JOURNAL OF
FUNCTION SPACES AND APPLICATIONS
Volume 7, Number 2 (2009), 167-186

http://www.jfsa.net

Some new refinements of strengthened Hardy and Pólya–Knopp's inequalities

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(*Communicated by Lars-Erik Persson*)

2000 Mathematics Subject Classification. Primary 26D10, Secondary 26D15.

Keywords and phrases. Integral inequalities, Boas's inequality, Hardy–Littlewood average, Hardy's inequality, Pólya–Knopp's inequality, weights, power weights, convex functions.

Abstract. We prove a new general one-dimensional inequality for convex functions and Hardy–Littlewood averages. Furthermore, we apply this result to unify and refine the so-called Boas's inequality and the strengthened inequalities of the Hardy–Knopp–type, deriving their new refinements as special cases of the obtained general relation. In particular, we get new refinements of strengthened versions of the well-known Hardy and Pólya–Knopp's inequalities.

1. Introduction

To begin with, we recall some well-known classical integral inequalities. If $p > 1$, $k \neq 1$, and the function F is defined on $\mathbb{R}_+ = \langle 0, \infty \rangle$ by

$$F(x) = \begin{cases} \displaystyle\int_0^x f(t)\,dt, & k > 1, \\ \displaystyle\int_x^\infty f(t)\,dt, & k < 1, \end{cases}$$

then the highly important Hardy's integral inequality

$$\int_0^\infty x^{-k} F^p(x)\,dx \le \left(\frac{p}{|k-1|}\right)^p \int_0^\infty x^{p-k} f^p(x)\,dx \tag{1.1}$$

holds for all non-negative functions f, such that $x^{1-\frac{k}{p}} f \in L^p(\mathbb{R}_+)$. This relation was obtained by G. H. Hardy [12] in 1928, although he announced its version with $k = p > 1$ already in 1920, [10], and then proved it in 1925, [11]. In [12], Hardy also pointed out that if k and F fulfill the conditions of the above result, but $0 < p < 1$, then the sign of inequality in (1.1) is reversed, that is,

$$\int_0^\infty x^{-k} F^p(x)\,dx \ge \left(\frac{p}{|k-1|}\right)^p \int_0^\infty x^{p-k} f^p(x)\,dx \tag{1.2}$$

holds. On the other hand, the first unweighted Hardy–type inequality for $p < 0$ was considered by K. Knopp [20] in 1928, but in a discrete setting, for sequences of positive real numbers, while general weighted integral Hardy–type inequalities for exponents $p, q < 0$ and $0 < p, q < 1$ were first studied much later, by P. R. Beesack and H. P. Heinig [1] and H. P. Heinig [14].

Another important classical integral inequality is the so-called Pólya–Knopp's inequality,

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \log f(t)\,dt\right) dx < e\int_0^\infty f(x)\,dx\,, \tag{1.3}$$

which holds for all positive functions $f \in L^1(\mathbb{R}_+)$. This result was first published by K. Knopp [20] in 1928, but it was certainly known before since Hardy himself (see [11, p. 156]) claimed that it was G. Pólya who pointed it out to him earlier. Note that the discrete version of (1.3) is surely due to T. Carleman, [3].

It is important to observe that relations (1.1) and (1.3) are closely related since (1.3) can be obtained from (1.1) by rewriting it with the function f replaced with $f^{1/p}$ and letting $p \to \infty$. Therefore, Pólya–Knopp's inequality may be considered as a limiting case of Hardy's inequality. Moreover, the constants $\left(\frac{p}{|k-1|}\right)^p$ and e, respectively appearing on the right-hand sides of (1.1) and (1.3), are the best possible, that is, neither of them can be replaced with any smaller constant.

Since Hardy and Pólya established inequalities (1.1), (1.2), and (1.3), they have been discussed by several authors, who either gave their alternative proofs using different techniques, or applied, refined and generalized them in various ways. Further information and remarks concerning the

rich history, development, generalizations, and applications of Hardy and Pólya–Knopp's integral inequalities can be found e.g. in the monographs [13, 22, 23, 25, 26, 27, 28], expository papers [6, 17, 21], and the references cited therein. Besides, here we also emphasize the papers [2, 4, 5, 7, 8, 9, 18, 19, 24, 29, 32, 33], all of which to some extent have guided us in the research we present here.

In particular, in 1970, R. P. Boas [2] proved that (1.1) and (1.3) are just special cases of a much more general inequality

$$\int_0^\infty \Phi\left(\frac{1}{M}\int_0^\infty f(tx)\,dm(t)\right)\frac{dx}{x} \le \int_0^\infty \Phi(f(x))\,\frac{dx}{x} \tag{1.4}$$

for continuous convex functions $\Phi : [0,\infty\rangle \longrightarrow \mathbb{R}$, measurable non-negative functions $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$, and non-decreasing and bounded functions $m : [0,\infty\rangle \longrightarrow \mathbb{R}$, where $M = m(\infty) - m(0) > 0$ and the inner integral on the left-hand side of (1.4) is a Lebesgue–Stieltjes integral with respect to m. After its author, the relation (1.4) was named Boas's inequality (see also [25, Chapter IV, p. 156] and [28, Chapter 8, Theorem 8.1]). In the case of a concave function Φ, (1.4) holds with the reversed sign of inequality.

On the other hand, obviously unaware of the mentioned more general Boas's result for Hardy–Littlewood averages, in 2002, S. Kaijser et al. [18] established the so-called general Hardy–Knopp–type inequality for positive functions $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$,

$$\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{dx}{x} \le \int_0^\infty \Phi(f(x))\,\frac{dx}{x}\,, \tag{1.5}$$

where Φ is a convex function on $\mathbb{R}_+$. By taking $\Phi(x) = x^p$ and $\Phi(x) = e^x$, they obtained an elegant new proof of inequalities (1.1) and (1.3) and showed that both Hardy and Pólya–Knopp's inequality can be derived by using only a convexity argument. Later on, A. Čižmešija et al. [9] generalized the relation (1.5) to the so-called strengthened Hardy–Knopp–type inequality by adding a weight function and truncating the range of integration to $\langle 0, b\rangle$. They also obtained a related dual inequality, that is, an inequality with the outer integrals taken over $\langle b,\infty\rangle$ and with the inner integral on the left-hand side taken over $\langle x,\infty\rangle$. These general inequalities provided an unified treatment of the strengthened Hardy and Pólya–Knopp's inequalities from [7, 8] and [32, 33].

