq q < \infty$:

$$
\|T \{u_k\}\|_{L^{p_1}(l^q)} \leq M \|\{u_k\}\|_{L^{p_0}(l^q)}.
$$

Then, $T : F^{s}_{p_0,q} \rightarrow F^{s}_{p_1,q}$ is bounded for all $s \in \mathbb{R}$ and we have

$$
(2) \quad \|Tu\|_{F^{s}_{p_1,q}} \leq M \|u\|_{F^{s}_{p_0,q}}.
$$

**Proof.** By translation invariance of $T$

$$
\psi_k(D)Tu = T\psi_k(D)u
$$

For every $k \geq 0$ we have

$$
\|Tu_k\|_{L^{p_1}(l^\infty)} \leq M \|\psi_k\|_{L^{p_0}(l^\infty)} \|u_k\|_{L^{p_0}(l^\infty)}.
$$

Taking the $L^{p_1}$ norm and using the continuity assumption we get

$$
\|T \{u_k\}\|_{L^{p_1}(l^\infty)} = \left\| \sup_{k \geq 0} |Tu_k| \right\|_{L^{p_1}} \leq M \left\| P \left( \sup_{k \geq 0} |u_k| \right) \right\|_{L^{p_1}} \\
\leq CM \left\| \{u_k\}\right\|_{L^{p_1}(l^\infty)}
$$

and the proof is done. \qed
for every \( k \in \mathbb{N}_0 \) and \( u \in S \). Taking the \( L^p \) norm of the sequence \( \{ 2^s \psi_k(D) Tu \} \) we get
\[
\left\| \{ 2^s \psi_k(D) Tu \} \right\|_{L^p \ell^q(n)} = \left\| T \{ 2^s \psi_k(D) u \} \right\|_{L^p \ell^q(n)} \leq M \left\| \{ 2^s \psi_k(D) u \} \right\|_{L^p \ell^q(n)}
\]
hence the result follows from the density of \( S \) in \( F^s_{p_0,q} \) (that’s why \( q \) must be different from \( \infty \)).
\( \square \)

So, for a solution to Problem 2, we have:

**Corollary 3.** Suppose that \( T \) is a translation invariant linear operator with \( T = O(P) \) for some linear positive operator \( P \). Suppose that for some \( 1 \leq p_0, p_1 < \infty \), the operator \( P \) is continuous map from \( L^{p_0} \) into \( L^{p_1} \) with
\[
\| Pu \|_{L^{p_1}} \leq C \| u \|_{L^{p_0}}.
\]
Then, if \( 1 \leq q < \infty \), \( T : F^s_{p_0,q} \rightarrow F^s_{p_1,q} \) is bounded for all \( s \in \mathbb{R} \) and we have
\[
\| Tu \|_{F^s_{p_1,q}} \leq M \| u \|_{F^s_{p_0,q}}.
\]
The proof is clear.

The Corollary 3 applies to special types of Fourier multipliers:

**Corollary 4.** Suppose that \( m \in S' \) with \( F^{-1} m \in L^r(\mathbb{R}^n) \) for some \( 1 \leq r \leq \infty \). Then \( m \) is a Fourier multiplier from \( F^s_{p_0,q} \) to \( F^s_{p_1,q} \) for \( 0 \leq \frac{1}{p_0} - \frac{1}{p_1} = 1 - \frac{1}{r} \), \( 1 < p_0, p_1 < \infty \), \( 1 < q < \infty \) and all \( s \in \mathbb{R} \).

**Proof.** Since, for \( u \in S \),
\[
F^{-1} m F u = (F^{-1} m) \ast u,
\]
if \( F^{-1} m \in L^r \) for some \( 1 \leq r \leq +\infty \), by Young inequality, \( F^{-1} m F \) extends to a bounded operator \( L^{p_0} \rightarrow L^{p_1} \) for \( 1 \leq p_0 < \infty \), \( 1 \leq p_1 \leq \infty \) and \( \frac{1}{p} = 1 + \frac{1}{r} - \frac{1}{p_0} \) and the result follows from the theorem and the previous Corollary.
\( \square \)

**Remark 5.** In the results above, we can replace the target space \( F^s_{p_1,q} \) by the space \( F^{s_2}_{p_2,q_2} \), with \( q_2 \geq q \), \( 0 < s_2 \leq s \) and \( s - \frac{n}{p_1} = s_2 - \frac{n}{p_2} \), due to the embedding \( F^s_{p_1,q} \subset F^{s_2}_{p_2,q_2} \).

**Remark 6.** By suitably modification of the definition of the \( L^p \) spaces, we could prove a version of Corollary 3 for the homogenous versions of the spaces \( F^s_{p,q} \).

