Boundedness of several operators on weighted Herz spaces

Yasuo Komori and Katsuo Matsuoka

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Abstract. We consider the boundedness of singular integral operators and fractional integral operators on weighted Herz spaces. For this purpose we introduce generalized Herz space. Our results are the best possible.

1. Introduction

Since Beurling [1] introduced the Beurling algebras and Herz [5] generalized these spaces, many studies have been done for Herz spaces (see, for example, [2], [4], [7] and [11]). Weighted Herz spaces are also considered in [6], [8] [9] and [10]. But they consider only A_1 weights or power weights.

In this paper we consider the boundedness of singular integral operators and fractional integral operators on weighted Herz spaces with A_p weights. For this purpose, we introduce generalized Herz spaces, and as corollaries of our theorems we obtain the boundedness of these operators on weighted Herz spaces.

2. Definitions and Notation

The following notation is used: For a set $E \subset R^n$ we denote the Lebesgue measure of E by $|E|$. We denote the characteristic function of E by χ_E . We indicate a ball of radius 2^k centered at the origin by $B_k = \{x \in R^n; |x| \leq 2^k\}$ where $k \in Z$. We write $C_k = B_k \setminus B_{k-1}$ and $\chi_k = \chi_{C_k}$. For a nonnegative function w , we write $w(E) = \int_E w(x)dx$ and define

$$L^p(w)(R^n) = \left\{ f; \|f\|_{L^p(w)} = \left(\int_{R^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty \right\}.$$

First we define homogeneous Herz spaces.

Definition 1 (Herz spaces). Let $1 < p < \infty, 0 < q < \infty$ and $-\infty < \alpha < \infty$. The ordinary Herz space is defined by

$$K_p^{\alpha,q}(R^n) = \left\{ f; \|f\|_{K_p^{\alpha,q}} = \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha q} \|f\chi_k\|_{L^p}^q \right)^{1/q} < \infty \right\}.$$

For non-negative functions w_1 and w_2 , weighted Herz space is defined by

$$\begin{aligned} K_p^{\alpha,q}[w_1, w_2](R^n) &= \left\{ f; \|f\|_{K_p^{\alpha,q}[w_1, w_2]} \right. \\ &= \left. \left(\sum_{k=-\infty}^{\infty} w_1(B_k)^{\alpha q/n} \|f\chi_k\|_{L^p(w_2)}^q \right)^{1/q} < \infty \right\}. \end{aligned}$$

Next we define some classes of weight functions.

Definition 2. Let $1 < p < \infty$. We say that $w \in A_p$ if for all balls Q

$$\left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} \leq C.$$

Definition 3. We say that $w \in A_1$ if for all balls Q

$$\frac{1}{|Q|} \int_Q w(x) dx \leq C \operatorname{ess} \inf_{x \in Q} w(x).$$

Remark . $A_1 \subset A_p$ for all $p > 1$.

Definition 4. Let $1 < p_1, p_2 < \infty$. We say that $w \in A(p_1, p_2)$ if

$$\left(\frac{1}{|Q|} \int_Q w(x)^{p_2} dx \right)^{1/p_2} \left(\frac{1}{|Q|} \int_Q w(x)^{-p_1'} dx \right)^{1/p_1'} \leq C$$

for all balls Q , where $1/p_1 + 1/p'_1 = 1$.

Remarks . The following properties about weight functions are easily obtained from definitions. Especially the equivalence relation (1) is important for our theory (see Theorems 1 and 2).

- (1) $w \in A(p_1, p_2)$ if and only if $w^{p_2} \in A_{1+p_2/p'_1}$.
 $w \in A(p, p)$ if and only if $w^p \in A_p$.
 If $\tilde{p}_1 < p_1$ and $\tilde{p}_2 > p_2$, then $A(\tilde{p}_1, \tilde{p}_2) \subset A(p_1, p_2)$.

Definition 5 (centered reverse doubling). Let $\delta > 0$. We say $w \in RD(\delta)$ if

$$\frac{w(B_k)}{w(B_j)} \geq C2^{\delta(k-j)} \quad \text{for } k > j.$$

The following lemmas are well-known (see, for example, [3] and [12]).

Lemma 1 (strong doubling). If $w \in A_p$, then

$$\frac{w(B_k)}{w(B_j)} \leq C2^{np(k-j)} \quad \text{for } k > j.$$

Lemma 2. If $w \in A_p$, then $w \in RD(\delta)$ for some $\delta > 0$.

