Marcinkiewicz integrals with variable kernels on
Hardy and weak Hardy spaces*

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Abstract. In this article, we consider the Marcinkiewicz integrals with variable kernels defined by

$$
\mu_{\Omega}(f)(x) = \left( \int_{0}^{\infty} \left| \int_{|x-y| \leq t} \frac{\Omega(x, x-y)}{|x-y|^{n-1}} f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2},
$$

where $\Omega(x, z) \in L^{\infty}(\mathbb{R}^n) \times L^q(S^{n-1})$ for $q > 1$. We prove that the operator $\mu_{\Omega}$ is bounded from Hardy space, $H^p(\mathbb{R}^n)$, to $L^p(\mathbb{R}^n)$ space; and is bounded from weak Hardy space, $H^{p,\infty}(\mathbb{R}^n)$, to weak $L^p(\mathbb{R}^n)$ space for $\max\left\{ \frac{2n+1}{n+1}, \frac{n+\alpha}{n} \right\} < p < 1$, if $\Omega$ satisfies the $L^{1,\alpha}$-Dini condition with any $0 < \alpha \leq 1$.

1. Introduction

Let $\mathbb{R}^n (n \geq 2)$ be the $n$-dimensional Euclidean space and $S^{n-1}$ denote the unit sphere in $\mathbb{R}^n$ equipped with induced Lebesgue measure $d\sigma$, and let $x' = \frac{x}{|x|}$ for any $x \neq 0$.

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In 1958, E. M. Stein [12] first introduced the following Marcinkiewicz integral $\mu_\omega$ of higher dimension with convolution kernel,

$$
\mu_\omega(f) = \left( \int_0^\infty \left| \int_{|x-y| \leq t} \frac{\omega(x-y)}{|x-y|^{n-1}} f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2},
$$

where $\omega(x)$ is a homogeneous function of degree zero with $\omega \in L^1(S^{n-1})$ and $\int_{S^{n-1}} \omega(x') d\sigma(x') = 0$.

E. M. Stein proved that $\mu_\omega$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \leq 2$ and is of weak type $(1,1)$ if $\omega \in \text{Lip}_\alpha(S^{n-1})$ with $0 < \alpha \leq 1$. Subsequently, A. Benedek, A. Calderon and R. Panzone [3] showed that if $\omega \in C^1(S^{n-1})$, then $\mu_\omega$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Later on, the above results were improved by many authors under some weaker smoothness conditions on $\omega$, see [1], [7, 8], [11], [14, 15, 16] for instance.

We remark that the Marcinkiewicz integral is essentially a Littlewood-Paley $g$-function. If let $\phi(x) = \omega(x)|x|^{-n+1}\chi_B(x)$ and $\phi_t(x) = t^{-n} \phi(x/t)$, where $B$ denotes the unit ball of $\mathbb{R}^n$ and $\chi_B$ denotes the characteristic function of $B$, then

$$
\mu_\omega(f)(x) = \left( \int_0^\infty |\phi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2} = g_\phi(f)(x).
$$

In order to study non-smoothness partial differential equations with variable coefficients, mathematicians pay more attention to the singular integral with variable kernels, see [2], [4], [5] and [6] among others. Specially, in 1955 Calderón and Zygmund [4] considered the singular integral with variable kernel defined by

$$
T_\Omega(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x, x-y)}{|x-y|^n} f(y) dy.
$$

In this paper, we study the Marcinkiewicz integral with variable kernel defined by

$$
\mu_\Omega(f)(x) = \left( \int_0^\infty \left| \int_{|x-y| \leq t} \frac{\Omega(x, x-y)}{|x-y|^{n-1}} f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2}.
$$

We point out that $\mu_\Omega$ can be interpreted as a Hilbert-valued function. In fact, denote the Hilbert space $\mathcal{H}$ by
Since the condition (2) implies (2′), so the \( n \) that for no
proved that if \( \Omega \) under certain Dini condition.

Then we obtain that \( \mu_\Omega (f) (x) = \| h_f (\cdot, x) \|_\mathcal{H} \).