Finally, we mention a recent paper [29] by L.-E. Persson and J. A. Oguntuase. They obtained a class of refinements of Hardy's inequality (1.1) related to an arbitrary $b \in \mathbb{R}_+$ and the outer integrals on both hand sides of (1.1) taken over $\langle 0, b\rangle$ or $\langle b,\infty\rangle$. These results extend those of

D. T. Shum [31] and C. O. Imoru [15, 16] and cover all admissible parameters $p, k \in \mathbb{R}$, $p \neq 0$, $k \neq 1$. Namely, let f be a non-negative integrable function on $\langle 0, b\rangle$, $F(x) = \int_0^x f(t)\,dt$, and let $\frac{p}{k-1} > 0$. If $p \in \langle -\infty, 0\rangle \cup [1, \infty\rangle$, then

$$\int_0^b x^{-k} F^p(x)\,dx + \frac{p}{k-1} b^{1-k} F^p(b) \leq \left(\frac{p}{k-1}\right)^p \int_0^b x^{p-k} f^p(x)\,dx\ , \tag{1.6}$$

while for $p \in \langle 0, 1]$ inequality (1.6) holds in the reversed direction. On the contrary, if f is a non-negative integrable function on $\langle b, \infty\rangle$, $\tilde{F}(x) = \int_x^\infty f(t)\,dt$, and $\frac{p}{k-1} < 0$, then

$$\int_b^\infty x^{-k} \tilde{F}^p(x)\,dx + \frac{p}{1-k} b^{1-k} \tilde{F}^p(b) \leq \left(\frac{p}{1-k}\right)^p \int_b^\infty x^{p-k} f^p(x)\,dx \tag{1.7}$$

holds for $p \in \langle -\infty, 0\rangle \cup [1, \infty\rangle$, while for $p \in \langle 0, 1]$ the sign of inequality in (1.7) is reversed. The constant $\left(\frac{p}{|k-1|}\right)^p$ is the best possible for all cases and both inequalities (1.6) and (1.7).

Motivated by the above observations, in this paper we consider a general positive Borel measure λ on $\mathbb{R}_+$, such that

$$L = \lambda(\mathbb{R}_+) = \int_0^\infty d\lambda(t) < \infty\,, \tag{1.8}$$

and prove a new weighted Boas–type inequality for this setting. Further, we point out that our result unifies, generalizes and refines relations (1.4) and (1.5), as well as the strengthened Hardy–Knopp–type inequalities from [9]. More precisely, applying the obtained general relation with some particular weights and a measure λ, we derive new refinements of the above inequalities. Finally, as their special cases we get new refinements of the strengthened versions of Hardy and Pólya–Knopp's inequalities, completely different from (1.6) and (1.7) and even hardly comparable with these inequalities.

The paper is organized in the following way. After this Introduction, in Section 2 we introduce some necessary notation and state, prove and discuss a general refined weighted Boas–type inequality. As its particular cases, in the same section we obtain a new refinement of inequality (1.4), as well as refinements of (1.5) and of the strengthened weighted Hardy–Knopp–type inequalities. Refinements of the strengthened Hardy and Pólya–Knopp's inequalities are presented in the concluding Section 3 of the paper, along with some final remarks.

Conventions. Throughout this paper, all measures are assumed to be positive, all functions are assumed to be measurable, and expressions of the

form $0 \cdot \infty$, $\frac{0}{0}$, $\frac{a}{\infty}$ $(a \in \mathbb{R})$, and $\frac{\infty}{\infty}$ are taken to be equal to zero. As usual, by dx we denote the Lebesgue measure on $\mathbb{R}$, by a weight function (shortly: a weight) we mean a non-negative measurable function on the actual interval, while an interval in $\mathbb{R}$ is any convex subset of $\mathbb{R}$. Finally, by $\operatorname{Int} I$ we denote the interior of an interval $I \subseteq \mathbb{R}$.

2. The main results

First, we introduce some necessary notation and, for reader's convenience, recall some basic facts about convex functions. Let I be an interval in $\mathbb{R}$ and $\Phi : I \longrightarrow \mathbb{R}$ be a convex function. For $x \in I$, by $\partial\Phi(x)$ we denote the subdifferential of Φ at x, that is, the set $\partial\Phi(x) = \{\alpha \in \mathbb{R} : \Phi(y) - \Phi(x) - \alpha(y - x) \geq 0,\, y \in I\}$. It is well-known that $\partial\Phi(x) \neq \emptyset$ for all $x \in \operatorname{Int} I$. More precisely, at each point $x \in \operatorname{Int} I$ we have $-\infty < \Phi'_-(x) \leq \Phi'_+(x) < \infty$ and $\partial\Phi(x) = [\Phi'_-(x),\, \Phi'_+(x)]$, while the set on which Φ is not differentiable is at most countable. Moreover, every function $\varphi : I \longrightarrow \mathbb{R}$ for which $\varphi(x) \in \partial\Phi(x)$, whenever $x \in \operatorname{Int} I$, is increasing on $\operatorname{Int} I$. For more details about convex functions see e.g. a recent monograph [26].

On the other hand, for a finite Borel measure λ on $\mathbb{R}_+$, that is, having property (1.8), and a Borel measurable function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$, by Af we denote its Hardy–Littlewood average, defined in terms of the Lebesgue integral as

$$Af(x) = \frac{1}{L} \int_0^\infty f(tx)\, d\lambda(t)\,, \qquad x \in \mathbb{R}_+\,, \tag{2.1}$$

where L is defined by (1.8).

Now, we can state and prove the main result of this paper. It is given in the following theorem.