Lemma 3. Let $1 < p < \infty$. If $w \in A_p$, then $w \in A_{\tilde{p}}$ for some $\tilde{p} < p$.

Lemma 4. Let $1 < p < \infty$ and $w = |x|^\beta$. Then $w \in A_p$ if and only if $-n < \beta < n(p-1)$.

Lemma 5. If $w \in A(p_1, p_2)$, then $w^{-p'_1}(Q)^{1/p'_1} w^{p_2}(Q)^{1/p_2} \leq C|Q|^{1/p'_1 + 1/p_2}$.

Remark . Note that $w^p(Q)^{1/p} = (\int_Q w^p(x)dx)^{1/p}$.

Next we define singular integral operators and fractional integral operators.

Definition 6. We say that T is a standard singular integral operator if there exists a function K which satisfies the following conditions.

$$\begin{aligned} Tf(x) &= \text{p.v.} \int_{R^n} K(x-y)f(y)dy. \\ |K(x)| &\leq \frac{C_K}{|x|^n} \quad \text{and} \quad |\nabla K(x)| \leq \frac{C_K}{|x|^{n+1}}, \quad x \neq 0. \\ \int_{\varepsilon < |x| < N} K(x)dx &= 0 \quad \text{for all } 0 < \varepsilon < N. \end{aligned}$$

Remark . We can weaken the conditions about K , but for the simplicity we assume these conditions.

Definition 7. The fractional integral operator is define by

$$I_s f(x) = \int_{R^n} \frac{f(y)}{|x-y|^{n-s}} dy, \quad \text{where } 0 < s < n.$$

3. Known results

Let T be a standard singular integral operator and I_s be the fractional integral operator. The following two propositions are well-known (see, for example, [12]).

Proposition 1. *Let $1 < p < \infty$. If $w \in A_p$, then T is bounded on $L^p(w)$.*

Proposition 2. *Let $1 < p_1 < n/s$ and $1/p_2 = 1/p_1 - s/n$. If $w \in A(p_1, p_2)$, then I_s is bounded from $L^{p_1}(w^{p_1})$ to $L^{p_2}(w^{p_2})$.*

Li and Yang [7] and Lu and Yang [9], [10] proved the next theorems about the boundedness on Herz space.

Theorem A ([7]). *Let $1 < p < \infty, 0 < q < \infty$ and $-n/p < \alpha < n(1 - 1/p)$. Then T is bounded on $K_p^{\alpha, q}$.*

Theorem B ([9]). *Let $1 < p < \infty, 0 < q < \infty$ and $0 \leq \alpha < n(1 - 1/p)$. We assume that $w_1, w_2 \in A_1$. Then T is bounded on $K_p^{\alpha, q}[w_1, w_2]$.*

Theorem C ([7]). *Let $1 < p_1 < n/s, 0 < q < \infty, 1/p_2 = 1/p_1 - s/n$ and $-n/p_1 + s < \alpha < n(1 - 1/p_1)$. Then I_s is bounded from $K_{p_1}^{\alpha, q}$ to $K_{p_2}^{\alpha, q}$.*

Theorem D ([10]). *Let $p_1 \geq n/s, 0 < q < n/s$ and*

$$(2) \quad -n/p_1 + s < \alpha_1 < n(1 - 1/p_1) + s(1 - q/p_1).$$

We set $1/p_2 = (1/p_1) \cdot (1 - qs/n)$ and $\alpha_2 = \alpha_1 + s(q/p_1 - 1)$. Then I_s is bounded from $K_{p_1}^{\alpha_1, q}$ to $K_{p_2}^{\alpha_2, q}$.

Lu and Yang also proved weighted estimates for Theorems C and D.

Remark . In [10] p. 365, Lu and Yang assume the condition that $0 < \alpha_1 < n(1 - 1/p_1)$ in stead of (2), but this is incorrect. We shall show our condition (2) is the best possible in Section 7. Note that the condition (2) is equivalent to the following:

$$(2') \quad -n/p_2 < \alpha_2 < n(1 - 1/p_1).$$

In this paper we consider weighted estimates for these theorems with A_p weights. For this purpose we introduce generalized Herz spaces in the next section, and as corollaries of our general theory, we can prove all the theorems above. We also prove that our results are the best possible by showing counterexamples in Section 7.