Before stating our theorems, we first introduce some definitions about the
variable kernel \( \Omega (x, z) \). A function \( \Omega (x, z) \) defined on \( \mathbb{R}^n \times \mathbb{R}^n \) is said to
be in \( L^\infty (\mathbb{R}^n) \times L^q (S^{n-1}) \), \( q \geq 1 \), if \( \Omega (x, z) \) satisfies the following three
conditions:

1. \( \Omega (x, \lambda z) = \Omega (x, z) \), for any \( x, z \in \mathbb{R}^n \) and any \( \lambda > 0 \);
2. \( \| \Omega \|_{L^\infty (\mathbb{R}^n) \times L^q (S^{n-1})} = \sup_{r \geq 0, y \in \mathbb{R}^n} \left( \int_{S^{n-1}} |\Omega (rz + y, z')|^q d\sigma (z') \right)^{1/q} < \infty \);
3. \( \int_{S^{n-1}} \Omega (x, z') d\sigma (z') = 0 \) for any \( x \in \mathbb{R}^n \).

In [4], Calderón and Zygmund proved that if \( \Omega \) satisfies (1), (3) and

\[
(2') \quad \sup_{y \in \mathbb{R}^n} \left( \int_{S^{n-1}} |\Omega (y, z')|^q d\sigma (z') \right)^{1/q} < \infty,
\]

then \( T_\Omega \) is bounded on \( L^2 (\mathbb{R}^n) \) provided \( q \geq 2(n-1)/n \). They also found
that for no \( n \) can we replace the exponent \( 2(n-1)/n \) by a smaller one.

Since the condition (2) implies (2′), so the \( L^2 (\mathbb{R}^n) \) boundedness of \( T_\Omega \) holds
if \( \Omega \in L^\infty (\mathbb{R}^n) \times L^q (S^{n-1}) \) with \( q \geq 2(n-1)/n \). Recently in [7], the authors
proved that if \( \Omega \in L^\infty (\mathbb{R}^n) \times L^q (S^{n-1}) \) with \( q \geq 2(n-1)/n \), then \( \mu_\Omega \) is
bounded on \( L^2 (\mathbb{R}^n) \); and they also showed the \( H^1 - L^1 \) boundedness of \( \mu_\Omega \)
under certain Dini condition.

For \( 0 < \alpha \leq 1 \), a function \( \Omega \in L^\infty (\mathbb{R}^n) \times L^1 (S^{n-1}) \) is called to satisfy
the \( L^{1,\alpha} \)-Dini condition if

\[
(1.1) \quad \int_0^1 \frac{\varpi (\delta)}{\delta^{1+\alpha}} d\delta < \infty,
\]

where

\[
(1.2) \quad \varpi (\delta) = \sup_{r > 0, y \in \mathbb{R}^n \mid |r| \leq \delta} \int_{S^{n-1}} |\Omega (rz + y, O z') - \Omega (rz' + y, z')| d\sigma (z'),
\]

and \( O \) is a rotation in \( \mathbb{R}^n \) with \( |O| = \| O - I \| \), where \( I \) is the identity
operator. For the special case \( \alpha = 0 \), it reduces to the \( L^1 \)-Dini condition.
Our first aim is to show that the Marcinkiewicz integral $\mu_{\Omega}$ with variable kernel is bounded on Hardy spaces $H^p(\mathbb{R}^n)$ with some $p < 1$.

**Theorem 1.1.** Let $\Omega(x, z) \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ with $q > 2(n-1)/n$, and let $\Omega(x, z')$ satisfy the $L^{1,\alpha}$-Dini condition with $0 < \alpha \leq 1$. Then, if $\max\{\frac{2n}{2n+1}, \frac{n}{n+\alpha}\} < p < 1$, there exists an absolute constant $C$ independent of $f$ such that

$$\|\mu_{\Omega}(f)\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{H^p(\mathbb{R}^n)}.$$  

**Corollary 1.2.** Let $\omega(z) \in L^q(S^{n-1})$ with $q > 2(n-1)/n$, and let $\omega(z')$ satisfy $L^{1,\alpha}$-Dini condition with $0 < \alpha \leq 1$. Then, $\mu_{\omega}$ is bounded from Hardy space $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $\max\{\frac{2n}{2n+1}, \frac{n}{n+\alpha}\} < p < 1$.