Theorem 2.1. *Let λ be a finite Borel measure on $\mathbb{R}_+$, L be defined by* (1.8)*, and let u and v be non-negative measurable functions on $\mathbb{R}_+$, where*

$$v(x) = \int_0^\infty u\left(\frac{x}{t}\right) d\lambda(t) < \infty\,, \qquad x \in \mathbb{R}_+\,. \tag{2.2}$$

Let Φ be a continuous convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \longrightarrow \mathbb{R}$ be any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$. If $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a measurable function, such that $f(x) \in I$ for all $x \in \mathbb{R}_+$, and $Af(x)$ is defined by (2.1)*, then $Af(x) \in I$, for all $x \in \mathbb{R}_+$, and the inequality*

$$\begin{aligned}
&\frac{1}{L}\int_0^\infty v(x)\Phi(f(x))\,\frac{dx}{x} - \int_0^\infty u(x)\Phi(Af(x))\,\frac{dx}{x} \\
&\qquad \geq \frac{1}{L}\left|\int_0^\infty\int_0^\infty u(x)\,|\Phi(f(tx)) - \Phi(Af(x))|\;d\lambda(t)\,\frac{dx}{x}\right. \\
&\qquad\qquad \left. - \int_0^\infty\int_0^\infty u(x)\,|\varphi(Af(x))|\,|f(tx) - Af(x)|\;d\lambda(t)\,\frac{dx}{x}\right| \qquad (2.3)
\end{aligned}$$

holds.

Proof. For a fixed $x \in \mathbb{R}_+$, let $h_x : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be defined by $h_x(t) = f(tx) - Af(x)$. Then (1.8) and (2.1) imply

$$\int_0^\infty h_x(t)\,d\lambda(t) = \int_0^\infty f(tx)\,d\lambda(t) - Af(x)\int_0^\infty d\lambda(t) = 0\,. \qquad (2.4)$$

Now, suppose $x \in \mathbb{R}_+$ is such that $Af(x) \notin I$. Observing that $f(\mathbb{R}_+) \subseteq I$ and that I is an interval in $\mathbb{R}$, we have $h_x(t) > 0$ for all $t \in \mathbb{R}_+$, or $h_x(t) < 0$ for all $t \in \mathbb{R}_+$, that is, the function h_x is either strictly positive or strictly negative. Since this contradicts (2.4), we have proved that $Af(x) \in I$, for all $x \in \mathbb{R}_+$. Note that if $Af(x)$ is an endpoint of I for some $x \in \mathbb{R}_+$ (in cases when I is not an open interval), then h_x (or $-h_x$) will be a non-negative function whose integral over $\mathbb{R}_+$, with respect to the measure λ, is equal to 0. Therefore, $h_x \equiv 0$, that is, $f(tx) = Af(x)$ holds for λ–a.e. $t \in \mathbb{R}_+$. To prove inequality (2.3), observe that for all $r \in \operatorname{Int} I$ and $s \in I$ we have

$$\Phi(s) - \Phi(r) - \varphi(r)(s-r) \geq 0,$$

where $\varphi : I \longrightarrow \mathbb{R}$ is any function such that $\varphi(x) \in \partial\Phi(x)$ for $x \in \operatorname{Int} I$, and hence

$$\begin{aligned}
\Phi(s) - \Phi(r) - \varphi(r)(s-r) &= |\Phi(s) - \Phi(r) - \varphi(r)(s-r)| \\
&\geq ||\Phi(s) - \Phi(r)| - |\varphi(r)|\,|s-r||\,. \qquad (2.5)
\end{aligned}$$

Especially, in the case when $Af(x) \in \operatorname{Int} I$, by substituting $r = Af(x)$ and $s = f(tx)$ in (2.5), for all $t \in \mathbb{R}_+$ we get

$$\begin{aligned}
&\Phi(f(tx)) - \Phi(Af(x)) - \varphi(Af(x))\,[f(tx) - Af(x)] \\
&\qquad \geq ||\Phi(f(tx)) - \Phi(Af(x))| - |\varphi(Af(x))|\,|f(tx) - Af(x)||\,. \qquad (2.6)
\end{aligned}$$

On the other hand, the above analysis provides (2.6) to hold even if $Af(x)$ is an endpoint of I, since in that case both sides of inequality (2.6) are equal to 0 for λ–a.e. $t \in \mathbb{R}_+$. Multiplying (2.6) by $\frac{u(x)}{x}$, then integrating it

over $\mathbb{R}_+^2$ with respect to the measures $d\lambda(t)$ and $\frac{dx}{x}$, and applying Fubini's theorem, we obtain the following sequence of inequalities:

$$\begin{aligned}
&\int_0^\infty \int_0^\infty u(x)\Phi(f(tx))\, d\lambda(t)\, \frac{dx}{x} - \int_0^\infty \int_0^\infty u(x)\Phi(Af(x))\, d\lambda(t)\, \frac{dx}{x} \\
&\qquad - \int_0^\infty \int_0^\infty u(x)\varphi(Af(x))\, [f(tx) - Af(x)]\, d\lambda(t)\, \frac{dx}{x} \\
&\geq \int_0^\infty \int_0^\infty u(x)||\Phi(f(tx)) - \Phi(Af(x))| \\
&\qquad -|\varphi(Af(x))||f(tx) - Af(x)||\, d\lambda(t)\, \frac{dx}{x} \\
&\geq \int_0^\infty u(x) \left| \int_0^\infty |\Phi(f(tx)) - \Phi(Af(x))|\, d\lambda(t) \right. \\
&\qquad \left. - |\varphi(Af(x))| \int_0^\infty |f(tx) - Af(x)|\, d\lambda(t) \right| \frac{dx}{x} \\
&\geq \left| \int_0^\infty \int_0^\infty u(x)\, |\Phi(f(tx)) - \Phi(Af(x))|\, d\lambda(t)\, \frac{dx}{x} \right. \\
&\qquad \left. - \int_0^\infty \int_0^\infty u(x)\, |\varphi(Af(x))|\, |f(tx) - Af(x)|\, d\lambda(t)\, \frac{dx}{x} \right| . \qquad (2.7)
\end{aligned}$$