4. Theorems

For a sequence $\varphi = \{\varphi(k)\}_{k=-\infty}^{\infty}$, $\varphi(k) > 0$, we define generalized Herz space as follows.

Definition 8.

$$K_p^q(\varphi, w)(R^n) = \left\{ f; \|f\|_{K_p^q(\varphi, w)} = \left(\sum_{k=-\infty}^{\infty} \varphi(k)^q \|f \chi_k\|_{L^p(w)}^q \right)^{1/q} < \infty \right\}.$$

We define some conditions about φ .

Definition 9. Let $-\infty < a \leq b < \infty$. We say that φ satisfies doubling condition of order (a, b) and write $\varphi \in D(a, b)$, if there exists $C \geq 1$ such that

$$C^{-1}2^{a(k-j)} \leq \frac{\varphi(k)}{\varphi(j)} \leq C2^{b(k-j)} \quad \text{for } k > j.$$

Definition 10. We say that φ satisfies doubling condition, if $\varphi(k+1) + \varphi(k-1) \leq C\varphi(k)$.

Remark . If $\varphi \in D(a, b)$, then φ satisfies doubling condition.

Our results are the following.

Theorem 1. Let $1 < p_1 < n/s$, $0 < q < \infty$, $\delta > 0$ and $1/p_2 = 1/p_1 - s/n$. We assume that

$$(3) \quad \varphi \in D(a, b), \quad \text{where} \quad -\frac{\delta}{p_2} < a \leq b < n\left(1 - \frac{1}{p_1}\right),$$

$$(4) \quad w^{p_2} \in A_r, \quad \text{where} \quad r = \min\left(1 + \frac{p_2}{p'_1}, 1 + \frac{p_2}{p'_1} - \frac{bp_2}{n}\right),$$

$$(5) \quad w^{p_2} \in RD(\delta).$$

Then I_s is bounded from $K_{p_1}^q(\varphi, w^{p_1})$ to $K_{p_2}^q(\varphi, w^{p_2})$.

Remark . When $b \leq 0$, the condition (4) is equivalent to the condition that $w \in A(p_1, p_2)$, but when $b > 0$, (4) is stronger than $A(p_1, p_2)$ -condition. We shall show that the condition (4) is the best possible by showing a counterexample in Section 7.

If we do not assume the condition (5), then in (3), we need to assume

$$0 \leq a \leq b < n \left(1 - \frac{1}{p_1}\right).$$

Theorem 2. *Let $p_1 \geq n/s, 0 < q < n/s$ and $1/p_2 = (1/p_1) \cdot (1 - sq/n)$. We assume (3), (5) and the following:*

$$(6) \quad w^{p_2} \in A_r, \quad \text{where} \quad r = \min \left(1 + \frac{p_2}{p'_1}, 1 + \frac{p_2}{p'_1} - \frac{p_2}{n} \left(b + s \left(1 - \frac{q}{p_1} \right) \right) \right).$$

Then I_s is bounded from $K_{p_1}^q(\psi, w^{p_1})$ to $K_{p_2}^q(\varphi, w^{p_2})$, where $\psi(k) = 2^{ks(1-q/p_1)}\varphi(k)$.

We also obtain the boundedness of singular integral operators.

Theorem 3. *Let T be a standard singular integral operator. Suppose that $1 < p < \infty, 0 < q < \infty$ and $\delta > 0$. We assume that*

$$(3') \quad \varphi \in D(a, b), \quad \text{where} \quad -\frac{\delta}{p} < a \leq b < n \left(1 - \frac{1}{p}\right),$$

$$(4') \quad w \in A_r, \quad \text{where} \quad r = \min(p, p(1 - b/n)),$$

$$(5') \quad w \in RD(\delta).$$

Then T is bounded from $K_p^q(\varphi, w)$ to $K_p^q(\varphi, w)$.

In Theorem 1, when we set $s = 0$ formally, we can obtain Theorem 3. The proof of Theorem 3 is similar to that of Theorem 1, therefore we omit the proof.