On the remaining sections of this paper we give some miscellaneous applications of Theorem 1 and Corollary 3.
4. Powers of the resolvent of the Laplacian

Consider the operators $T_{\lambda, z} = (z - \Delta)^{-\lambda}$, for $\Re z > 0$ and $\lambda > 0$. We have $(z - \Delta)^{-\lambda}u = G_{\lambda, z} * u$ where

$$G_{\lambda, z}(x) = \frac{(4\pi t)^{-\frac{n}{2}}}{T(\lambda)} \int_0^\infty e^{-|x|^2/4t} e^{-zt} t^{\lambda - 1} dt.$$ 

$T_{\lambda, z}$ is a translation invariant operator and $|G_{\lambda, z}(x)| \leq G_{\lambda, \Re z}(x)$ which is the kernel of the positive operator $T_{\lambda, \Re z}$.

It is known ([5]) that for $p_0 \leq p_1$ such that $\frac{1}{p_0} - \frac{1}{p_1} \leq \frac{2\lambda}{n}$ we have $(z - \Delta)^{-\lambda} : L^{p_0}(\mathbb{R}^n) \to L^{p_1}(\mathbb{R}^n)$ with

$$\left\| (z - \Delta)^{-\lambda}u \right\|_{L^{p_1}} \leq C(\Re z)^{-\lambda} \left\| u \right\|_{L^{p_0}}$$

so by the Corollary 3 we have $(z - \Delta)^{-\lambda} : F^s_{p_0, q} \to F^s_{p_1, q}$ with

$$\left\| (z - \Delta)^{-\lambda}u \right\|_{F^s_{p_1, q}} \leq C(\Re z)^{-\lambda} \left\| u \right\|_{F^s_{p_0, q}}$$

for complex $z$ with $\Re z > 0$, $\lambda > 0$, $0 \leq \frac{1}{p_0} - \frac{1}{p_1} \leq \frac{2\lambda}{n}$.

An interesting application occurs in the limiting case $z = 0$ where we may still consider the operator $T_{\beta} = (-\Delta)^{-\beta/2}$ for $0 < \beta < n$. This is a positive operator, bounded from $L^{p_0}$ to $L^{p_1}$ for $\frac{1}{p_1} = \frac{1}{p_0} - \frac{\beta}{n}$, a fact known as the fractional integration theorem. $T_{\beta}$ is given on $\mathcal{S}$ as convolution with the $L^1_{loc}$ function $|y|^{-\beta + n}$ and an application of Corollary 3 leads to the fact that $(-\Delta)^{-\beta/2} : F^s_{p_0, q} \to F^s_{p_1, q}$ for all $s > 0$ or, since $(-\Delta)^{\beta/2} F^s_{p_0, q} = F^{s+\beta}_{p_0, q}$,

$$F^{s+\beta}_{p_0, q} \subset F^s_{p_1, q}$$

for $s + \beta - \frac{n}{p_0} = s - \frac{n}{p_1}$. This is the known expression for the embedding theorem for Triebel-Lizorkin spaces and this shows how this embedding theorem follows easily from the fractional integration theorem.

5. Parabolic equations

We consider now the parabolic equation

$$\frac{\partial u}{\partial t} = Lu$$

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where

$$L = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} (x, t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i (x, t) \frac{\partial}{\partial x_i} + c(x, t)$$

is a elliptic operator. Under some mild conditions on the coefficients $a_{ij}$, $b_i$, $c$ ([3, Chap. 9]), the solution to (3), which satisfies $u(0, x) = u_0(x)$, is given by

$$u(x, t) = \int K(x, y, t) u_0(y) dy = T(t) u$$

with

$$|K(x, y, t)| \leq C t^{-n/2} e^{-\frac{\|x-y\|^2}{4t}}$$

for some positive constants $C, \mu > 0$ ([3, Chapter 9]). An easy computation shows that $T(t)$ is majorized by $\left(\frac{\pi}{\sqrt{\mu}}\right)^{n/2} S \left(\frac{t}{4\mu}\right)$, where $S(t)$ is the semigroup that gives the solution of the heat equation on $\mathbb{R}^n$, $\frac{\partial u}{\partial t} = \Delta u$.

It is known ([5]) that, for each $t > 0$, $S(t)$ is bounded from $L^{p_0}(\mathbb{R}^n) \to L^{p_1}(\mathbb{R}^n)$ for $1 \leq p_0 < \infty, 1 \leq p_1 \leq \infty$ and $p_0 \leq p_1$ with

$$\|S(t)u\|_{L^{p_1}} \leq C t^{-\frac{n}{2} \left(\frac{1}{p_0} - \frac{1}{p_1}\right)} \|u\|_{L^{p_0}}.$$ 