Again, by using Fubini's theorem and the substitution $y = tx$, the first integral on the left-hand side of (2.7) becomes

$$\begin{aligned}
&\int_0^\infty \int_0^\infty u(x)\Phi(f(tx))\, d\lambda(t)\, \frac{dx}{x} \\
&= \int_0^\infty \int_0^\infty u(x)\Phi(f(tx))\, \frac{dx}{x}\, d\lambda(t) \\
&= \int_0^\infty \int_0^\infty u\left(\frac{y}{t}\right) \Phi(f(y))\, \frac{dy}{y}\, d\lambda(t) \\
&= \int_0^\infty \Phi(f(y)) \int_0^\infty u\left(\frac{y}{t}\right)\, d\lambda(t)\, \frac{dy}{y} \\
&= \int_0^\infty v(y)\Phi(f(y))\, \frac{dy}{y}\,, \qquad (2.8)
\end{aligned}$$

while for the second integral we have

$$\int_0^\infty \int_0^\infty u(x)\Phi(Af(x))\, d\lambda(t)\, \frac{dx}{x} = L \int_0^\infty u(x)\Phi(Af(x))\, \frac{dx}{x}\,. \qquad (2.9)$$

Finally, considering (2.4), we similarly get

$$\begin{aligned}&\int_0^\infty \int_0^\infty u(x)\varphi(Af(x))\,[f(tx)-Af(x)]\;d\lambda(t)\,\frac{dx}{x}\\ &\qquad = \int_0^\infty u(x)\varphi(Af(x))\left(\int_0^\infty h_x(t)\,d\lambda(t)\right)\frac{dx}{x}=0\,,\end{aligned}\tag{2.10}$$

so (2.3) holds by combining (2.7), (2.8), (2.9), and (2.10). □

Remark 2.1. Observe that (2.7) provides a pair of inequalities interpolated between the left-hand side and the right-hand side of (2.3), that is, further new refinements of (2.3).

Remark 2.2. If Φ is a concave function (that is, if $-\Phi$ is convex), then (2.5) reads

$$\begin{aligned}\Phi(r)-\Phi(s)-\varphi(r)(r-s) &= |\Phi(r)-\Phi(s)-\varphi(r)(r-s)|\\ &\geq ||\Phi(s)-\Phi(r)|-|\varphi(r)|\,|s-r||\,,\end{aligned}$$

where φ is a real function on I such that $\varphi(x)\in\partial\Phi(x)=[\Phi'_+(x),\,\Phi'_-(x)]$, for all $x\in \operatorname{Int} I$. Therefore, in this setting (2.3) holds by its left-hand side replaced with

$$\int_0^\infty u(x)\Phi(Af(x))\,\frac{dx}{x}-\frac{1}{L}\int_0^\infty v(x)\Phi(f(x))\,\frac{dx}{x}\,.$$

Moreover, if Φ is an affine function, then (2.3) becomes equality.

Since the right-hand side of (2.3) is non-negative, as an immediate consequence of Theorem 2.1 and Remark 2.2 we get the following result, a weighted Boas's inequality.

Corollary 2.1. *Suppose λ is a finite Borel measure on $\mathbb{R}_+$, L is defined by* (1.8)*, u is a non-negative measurable function on $\mathbb{R}_+$, and the function v is defined on $\mathbb{R}_+$ by* (2.2)*. If Φ is a continuous convex function on an interval $I\subseteq\mathbb{R}$, then the inequality*

$$\int_0^\infty u(x)\Phi(Af(x))\,\frac{dx}{x}\leq\frac{1}{L}\int_0^\infty v(x)\Phi(f(x))\,\frac{dx}{x}\tag{2.11}$$

holds for all measurable functions $f:\mathbb{R}_+\longrightarrow\mathbb{R}$, such that $f(x)\in I$ for all $x\in\mathbb{R}_+$, where $Af(x)$ is defined by (2.1)*. For a concave function Φ, the sign of inequality in relation* (2.11) *is reversed.*

In the sequel, we analyze some important particular cases of Theorem 2.1 and Corollary 2.1 and compare them with some results previously known

from the literature. Namely, by setting $u(x) \equiv 1$, we obtain a refined Boas-type inequality with $Af(x)$ defined by the Lebesgue integral.

Corollary 2.2. *Let λ be a finite Borel measure on $\mathbb{R}_+$ and L be defined by* (1.8)*. Then the inequality*

$$\begin{aligned}\int_0^\infty \Phi(f(x))\,\frac{dx}{x} - \int_0^\infty \Phi(Af(x))\,\frac{dx}{x} \\ \geq \frac{1}{L}\Bigg|\int_0^\infty\int_0^\infty |\Phi(f(tx)) - \Phi(Af(x))|\,d\lambda(t)\,\frac{dx}{x} \\ - \int_0^\infty\int_0^\infty |\varphi(Af(x))|\,|f(tx) - Af(x)|\,d\lambda(t)\,\frac{dx}{x}\Bigg|\end{aligned} \tag{2.12}$$

holds for all continuous convex functions Φ on an interval $I \subseteq \mathbb{R}$, real functions φ on I, such that $\varphi(x) \in \partial\Phi(x)$ for $x \in \operatorname{Int} I$, and all measurable real functions f on $\mathbb{R}_+$, such that $f(x) \in I$ for all $x \in \mathbb{R}_+$, and $Af(x)$ defined by (2.1)*. If the function Φ is concave, then* (2.1) *holds with*

$$\int_0^\infty \Phi(Af(x))\,\frac{dx}{x} - \int_0^\infty \Phi(f(x))\,\frac{dx}{x}$$

on its left-hand side.

Evidently, Corollary 2.2 implies the following analogue of (1.4).

Corollary 2.3. *If λ is a finite Borel measure on $\mathbb{R}_+$, L is defined by* (1.8)*, Φ is a continuous convex function on an interval $I \subseteq \mathbb{R}$, f is a measurable real function on $\mathbb{R}_+$ with values in I, and $Af(x)$ is defined by* (2.1)*, then*

$$\int_0^\infty \Phi(Af(x))\,\frac{dx}{x} \leq \int_0^\infty \Phi(f(x))\,\frac{dx}{x}\,. \tag{2.13}$$

If Φ is concave, then the sign of inequality in (2.13) *is reversed.*

Remark 2.3. Let $m : [0,\infty\rangle \longrightarrow \mathbb{R}$ be a non-decreasing bounded function and $M = m(\infty) - m(0) > 0$. It is well-known that m induces a finite Borel measure λ on $\mathbb{R}_+$ (and vice versa), such that the related Lebesgue and Lebesgue–Stieltjes integrals are equivalent. Thus, all the above results from this section can be interpreted as for $Af(x)$ defined by the Lebesgue-Stieltjes integral with respect to m, that is, as

$$Af(x) = \frac{1}{M}\int_0^\infty f(tx)\,dm(t)\,, \qquad x \in \mathbb{R}_+\,.$$

Therefore, our results refine and generalize Boas's inequality (1.4). Namely, we obtained a refinement of its weighted version.