Another aim of the paper is to derive that $\mu_{\Omega}$ is bounded from weak Hardy space $H^{p,\infty}(\mathbb{R}^n)$ to weak $L^p(\mathbb{R}^n)$ space, $L^{p,\infty}(\mathbb{R}^n)$, for some $p < 1$. Let us first recall the definition of weak Hardy space $H^{p,\infty}(\mathbb{R}^n)$.

**Definition 1.3.** Let $\varphi \in C_0^\infty$ with $\int \varphi(x)dx \neq 0$. Denote by $f^*_\ast(x) = \sup_{t > 0} |(\varphi_t * f)(x)|$, where $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$. A distribution $f$ is said to belong to the weak Hardy space $H^{p,\infty}(\mathbb{R}^n)$ if $f^*_\ast(x)$ belongs to the weak $L^p(\mathbb{R}^n)$ space, $L^{p,\infty}(\mathbb{R}^n)$, i.e., there is a constant $C > 0$ such that

$$|\{x \in \mathbb{R}^n : f^*_\ast(x) > \beta\}| \leq \frac{Cp}{\beta^p}, \quad \forall \ \beta > 0.$$  

The smallest constant $C$ satisfying the above inequality is called the $H^{p,\infty}$ norm of $f$, and is denoted by $\|f\|_{H^{p,\infty}}$.

**Theorem 1.4.** Let $\Omega(x, z) \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ with $q > 2(n-1)/n$, and let $\Omega(x, z')$ satisfy the $L^{1,\alpha}$-Dini condition with $0 < \alpha \leq 1$. Then, if $\max\{\frac{2n}{2n+1}, \frac{n}{n+\alpha}\} < p < 1$, there exists a constant $C$ independent of $f$ and $\beta$ such that

$$|\{x : |\mu_{\Omega}(f)(x)| > \beta\}| \leq C \frac{\|f\|^p_{H^{p,\infty}}}{\beta^p}.$$  

**Corollary 1.5.** Let $\omega(z) \in L^q(S^{n-1})$ for $q > 2(n-1)/n$, and let $\omega(z')$ satisfy $L^{1,\alpha}$-Dini condition with $0 < \alpha \leq 1$. Then $\mu_{\omega}$ is bounded from weak Hardy space $H^{p,\infty}(\mathbb{R}^n)$ to weak space $L^{p,\infty}(\mathbb{R}^n)$ for $\max\{\frac{2n}{2n+1}, \frac{n}{n+\alpha}\} < p < 1$.

**Remark 1.6.** It’s easy to see that $\text{Lip}_\alpha \subset L^{1,\alpha}$-Dini for any $0 < \alpha \leq 1$. Thus, the conclusions of Corollary 1.2 and 1.5 may be regarded as an improvement and extension of Stein’s results about the Marcinkiewicz integrals with convolution kernels in [12] and [8].
Remark 1.7. It is worthy noting that the $H^1 - L^1$ boundedness of $\mu_\Omega$ may be regarded as the limit case of Theorem 1.1 by choosing $p = 1$ and letting $\alpha \to 0$. Hence, Theorem 1.3 and Corollary 1.7 in [7] are the special cases of above Theorem 1.1.

Throughout the paper, $\mathcal{C}$ always denotes a positive constant not necessarily the same at each occurrence. We use $a \sim b$ to mean the equivalence of $a$ and $b$; that is, there exists a positive constant $C$ independent of $a, b$ such that $C^{-1}a \leq b \leq Ca$.

2. Proof of theorem 1.1

In order to show the $H^p - L^p$ boundedness of $\mu_\Omega$, we will use the atomic decomposition theory of the real Hardy space $H^p(\mathbb{R}^n)$ for $\frac{n}{n+1} < p \leq 1$, see for instance [13]. A function $a(x)$ is said to be $(p, 2, 0)$ atom if it satisfies the following three conditions:

(i) $\text{supp}(a) \subset B(x_0, r)$, where $B(x_0, r) = \{y \in \mathbb{R}^n : |y - x_0| \leq r\}$ is a ball in $\mathbb{R}^n$;
(ii) $\|a\|_{L^2} \leq |B(x_0, r)|^{1/2 - 1/p}$;
(iii) $\int_{\mathbb{R}^n} a(x)dx = 0$.