Thus, it follows from Theorem 1, under the above assumptions on the parabolic operator $L$, that $T(t) : L^{p_0}(l^n) \to L^{p_1}(l^n)$ is bounded for all $1 \leq p_0 \leq \infty$, $1 \leq p_1 \leq \infty$, $p_0 \leq p_1$ and $1 \leq q \leq \infty$ with

$$\|T(t)u\|_{L^{p_1}(l^n)} \leq C t^{-\frac{n}{2} \left(\frac{1}{p_0} - \frac{1}{p_1}\right)} \|u\|_{L^{p_0}(l^n)}, \quad t > 0$$

and $C = O\left(\mu^{-\frac{1}{2}n(1-\frac{1}{p_0}+\frac{1}{p_1})}\right)$. If the coefficients of the parabolic operator $L$ are constant, $T(t)$ are translation invariant operators and so, with $1 \leq p_0 < \infty$, $1 \leq p_1 < \infty$, $p_0 \leq p_1$ and $1 \leq q < \infty$, $T : F_{p_0, q} \to F_{p_1, q}$ is bounded, for all $s \in \mathbb{R}$,

$$\|T(t)u\|_{F_{p_1, q}} \leq C t^{-\frac{n}{2} \left(\frac{1}{p_0} - \frac{1}{p_1}\right)} \|u\|_{F_{p_0, q}}, \quad t > 0.$$ 

Of course, this last estimate applies in particular to $S(t)$. In this case, for $p_1 = p_0$, we can refine it in the following way: let $P(-i\nabla)$ be a (constant coefficient) p.d.o. of order $2m$ and $g(\xi) = P(\xi)e^{-|\xi|^2 t}$, $t > 0$. The kernel for $P(-i\nabla) S(t)$ is given by $K = F^{-1} g \in L^{r}(\mathbb{R}^n)$ for $r \geq 1$ and it is
known ([2]) that
\[ \| P(-i\nabla) S(t)u \|_{L^p} \leq C \max (t^{-m}, 1) \| u \|_{L^p} \]
for \( u \in L^p(\mathbb{R}^n) \) and \( 1 \leq p \leq \infty \); if \( P \) is homogeneous of degree \( 2m \) this simplifies to
\[ \| P(-i\nabla) S(t)u \|_{L^p} \leq C t^{-m} \| u \|_{L^p} . \]
In particular, for \( P(-i\nabla) = (1 - \Delta)^m \) we obtain
\[ \| (1 - \Delta)^m S(t)u \|_{F^{p,q}_{s,1}} \leq C \max \left( t^{-\frac{1}{2}(s_1 - s_0)}, 1 \right) \| u \|_{F^{p,q}_{s_0}} \]
for \( s_1 \geq s_0 > 0 \), \( 1 \leq p < \infty \) and \( t > 0 \).

6. Schrödinger semigroups

A final example, which is closely related with the one of the previous section, is the case of the so called Schrödinger operators. This is the name given to a partial differential operator on \( \mathbb{R}^n \) of the form \( H = -\Delta + V \), where \( V \) is a (real-valued) function on \( \mathbb{R}^n \) which is not supposed to be smooth or, in fact, to be defined at all points. In nonrelativistic quantum mechanics the operator \( H \) is the Hamiltonian (energy) operator, \( V \) being the potential function. We will consider \( V = V_+ - V_- \), where \( V_+(V_-) \) is the positive(negative) part of \( V \) and suppose that \( V_- \in K_n, V_+ \in K_{n, loc} \), \( K_n(K_{n,loc}) \) being the Kato(local Kato) class of potentials; see ([5]) for the meaning of these terms as well as for the results that we will use below.

Now, it is known that, with the above hypothesis on \( V \), the Schrödinger semigroup \( e^{-tH} \) is a bounded operator from \( L^{p_0} \) to \( L^{p_1} \), for all \( 1 \leq p_0 \leq p_1 \leq \infty \) and all \( t > 0 \), and that the integral kernel of \( e^{-tH} \), \( K(x,y) \), satisfies an estimate similar to (4):
\[ |K(x,y)| \leq C_1 t^{-n/2} e^{\frac{k}{4t}} \exp \left[ -\frac{(y-x)^2}{2t} \right], \quad t > 0, \]
where \( C_1 \) is a positive constant and \( k > -\inf \sigma(H) \). In terms of the results that we have proven above, this means that \( e^{-tH} : L^{p_0}(l^q) \rightarrow L^{p_1}(l^q) \) is
bounded for all $1 \leq p_0 \leq p_1 < \infty$ and $1 \leq q \leq \infty$ with
\[ \left\| e^{-tH}u \right\|_{L^{p_1}(l^q)} \leq C t^{-\frac{1}{2} n \left( \frac{1}{p_0} - \frac{1}{p_1} \right)} e^{\frac{k^2}{2} t} \left\| u \right\|_{L^{p_0}(l^q)} , \quad t > 0. \]

The difference from the previous example is that $V$ can be more singular than the coefficient $c$ in the operator $L$.

References