To conclude this section, we consider measures λ which yield refinements of the Hardy–Knopp–type inequalities mentioned in the Introduction. Especially, for $d\lambda(t) = \chi_{[0,1]}(t)\,dt$ we obtain a refinement of a weighted version of (1.5).

Theorem 2.2. *Let u be a non-negative function on $\mathbb{R}_+$, such that the function $t \mapsto \frac{u(t)}{t^2}$ is locally integrable in $\mathbb{R}_+$, and let*

$$w(x) = x \int_x^\infty u(t)\,\frac{dt}{t^2}\,, \quad t \in \mathbb{R}_+.$$

If a real-valued function Φ is convex on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \longrightarrow \mathbb{R}$ is such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$, then the inequality

$$\begin{aligned}\int_0^\infty w(x)\Phi(f(x))\,\frac{dx}{x} - \int_0^\infty u(x)\Phi(Hf(x))\,\frac{dx}{x} \\ \geq \Bigg|\int_0^\infty u(x)\int_0^x |\Phi(f(t)) - \Phi(Hf(x))|\,dt\,\frac{dx}{x^2} \\ - \int_0^\infty u(x)\,|\varphi(Hf(x))|\int_0^x |f(t) - Hf(x)|\,dt\,\frac{dx}{x^2}\Bigg|\end{aligned} \tag{2.14}$$

holds for all functions f on $\mathbb{R}_+$ with values in I and for $Hf(x)$ defined by

$$Hf(x) = \frac{1}{x}\int_0^x f(t)\,dt \tag{2.15}$$

for $x \in \mathbb{R}_+$. If Φ is a concave function, then (2.14) *holds with*

$$\int_0^\infty u(x)\Phi(Hf(x))\,\frac{dx}{x} - \int_0^\infty w(x)\Phi(f(x))\,\frac{dx}{x}$$

on its left-hand side.

Proof. Follows directly from Theorem 2.1 and Remark 2.2, rewritten with the measure $d\lambda(t) = \chi_{[0,1]}(t)\,dt$. In this setting, we have $L = 1$,

$$Af(x) = \int_0^1 f(tx)\,dt = Hf(x) \;\text{ and }\; v(x) = \int_0^1 u\left(\frac{x}{t}\right)\,dt = w(x), \;\; x \in \mathbb{R}_+\,,$$

so (2.14) holds. □

Remark 2.4. Let a convex function Φ and functions u, w, f, and Hf be as in Theorem 2.2. Observing that the right-hand side of relation (2.14)

is non-negative, we get

$$\int_0^\infty u(x)\Phi(Hf(x))\,\frac{dx}{x} \le \int_0^\infty w(x)\Phi(f(x))\,\frac{dx}{x}\,. \tag{2.16}$$

Moreover, for a concave function Φ relation (2.16) holds with the reversed sign of inequality. This result, the so-called weighted Hardy–Knopp–type inequality, was already obtained in [9, Theorem 1], while its particular case (1.5), originally proved in [18], follows by setting $u(x) \equiv 1$. Therefore, (2.14) may be regarded as a refined weighted inequality of the Hardy–Knopp type and relation (2.3) as its generalization.

On the other hand, a dual result to Theorem 2.2 can be derived by considering (2.3) with $d\lambda(t) = \chi_{[1,\infty\rangle}(t)\,\frac{dt}{t^2}$.

Theorem 2.3. *Suppose $u : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a non-negative function, locally integrable in $\mathbb{R}_+$, and w is defined on $\mathbb{R}_+$ by*

$$w(x) = \frac{1}{x}\int_0^x u(t)\,dt\,.$$

If Φ is a convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \longrightarrow \mathbb{R}$ is such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$, then the inequality

$$\begin{aligned}\int_0^\infty w(x)\Phi(f(x))\,\frac{dx}{x} &- \int_0^\infty u(x)\Phi(\tilde{H}f(x))\,\frac{dx}{x}\\ &\ge \Bigg|\int_0^\infty u(x)\int_x^\infty \Big|\Phi(f(t)) - \Phi(\tilde{H}f(x))\Big|\,\frac{dt}{t^2}\,dx\\ &\quad - \int_0^\infty u(x)\,\Big|\varphi(\tilde{H}f(x))\Big|\int_x^\infty \Big|f(t) - \tilde{H}f(x)\Big|\,\frac{dt}{t^2}\,dx\Bigg|\end{aligned} \tag{2.17}$$

holds for all functions f on $\mathbb{R}_+$ with values in I and for $\tilde{H}f(x)$ defined by

$$\tilde{H}f(x) = x\int_x^\infty f(t)\,\frac{dt}{t^2} \tag{2.18}$$

for $x \in \mathbb{R}_+$. In the case when Φ is concave, (2.17) *holds if its left-hand side is replaced with*

$$\int_0^\infty u(x)\Phi(\tilde{H}f(x))\,\frac{dx}{x} - \int_0^\infty w(x)\Phi(f(x))\,\frac{dx}{x}\,.$$

Proof. Set $d\lambda(t) = \chi_{[1,\infty\rangle}(t)\,\frac{dt}{t^2}$ in Theorem 2.1 and Remark 2.2. Then

$$Af(x) = \int_1^\infty f(tx)\,\frac{dt}{t^2} = \tilde{H}f(x), \quad v(x) = \int_1^\infty u\left(\frac{x}{t}\right)\,\frac{dt}{t^2} = w(x), \quad x \in \mathbb{R}_+\,,$$

and $L = 1$, so (2.17) holds. $\square$

Remark 2.5. As in Remark 2.4, note that for a convex function Φ and functions u, w, f, and $\tilde{H}f$ from the statement of Theorem 2.3, we have

$$\int_0^\infty u(x)\Phi(\tilde{H}f(x))\,\frac{dx}{x} \le \int_0^\infty w(x)\Phi(f(x))\,\frac{dx}{x}\,, \tag{2.19}$$

while for a concave Φ relation (2.19) holds with the inequality sign $\ge$. Since as a consequence of Theorem 2.1 and Theorem 2.3 we derived a dual inequality to (2.16), relation (2.17) can be considered as a refined dual weighted Hardy–Knopp–type inequality and (2.3) as its generalization.