5. Applications

5.1 Applications to Theorems A, C and D. Let $\varphi(k) = 2^{k\alpha_2}$ and $w \equiv 1$. Then $K_p^q(\varphi, w) = K_p^{\alpha_2, q}$, $\varphi \in D(\alpha_2, \alpha_2)$ and $w \in RD(n)$. Therefore the condition (3) in Theorem 1 is equivalent to the following:

$$-\frac{n}{p_2} < \alpha_2 < n \left(1 - \frac{1}{p_1}\right).$$

Using Theorem 2, we obtain Theorem D. Let $\alpha_1 = \alpha_2 = \alpha$. Then we can prove Theorem C by Theorem 1. By Theorem 3, we obtain Theorem A.

5.2 Application to Theorem B. Suppose that $0 \leq \alpha < n(1 - 1/p)$. For two weight functions $w_1, w_2 \in A_1$, let $\varphi(k) = w_1(B_k)^{\alpha/n}$ and $w = w_2$.

Then $K_p^q(\varphi, w) = K_p^{\alpha, q}[w_1, w_2]$, $\varphi \in D(0, \alpha)$ and $w \in RD(\delta)$ for some $\delta > 0$.

Since $\alpha (= b)$ satisfies the condition (3') in Theorem 3, we can prove Theorem B. Furthermore we obtain the following.

Corollary 1. *Let $1 < p < \infty$, $0 < q < \infty$ and $0 \leq \alpha < n(1 - 1/p)$. If $w \in A_{p(1-\alpha/n)}$, then T is bounded on $K_p^{\alpha, q}[1, w]$.*

Remark . We shall prove that the condition that $w \in A_{p(1-\alpha/n)}$ is the best possible in Section 7 (see Counterexample 3).

Corollary 2. *Let $1 < p < \infty$, $0 < q < \infty$. We assume that*

$$(7) \quad -n < \alpha \leq 0 \quad \text{and} \quad -n - \alpha p < \beta < n(p-1)$$

or

$$(8) \quad 0 < \alpha < n(1 - 1/p) \quad \text{and} \quad -n < \beta < n(p-1) - \alpha p.$$

Then T is bounded on $K_p^{\alpha, q}[1, |x|^\beta]$.

Proof of Corollary 2. Let $\varphi(k) = 2^{k\alpha}$ and $w = |x|^\beta$. Then $\varphi \in D(\alpha, \alpha)$ and $w \in RD(n + \beta)$. Note that $w \in A_r$ if and only if $-n < \beta < n(r-1)$. Therefore the conditions (3') and (4') in Theorem 3 are equivalent to the following:

$$(9) \quad -\frac{n + \beta}{p} < \alpha < n\left(1 - \frac{1}{p}\right),$$

$$(10) \quad -n < \beta < n(r-1) \quad \text{where} \quad r = \min(p, p - \alpha p/n).$$

These conditions are equivalent to (7) and (8). $\square$

5.3 Estimates for fractional integral operators (one weight case).

The next two corollaries are easily obtained from Theorems 1 and 2 respectively.

Corollary 3. *Let $1 < p_1 < n/s$, $0 < q < \infty$, $0 \leq \alpha < n(1 - 1/p_1)$ and $1/p_2 = 1/p_1 - s/n$. If $w^{p_2} \in A_{1+p_2/p_1' - \alpha p_2/n}$, then I_s is bounded from $K_{p_1}^{\alpha, q}[1, w^{p_1}]$ to $K_{p_2}^{\alpha, q}[1, w^{p_2}]$.*

Corollary 4. *Let $p_1 \geq n/s$, $0 < q < n/s$, $0 \leq \alpha_2 < n(1 - 1/p_1)$, and we set $1/p_2 = 1/p_1 \cdot (1 - sq/n)$, $\alpha_2 = \alpha_1 + s(q/p_1 - 1)$. If $w^{p_2} \in A_{1+p_2/p_1' - \alpha_1 p_2/n}$, then I_s is bounded from $K_{p_1}^{\alpha_1, q}[1, w^{p_1}]$ to $K_{p_2}^{\alpha_2, q}[1, w^{p_2}]$.*

6. Proofs

To prove Theorems 1 and 2, we use the same calculation, therefore we first estimate I_s for general indices p_1, p_2 and q .