It is well known that every $f \in H^p(\mathbb{R}^n)$, $\frac{n}{n+1} < p \leq 1$, has an atomic decomposition $f = \sum \lambda_k a_k$, which converges in $H^p$ norm and in the sense of distributions; where $\left(\sum |\lambda_k|^{1/p}\right) \approx \|f\|_{H^p}$, and all $a_k(x)$ are $(p, 2, 0)$ atoms.

Start with $f$ in a nice dense class of function, say $f \in H^p(\mathbb{R}^n) \cap C^\infty_0(\mathbb{R}^n)$. If we denote the kernel by $K(x, z) = \frac{\partial^2}{|x-z|^2}$ and set

$$\mu_{\Omega, \epsilon}(f)(x) = \left(\int_0^\infty |F_{\Omega, t, \epsilon}(x)|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}},$$

where

$$F_{\Omega, t, \epsilon}(x) = \int_{|x-y| \leq t} K(x, x-y) f(y) dy.$$

Now following [13] (Page 115), we write the distribution kernel $K = K_0 + K_\infty$, where $K_0$ has compact support, and thus the distribution $F_{\Omega, t, \epsilon}$ is well defined for every fixed $\epsilon$ and $t$, and

$$F_{\Omega, t, \epsilon}(x) = \sum_k \lambda_k \int_{\epsilon < |x-y| \leq t} K(x, x-y) a_k(y) dy.$$
We claim that, for almost every \( x \in \mathbb{R}^n \), \( \mu_\Omega(f)(x) = \lim_{\varepsilon \to 0} \mu_{\Omega,\varepsilon}(f)(x) \).

To see the claim, we use the cancellation condition of \( \Omega \) and the fact \( \Omega(x,z) \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1}) \) for any \( 1 \leq r < 2(n-1)/n \) to obtain

\[
F_{\Omega,t,\varepsilon}(x) = \int_{|x-y| \leq t} \frac{\Omega(x,x-y)(f(y) - f(x))}{|x-y|^{n-1}} dy 
\leq C \int_{|x-y| \leq t} \frac{|\Omega(x,x-y)|}{|x-y|^{n-2}} dy 
\leq Ct^2.
\]

On the other hand, Hölder inequality gives

\[
F_{\Omega,t,\varepsilon}(x) \leq C \left( \int_{|x-y| \leq t} \frac{|\Omega(x,x-y)|^r}{|x-y|^{r(n-1)}} dy \right)^{1/r} 
\leq Ct^{-(n-1)+r(n/r)},
\]

for any \( 1 < r < n/(n-1) \). Therefore

\[
t^{-3}|F_{\Omega,t,\varepsilon}(x)|^2 \leq C(t\chi_{(0,1)}(t) + t^{-1-2n+2n/(r)}\chi_{(1,\infty)}(t))
\]
uniformly on \( \varepsilon \). So by the Lebesgue dominated convergence theorem we get that \( \mu_\Omega(f)(x) = \lim_{\varepsilon \to 0} \mu_{\Omega,\varepsilon}(f)(x) \).

Thus, by similar approximation arguments as in [9] (Theorem 7.3) and in [13] (Page 115), we can obtain

\[
(2.1) \quad \mu_\Omega(f) \leq \sum_k |\lambda_k| \mu_\Omega(a_k).
\]

Therefore, to derive the inequality (1.3) for any \( f \in H^p(\mathbb{R}^n) \) and prove Theorem 1.1, it suffices to show that for any \( (p,2,0) \) atom \( a(x) \), there exists a constant \( C > 0 \) independent of \( a(x) \) such that

\[
(2.2) \quad \| \mu_\Omega(a) \|_{L^p} \leq C.
\]