Finally, as special cases of Theorem 2.2 and Theorem 2.3, we formulate refinements of the strengthened Hardy–Knopp-type inequalities.

Corollary 2.4. *Suppose $b \in \mathbb{R}_+$, $u : \langle 0, b\rangle \longrightarrow \mathbb{R}$ is a non-negative function, such that the function $t \mapsto \frac{u(t)}{t^2}$ is locally integrable in $\langle 0, b\rangle$, and the function w is defined by*

$$w(x) = x\int_x^b u(t)\,\frac{dt}{t^2}\,, \quad x \in \langle 0, b\rangle\,.$$

If Φ is a convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \longrightarrow \mathbb{R}$ is such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$, then the inequality

$$\begin{aligned}\int_0^b w(x)\Phi(f(x))\,\frac{dx}{x} &- \int_0^b u(x)\Phi(Hf(x))\,\frac{dx}{x}\\ &\ge \left|\int_0^b u(x)\int_0^x |\Phi(f(t)) - \Phi(Hf(x))|\,dt\,\frac{dx}{x^2}\right.\\ &\qquad \left. - \int_0^b u(x)\,|\varphi(Hf(x))|\int_0^x |f(t) - Hf(x)|\,dt\,\frac{dx}{x^2}\right|\end{aligned} \tag{2.20}$$

holds for all functions $f : \langle 0, b\rangle \longrightarrow \mathbb{R}$ with values in I and Hf defined on $\langle 0, b\rangle$ by (2.15)*. If Φ is a concave function, the order of integrals on the left-hand side of* (2.20) *is reversed.*

Proof. Let $\hat{u}$, $\hat{w}$, and $\hat{f}$ be defined on $\mathbb{R}_+$ by $\hat{u}(x) = u(x)\chi_{\langle 0,b\rangle}(x)$,

$$\hat{w}(x) = x\int_x^\infty \hat{u}(t)\,\frac{dt}{t^2} = w(x)\chi_{\langle 0,b\rangle}(x)\,,$$

and $\hat{f}(x) = f(x)\chi_{\langle 0,b\rangle}(x) + c\chi_{[b,\infty\rangle}(x)$, where $c \in I$ is arbitrary. Since these functions naturally extend u, w, and f to act on $\mathbb{R}_+$, they evidently fulfill

the conditions of Theorem 2.2, considered with $\hat{u}$, $\hat{w}$, and $\hat{f}$ instead of u, w, and f respectively. Therefore, (2.14) holds and in this setting it becomes (2.20). $\square$

Remark 2.6. Since the right-hand side of (2.20) is non-negative, Corollary 2.4 improves a result from [9, Theorem 1]. Hence, it can be considered as a refined strengthened Hardy–Knopp-type inequality.

Remark 2.7. For $u(x) \equiv 1$, we have $w(x) = 1 - \frac{x}{b}$ in Corollary 2.4, so (2.20) reads

$$\begin{aligned}\int_0^b \left(1-\frac{x}{b}\right)\Phi(f(x))\,\frac{dx}{x} - \int_0^b \Phi(Hf(x))\,\frac{dx}{x} \\ \geq \left|\int_0^b \int_0^x |\Phi(f(t)) - \Phi(Hf(x))|\,dt\,\frac{dx}{x^2}\right. \\ \left. - \int_0^b |\varphi(Hf(x))| \int_0^x |f(t) - Hf(x)|\,dt\,\frac{dx}{x^2}\right| . \end{aligned} \tag{2.21}$$

This relation provides a basis for results in the following section.

A dual result to inequality (2.20) is given in the next corollary.

Corollary 2.5. *For $b \in \mathbb{R}, b \geq 0$, let $u : \langle b, \infty\rangle \longrightarrow \mathbb{R}$ be a non-negative locally integrable function in $\langle b, \infty\rangle$ and the function w be given by*

$$w(x) = \frac{1}{x}\int_b^x u(t)\,dt\,, \quad x \in \langle b, \infty\rangle\,.$$

Let Φ be a convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \longrightarrow \mathbb{R}$ be such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$. Then the inequality

$$\begin{aligned}\int_b^\infty w(x)\Phi(f(x))\,\frac{dx}{x} - \int_b^\infty u(x)\Phi(\tilde{H}f(x))\,\frac{dx}{x} \\ \geq \left|\int_b^\infty u(x)\int_x^\infty \left|\Phi(f(t)) - \Phi(\tilde{H}f(x))\right|\,\frac{dt}{t^2}\,dx\right. \\ \left. - \int_b^\infty u(x)\left|\varphi(\tilde{H}f(x))\right| \int_x^\infty \left|f(t) - \tilde{H}f(x)\right|\,\frac{dt}{t^2}\,dx\right| \end{aligned} \tag{2.22}$$

holds for all functions f on $\langle b, \infty\rangle$ with values in I and $\tilde{H}f$ defined by (2.18). *For a concave function Φ, the order of integrals on the left-hand side of* (2.22) *is reversed.*

Proof. As in the proof of Corollary 2.4, inequality (2.22) follows by applying Theorem 2.3 to the functions $\hat{u}$, $\hat{w}$, and $\hat{f}$, where $\hat{u}(x) =$

$u(x)\chi_{\langle b,\infty\rangle}(x)$,

$$\hat{w}(x) = \frac{1}{x}\int_0^x \hat{u}(t)\,dt = w(x)\chi_{\langle b,\infty\rangle}(x)\,,$$

and $\hat{f}(x) = c\chi_{\langle 0,b]}(x) + f(x)\chi_{\langle b,\infty\rangle}(x)$, for an arbitrary $c \in I$. □

Remark 2.8. Note that (2.22) refines [9, Theorem 2] since the right-hand side of (2.22) is non-negative. Thus, we obtained a refined strengthened dual Hardy–Knopp–type inequality.