As in [7], we write

$$\begin{aligned}
\|I_s f\|_{K_p^q(\varphi, w)} &\leq C \left(\sum_{k=-\infty}^{\infty} \varphi(k)^q \left(\sum_{j=-\infty}^{k-2} \|(I_s(f\chi_j))\chi_k\|_{L^{p_2}(w^{p_2})} \right)^q \right)^{1/q} \\
&\quad + C \left(\sum_{k=-\infty}^{\infty} \varphi(k)^q \left(\sum_{j=k-1}^{k+1} \|(I_s(f\chi_j))\chi_k\|_{L^{p_2}(w^{p_2})} \right)^q \right)^{1/q} \\
&\quad + C \left(\sum_{k=-\infty}^{\infty} \varphi(k)^q \left(\sum_{j=k+2}^{\infty} \|(I_s(f\chi_j))\chi_k\|_{L^{p_2}(w^{p_2})} \right)^q \right)^{1/q} \\
&= I + II + III.
\end{aligned}$$

(I) When $j \leq k-2$ and $x \in C_k$, we have

$$|I_s(f\chi_j)(x)| \leq C 2^{k(-n+s)} \int_{B_j} |f(y)| dy \leq C 2^{k(-n+s)} \|f\chi_j\|_{L^{p_1}(w^{p_1})} w^{-p'_1}(B_j)^{1/p'_1},$$

and we obtain

$$\|(I_s(f\chi_j))\chi_k\|_{L^{p_2}(w^{p_2})} \leq C 2^{k(-n+s)} \|f\chi_j\|_{L^{p_1}(w^{p_1})} w^{-p'_1}(B_j)^{1/p'_1} w^{p_2}(B_k)^{1/p_2}.$$

(III) When $j \geq k+2$ and $x \in C_k$, we have

$$|I_s(f\chi_j)(x)| \leq C 2^{j(-n+s)} \int_{B_j} |f(y)| dy \leq C 2^{j(-n+s)} \|f\chi_j\|_{L^{p_1}(w^{p_1})} w^{-p'_1}(B_j)^{1/p'_1},$$

and we obtain

$$\|(I_s(f\chi_j))\chi_k\|_{L^{p_2}(w^{p_2})} \leq C 2^{j(-n+s)} \|f\chi_j\|_{L^{p_1}(w^{p_1})} w^{-p'_1}(B_j)^{1/p'_1} w^{p_2}(B_k)^{1/p_2}.$$

(I) By Lemma 3, $w^{p_2} \in A_{\tilde{r}}$ for some $\tilde{r} < r$. Therefore by Lemmas 1 and 5, we have

$$\begin{aligned}
&\|(I_s(f\chi_j))\chi_k\|_{L^{p_2}(w^{p_2})} \\
&\leq C 2^{k(-n+s)} \|f\chi_j\|_{L^{p_1}(w^{p_1})} w^{-p'_1}(B_j)^{1/p'_1} w^{p_2}(B_j)^{1/p_2} \left(\frac{w^{p_2}(B_k)}{w^{p_2}(B_j)} \right)^{1/p_2} \\
&\leq C 2^{k(-n+s+n\tilde{r}/p_2)} 2^{nj(1/p'_1+1/p_2-\tilde{r}/p_2)} \|f\chi_j\|_{L^{p_1}(w^{p_1})}.
\end{aligned}$$

(III) By Lemma 2, $w^{p_2} \in RD(\delta)$ for some $\delta > 0$. Therefore by the same argument as above we have

$$\|(I_s(f\chi_j))\chi_k\|_{L^{p_2}(w^{p_2})} \leq C 2^{k\delta/p_2} 2^{j(-n+s+n/p'_1+n/p_2-\delta/p_2)} \|f\chi_j\|_{L^{p_1}(w^{p_1})}.$$

Using the condition that $\varphi \in D(a, b)$, we obtain the following inequalities.

$$\begin{aligned}
 (11) \quad I^q &\leq C \sum_k 2^{k(-n+s+n\tilde{r}/p_2+b)q} \\
 &\quad \times \left(\sum_{j \leq k-2} 2^{nj(1/p'_1+1/p_2-\tilde{r}/p_2-b/n)} \varphi(j) \|f\chi_j\|_{L^{p_1}(w^{p_1})} \right)^q, \\
 (12) \quad III^q &\leq C \sum_k 2^{k(a+\delta/p_2)q} \\
 &\quad \times \left(\sum_{j \geq k+2} 2^{j(-n+s+n/p'_1+n/p_2-\delta/p_2-a)} \varphi(j) \|f\chi_j\|_{L^{p_1}(w^{p_1})} \right)^q.
 \end{aligned}$$

These are elementary estimates for I and III. From here we shall prove two theorems respectively.

Proof of Theorem 1. The estimate of II is easy. By using Proposition 2, we obtain $II \leq C \|f\|_{K_{p_1}^q(\varphi, w^{p_1})}$.

Next we estimate I. When $q \leq 1$, we have

$$\begin{aligned}
 I^q &\leq C \sum_j 2^{nj(1/p'_1+1/p_2-\tilde{r}/p_2-b/n)q} \varphi(j)^q \\
 &\quad \times \|f\chi_j\|_{L^{p_1}(w^{p_1})}^q \left(\sum_{k=j+2}^{\infty} 2^{k(-n+s+n\tilde{r}/p_2+b)q} \right) \\
 &\leq C \|f\|_{K_p^q(\varphi, w^{p_1})},
 \end{aligned}$$

because $-n+s+n\tilde{r}/p_2+b \leq 0$, that is, $-n+s+n\tilde{r}/p_2+b < 0$.

When $q > 1$, we take $\varepsilon > 0$ such that $-n+s+n\tilde{r}/p_2+b+n\varepsilon < 0$. Then we have

$$\begin{aligned}
 I^q &\leq C \sum_k 2^{k(-n+s+n\tilde{r}/p_2+b)q} \\
 &\quad \times \left(\left(\sum_{j=-\infty}^{k-2} 2^{nj(1/p'_1+1/p_2-\tilde{r}/p_2-b/n-\varepsilon)q} \varphi(j)^q \|f\chi_j\|_{L^{p_1}(w^{p_1})}^q \right) \right. \\
 &\quad \times \left. \left(\sum_{j=-\infty}^{k-2} 2^{n\varepsilon q' j} \right)^{q/q'} \right) \\
 &\leq C \sum_k 2^{k(-n+s+n\tilde{r}/p_2+b+n\varepsilon)q} \\
 &\quad \times \left(\sum_{j=-\infty}^{k-2} 2^{nj(1/p'_1+1/p_2-\tilde{r}/p_2-b/n-\varepsilon)q} \varphi(j)^q \|f\chi_j\|_{L^{p_1}(w^{p_1})}^q \right)
 \end{aligned}$$

$$\leq C\|f\|_{K_p^q(\varphi, w^{p_1})}.$$

Finally we estimate *III*. When $q \leq 1$, we have

$$\begin{aligned} III^q &\leq C \sum_j 2^{j(-n+s+n/p'_1+n/p_2-\delta/p_2-a)q} \varphi(j)^q \|f\chi_j\|_{L^{p_1}(w^{p_1})}^q \\ &\quad \times \left(\sum_{k=-\infty}^{j-2} 2^{k(a+\delta/p_2)q} \right) \\ &\leq C\|f\|_{K_p^q(\varphi, w^{p_1})}, \end{aligned}$$

because $a + \delta/p_2 > 0$.

When $q > 1$, we take $\varepsilon > 0$ such that $a + \delta/p_2 - \varepsilon > 0$. Then we have

$$\begin{aligned} III^q &\leq C \sum_k 2^{k(a+\delta/p_2)q} \\ &\quad \times \left(\left(\sum_{j \geq k+2}^{\infty} 2^{j(-n+s+n/p'_1+n/p_2-\delta/p_2-a+\varepsilon)q} \varphi(j)^q \|f\chi_j\|_{L^{p_1}(w^{p_1})}^q \right) \right. \\ &\quad \times \left. \left(\sum_{j=k+2}^{\infty} 2^{-jq'\varepsilon} \right)^{q/q'} \right) \\ &\leq C\|f\|_{K_p^q(\varphi, w^{p_1})}. \end{aligned}$$