Remark 2.9. For $u(x) \equiv 1$, (2.22) reads

$$\begin{aligned}\int_b^\infty \left(1-\frac{b}{x}\right)\Phi(f(x))\,\frac{dx}{x} - \int_b^\infty \Phi(\tilde{H}f(x))\,\frac{dx}{x} \\ \geq \left|\int_b^\infty\int_x^\infty \left|\Phi(f(t)) - \Phi(\tilde{H}f(x))\right|\,\frac{dt}{t^2}\,dx\right. \\ \left. - \int_b^\infty \left|\varphi(\tilde{H}f(x))\right| \int_x^\infty \left|f(t) - \tilde{H}f(x)\right|\,\frac{dt}{t^2}\,dx\right|\,. \end{aligned} \tag{2.23}$$

Together with (2.21), we shall use this inequality to derive refinements of the classical Hardy and Pólya–Knopp's inequalities.

3. Refinements of strengthened Hardy and Pólya–Knopp's inequalities

In the previous section, obtained inequalities were discussed with respect to a measure λ and a weight function u, while a convex function Φ remained unspecified. On the contrary, here we consider two particular convex (or concave) functions, namely $\Phi(x) = x^p$ and $\Phi(x) = e^x$, and derive some new refinements of the well-known Hardy and Pólya–Knopp's inequalities, as well as their strengthened versions. Moreover, we show that they are just special cases of the results mentioned.

We start with new refinements of Hardy's inequality, so let $p \in \mathbb{R}$, $p \neq 0$, and $\Phi(x) = x^p$. Obviously, $\varphi(x) = \Phi'(x) = px^{p-1}$, $x \in \mathbb{R}_+$, and the function Φ is convex for $p \in \mathbb{R} \setminus [0,1\rangle$, concave for $p \in \langle 0,1]$, and affine for $p = 1$. On the other hand, for a locally integrable function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$, as in the Introduction, we denote

$$F(x) = \int_0^x f(t)\,dt \quad \text{and} \quad \tilde{F}(x) = \int_x^\infty f(t)\,dt\,, \quad x \in \mathbb{R}_+. \tag{3.1}$$

A new refined strengthened Hardy's inequality is given in the following corollary.

Corollary 3.1. *Let* $0 < b \leq \infty$ *and* $p, k \in \mathbb{R}$ *be such that* $p \neq 0$, $k \neq 1$, *and* $\frac{p}{k-1} > 0$. *Let* f *be a non-negative function on* $\langle 0, b\rangle$. *If* $p \in \langle -\infty, 0\rangle \cup [1, \infty\rangle$, *then the inequality*

$$\begin{aligned}
&\left(\frac{p}{k-1}\right)^p \int_0^b \left[1 - \left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] x^{p-k} f^p(x)\,dx - \int_0^b x^{-k} F^p(x)\,dx \\
&\geq \left| \left(\frac{p}{k-1}\right)^{p-1} \int_0^b x^{\frac{1-k}{p}-1} \int_0^x t^{\frac{k-1}{p}-1} \right. \\
&\qquad \times \left. \left| t^{p-k+1} f^p(t) - \left(\frac{k-1}{p}\right)^p x^{1-k} F^p(x) \right| dt\,dx \right. \\
&\qquad \left. -|p| \int_0^b x^{-k} F^{p-1}(x) \int_0^x \left| f(t) - \frac{k-1}{p} \cdot \frac{1}{t} \left(\frac{t}{x}\right)^{\frac{k-1}{p}} F(x) \right| dt\,dx \right| \qquad (3.2)
\end{aligned}$$

holds. In the case when $p \in \langle 0, 1\rangle$, *the order of integrals of the right-hand side of inequality* (3.2) *is reversed.*

Proof. First, let either $p \geq 1$, $k > 1$, or $p < 0$, $k < 1$, and let $\Phi(x) = x^p$ and $\varphi(x) = px^{p-1}$. According to Corollary 2.4 and Remark 2.7, then (2.21) holds. Rewriting it for $a = b^{\frac{k-1}{p}}$ and $x \mapsto f(x^{\frac{p}{k-1}})x^{\frac{p}{k-1}-1}$, instead of b and f respectively, we get

$$\begin{aligned}
&\int_0^a \left(1 - \frac{x}{a}\right) x^{p\left(\frac{p}{k-1}-1\right)} f^p(x^{\frac{p}{k-1}})\,\frac{dx}{x} - \int_0^a \left(\frac{1}{x}\int_0^x f(t^{\frac{p}{k-1}})\,t^{\frac{p}{k-1}-1} dt\right)^p \frac{dx}{x} \\
&\geq \left| \int_0^a \int_0^x \left| t^{p\left(\frac{p}{k-1}-1\right)} f^p(t^{\frac{p}{k-1}}) - \left(\frac{1}{x}\int_0^x f(r^{\frac{p}{k-1}})\,r^{\frac{p}{k-1}-1} dr\right)^p \right| dt\,\frac{dx}{x^2} \right. \\
&\qquad - \int_0^a \left| p\left(\frac{1}{x}\int_0^x f(r^{\frac{p}{k-1}})\,r^{\frac{p}{k-1}-1} dr\right)^{p-1} \right| \\
&\qquad \times \left. \int_0^x \left| f(t^{\frac{p}{k-1}})\,t^{\frac{p}{k-1}-1} - \frac{1}{x}\int_0^x f(r^{\frac{p}{k-1}})\,r^{\frac{p}{k-1}-1} dr \right| dt\,\frac{dx}{x^2} \right|,
\end{aligned}$$

so (3.2) follows by a sequence of substitutions such as $s = x^{\frac{p}{k-1}}$. The remaining case, that is, when $p \in \langle 0, 1\rangle$ and $k > 1$, is a direct consequence of Corollary 2.4 and Remark 2.7. □

Now, we state and prove a refined strengthened dual Hardy's inequality.

Corollary 3.2. *Suppose* $0 \le b < \infty$ *and* $p, k \in \mathbb{R}$ *are such that* $p \neq 0$, $k \neq 1$, *and* $\frac{p}{k-1} < 0$. *If* $p \in \langle -\infty, 0\rangle \cup [1, \infty\rangle$, *then the inequality*

$$\begin{aligned}
&\left(\frac{p}{1-k}\right)^p \int_b^\infty \left[1 - \left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right] x^{p-k} f^p(x)\,dx - \int_b^\infty x^{-k} \tilde{F}^p(x)\,dx \\
&\ge \Bigg| \left(\frac{p}{1-k}\right)^{p-1} \int_b^\infty x^{\frac{1-k}{p}-1} \\
&\quad \times \int_x^\infty t^{\frac{k-1}{p}-1} \left| t^{p-k+1} f^p(t) - \left(\frac{1-k}{p}\right)^p x^{1-k} \tilde{F}^p(x) \right| dt\,dx \\
&\quad -|p| \int_b^\infty x^{-k} \tilde{F}^{p-1}(x) \int_x^\infty \left| f(t) - \frac{1-k}{p} \cdot \frac{1}{t} \left(\frac{x}{t}\right)^{\frac{1-k}{p}} \tilde{F}(x) \right| dt\,dx \Bigg|
\end{aligned} \tag{3.3}$$

holds for all non-negative functions f *on* $\langle b, \infty\rangle$. *In the case when* $p \in \langle 0, 1\rangle$, *the order of integrals on the left-hand side of inequality* (3.3) *is reversed.*

Proof. As in the proof of Corollary 3.1, we use $\Phi(x) = x^p$, that is, $\varphi(x) = px^{p-1}$, and rewrite inequality (2.23) for b and f respectively replaced with $a = b^{\frac{1-k}{p}}$ and $x \mapsto f(x^{\frac{p}{1-k}})x^{\frac{p}{1-k}+1}$. Relation (3.3) then follows by a sequence of substitutions of the form $s = x^{\frac{p}{1-k}}$. Note that we again distinguish two cases. The first one, which yields (3.3), holds when either $p \ge 1$, $k < 1$, or $p < 0$, $k > 1$. In the other one, with $p \in \langle 0, 1\rangle$ and $k < 1$, the order of integrals on the left-hand side of inequality (3.3) is reversed. □

Remark 3.1. Observe that for $b = \infty$ the left-hand side of (3.2) reads

$$\left(\frac{p}{k-1}\right)^p \int_0^\infty x^{p-k} f^p(x)\,dx - \int_0^\infty x^{-k} F^p(x)\,dx\,,$$

while for $b = 0$ on the left-hand side of (3.3) we have

$$\left(\frac{p}{1-k}\right)^p \int_0^\infty x^{p-k} f^p(x)\,dx - \int_0^\infty x^{-k} \tilde{F}^p(x)\,dx\,.$$

Therefore, we obtained a refinement of the classical Hardy's inequality (1.1). Also note that for $p = 1$ relations (3.2) and (3.2) are trivial since their both sides are equal to 0.

Remark 3.2. Observe that Corollary 3.1 and Corollary 3.2 provide new and original refinements of Hardy's inequality although the idea to strengthen and refine (1.1) is not new and results of such type already exist in the literature. As in the Introduction, here we just mention the papers

[15, 16], [31], and a recent paper [29]. It is important to emphasize that our results are completely different from those given in these papers and even hardly comparable with (1.6) and (1.7). A third type of refinements of a similar form can be found in another recent paper [30], where a fairly new concept of superquadratic function was used in a crucial way.

Finally, we consider $\Phi(x) = e^x$ to obtain refinements of the strengthened Pólya–Knopp's inequality and of its dual. For a positive function f on $\mathbb{R}_+$ and $x \in \mathbb{R}_+$, we denote

$$Gf(x) = \exp\left(\frac{1}{x}\int_0^x \log f(t)\,dt\right) \quad \text{and} \quad \tilde{G}f(x) = \exp\left(x\int_x^\infty \log f(t)\,\frac{dt}{t^2}\right).$$

Related results are given in the following two corollaries.

Corollary 3.3. *Let $0 < b \le \infty$ and f be a positive function on $\langle 0, b\rangle$. Then*

$$\begin{aligned} e\int_0^b \left(1-\frac{x}{b}\right) f(x)\,dx - \int_0^b Gf(x)\,dx \\ \ge \Big|\int_0^b \int_0^x |e\,tf(t) - x\,Gf(x)|\,dt\,\frac{dx}{x^2} \\ - \int_0^b Gf(x)\int_0^x \Big|\log\frac{e\,tf(t)}{x\,Gf(x)}\Big|\,dt\,\frac{dx}{x}\Big|. \end{aligned} \tag{3.4}$$

Corollary 3.4. *If $0 \le b < \infty$ and f is a positive function on $\langle b, \infty\rangle$, then*

$$\begin{aligned} \frac{1}{e}\int_b^\infty \left(1-\frac{b}{x}\right) f(x)\,dx - \int_b^\infty \tilde{G}f(x)\,dx \\ \ge \Big|\int_b^\infty \int_x^\infty \Big|\frac{1}{e}\,tf(t) - x\,\tilde{G}f(x)\Big|\,\frac{dt}{t^2}\,dx \\ - \int_b^\infty x\,\tilde{G}f(x)\int_x^\infty \Big|\log\frac{tf(t)}{ex\,\tilde{G}f(x)}\Big|\,\frac{dt}{t^2}\,dx\Big|. \end{aligned} \tag{3.5}$$

Remark 3.3. Note that for $b = \infty$ in (3.4) we have a refined Pólya–Knopp's inequality, while for $b = 0$ relation (3.5) becomes its refined dual inequality.

Acknowledgements. The research of the first and the third author was supported by the Croatian Ministry of Science, Education and Sports, under the Research Grants 058-1170889-1050 (first author) and 117-1170889-0888 (third author). The last two authors also acknowledge with thanks the facilities provided to them by the Abdus Salam School of Mathematical

Sciences, GC University, Lahore, Pakistan. We thank the careful referee and Professor Lars-Erik Persson for valuable comments and suggestions, which we have used to improve the final version of this paper.

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(*Received : April 2008*)