□

Proof of Theorem 2. We can estimate I and III same as above (note that $-n + s + nr/p_2 + b \leq 0$), therefore we omit the proof. We shall estimate II. We can take $\tilde{p}_1$ and $\tilde{p}_2$ such that $1/\tilde{p}_2 = 1/\tilde{p}_1 - s/n$, $\tilde{p}_1 < n/s$ and $\tilde{p}_2 > p_2$ (take $\tilde{p}_1$ sufficiently near n/s). Then $w \in A(\tilde{p}_1, \tilde{p}_2)$, therefore we can apply Proposition 2 to these indices. We have

$$\begin{aligned} \|(I_s(f\chi_j))\chi_k\|_{L^{p_2}(w^{p_2})} &\leq |B_k|^{1/p_2-1/\tilde{p}_2} \|(I_s(f\chi_j))\|_{L^{\tilde{p}_2}(w^{\tilde{p}_2})} \\ &\leq C|B_k|^{1/p_2-1/\tilde{p}_2} \|f\chi_j\|_{L^{\tilde{p}_1}(w^{\tilde{p}_1})} \\ &\leq C|B_k|^{1/p_2-1/\tilde{p}_2} |B_j|^{1/\tilde{p}_1-1/p_1} \|f\chi_j\|_{L^{p_1}(w^{p_1})} \\ &\leq C2^{js(1-q/p_1)} \|f\chi_j\|_{L^{p_1}(w^{p_1})}. \end{aligned}$$

This proves the theorem. □

7. Counterexamples

In this section we shall show that the condition (2') in Theorem D and the condition that $w \in A_{p(1-\alpha/n)}$ in Corollary 1 are the best possible.

For the simplicity we consider the case when $n = 1$. Let H be the Hilbert transform $Hf(x) = \int_{\mathbb{R}^1} f(y)/(x-y)dy$ and $I_s f(x) = \int_{\mathbb{R}^1} f(y)/|x-y|^{1-s}dy$.

Counterexample 1. *Theorem D is not true if $\alpha_2 \leq -1/p_2$.*

First, note that $\chi_{[0,1]} \notin K_{p_2}^{\alpha_2, q}(\mathbb{R}^1)$ when $\alpha_2 \leq -1/p_2$.

Let $f = \chi_{[1,2]}$, then $f \in K_{p_1}^{\alpha_1, q}(\mathbb{R}^1)$.

But $I_s f(x) \geq C > 0$ when $0 < x < 1$, therefore $I_s f \notin K_{p_2}^{\alpha_2, q}(\mathbb{R}^1)$.

Counterexample 2. *Theorem D is not true if $\alpha_2 \geq 1 - 1/p_1$.*

If $\alpha_2 \geq 1 - 1/p_1$, then $\alpha_1 \geq 1 - 1/p_1 + s(1 - q/p_1) > 0$, because $q < p_1$. Therefore $f = \chi_{[0,1]} \in K_{p_1}^{\alpha_1, q}(\mathbb{R}^1)$.

On the contrary, $I_s f(x) \geq C2^{k(-1+s)}$ where $2^{k-1} < x < 2^k$ and $k \geq 2$. We have

$$\|I_s f\|_{K_{p_2}^{\alpha_2, q}}^q \geq C \sum_{k=2}^{\infty} 2^{kq(\alpha_2 - 1 + s + 1/p_2)} = \infty,$$

because $\alpha_2 - 1 + s + 1/p_2 \geq s(1 - q/p_1) > 0$.

Counterexample 3. *Corollary 1 is not true if $w \in A_r$ where $r \geq p(1 - \alpha)$.*

Let $r = p(1 - \alpha) + \varepsilon$, where ε is a sufficiently small positive number, and we define $\beta = r - 1 - \varepsilon/2 = p(1 - \alpha) - 1 + \varepsilon/2$.

By Lemma 4, $w = |x|^\beta \in A_r$.

For $k < 1$, let $f_k(x) = \chi_{[2^{k-1}, 2^k]}(x)$. Then we have

$$(13) \quad \|f_k\|_{K_p^{\alpha, q}[1, w]} \leq C2^{k(\alpha + (\beta+1)/p)} = C2^{k(1 + \varepsilon/2p)}.$$

On the contrary, $Hf_k(x) \geq C2^k$ where $1 < x < 2$. Therefore we have

$$(14) \quad \|Hf_k\|_{K_p^{\alpha, q}[1, w]} \geq C2^k.$$

By (13) and (14), we know that H is not bounded on $K_p^{\alpha, q}[1, w](\mathbb{R}^1)$.

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School of High Technology for Human Welfare
 Tokai University
 317 Nishino Numazu
 Shizuoka 410-0395
 Japan
 (E-mail : komori@wing.ncc.u-tokai.ac.jp)

College of Economics, Nihon University
 1-3-2, Misaki-cho Chiyoda-ku
 Tokyo 101-8360
 Japan
 (E-mail : katsu@eco.nihon-u.ac.jp)

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