Without loss of generality, we let the support of the atom \( a(x) \) is \( B = B(0,r) \), and denote \( B^* = B(0,8r) \). Using the \( L^2 \) boundedness of \( \mu_\Omega \), we have

\[
(2.3) \quad \int_{B^*} |\mu_\Omega(a)(x)|^p dx \leq |B^*|^{1-p/2} \| \mu_\Omega(a) \|_{L^2}^p 
\leq C|B^*|^{1-p/2} \| a \|_{L^2}^p \leq C.
\]
We note that \((u + v)^s \leq u^s + v^s\) for any \(u, v \geq 0\) and \(0 \leq s \leq 1\). It is left to give the estimate for the integral

\[
I = \int_{(B^*)^c} |\mu_{\Omega}(a)(x)|^p dx
\]

\[
= \int_{(B^*)^c} \left( \int_0^\infty \int_{|x-y| \leq t} \frac{\Omega(x, x-y)}{|x-y|^{n-1}} a(y) dy \frac{1}{t^3} dt \right)^{p/2} dx
\]

\[
\leq \int_{(B^*)^c} \left( \int_0^{\|x\|+2r} \int_{|x-y| \leq t} \frac{\Omega(x, x-y)}{|x-y|^{n-1}} a(y) dy \frac{1}{t^3} dt \right)^{p/2} dx
\]

\[
+ \int_{(B^*)^c} \left( \int_{|x|+2r}^\infty \int_{|x-y| \leq t} \frac{\Omega(x, x-y)}{|x-y|^{n-1}} a(y) dy \frac{1}{t^3} dt \right)^{p/2} dx
\]

\[
:= I_1 + I_2.
\]

We note that, for \(y \in B\) and \(x \in (B^*)^c\), \(|x-y| \sim |x| \sim |x|+2r\). Thus by the mean value theorem we have

\[
\left| \frac{1}{|x-y|^2} - \frac{1}{(|x|+2r)^2} \right| \leq C \frac{r}{|x-y|^3}.
\]

Applying this inequality and the Minkowski’s inequality, we obtain that

\[
I_1 = \int_{(B^*)^c} \left( \int_0^{\|x\|+2r} \int_{|x-y| \leq t} \frac{\Omega(x, x-y)}{|x-y|^{n-1}} a(y) dy \frac{1}{t^3} dt \right)^{p/2} dx
\]

\[
\leq C \int_{(B^*)^c} \left( \int_B \left( \int_{|x-y|}^{\|x\|+2r} \frac{\Omega(x, x-y)}{|x-y|^{2(n-1)}} a(y) y^2 \frac{1}{t^3} dt \right)^{1/2} dy \right)^p dx
\]

\[
= C \int_{(B^*)^c} \left( \int_B \frac{\Omega(x, x-y)}{|x-y|^{n-1}} a(y) \left| \frac{1}{|x-y|^2} - \frac{1}{(|x|+2r)^2} \right|^{1/2} dy \right)^p dx
\]

\[
\leq C r^{\frac{p}{2}} \int_{(B^*)^c} \left( \int_B \frac{\Omega(x, x-y)}{|x-y|^{n+1}} a(y) dy \right)^p dx.
\]

Since \(1 > p > \frac{n}{n+\frac{1}{2}}\), so we can choose \(\varepsilon\) satisfying \(0 < \varepsilon < n + \frac{1}{2} - \frac{n}{p}\). Using Hölder inequality for integrals, we have

\[
I_1 \leq C r^{\frac{p}{2}} \int_{(B^*)^c} \left( \int_B \frac{\Omega(x, x-y)}{|x-y|^{n+\varepsilon}} \cdot |x-y|^{-\frac{\varepsilon}{2}} a(y) dy \right)^p dx
\]
\[
= C r^{\frac{p}{2}} \int_{(B^*)^c} \left( \int_B \frac{|\Omega(x, x - y)|}{|x - y|^{n+\varepsilon}} \cdot |a(y)|dy \right)^p \cdot |x|^{(\varepsilon - \frac{1}{2})p} dx \\
\leq C r^{\frac{p}{2}} \left( \int_{(B^*)^c} \int_B \frac{|\Omega(x, x - y)|}{|x - y|^{n+\varepsilon}} \cdot |a(y)|dy dx \right)^p \left( \int_{(B^*)^c} |x|^{(\varepsilon - \frac{1}{2})p} dx \right)^{1-p} \\
\leq C r^{\frac{p}{2}} \left( \int_B \int_{(B^*)^c} \frac{|\Omega(x, x - y)|}{|x - y|^{n+\varepsilon}} dx \cdot |a(y)|dy \right)^p \left( \int_{8r}^\infty t^{(\varepsilon - \frac{1}{2})p} \cdot t^{-n-1} dt \right)^{1-p} \\
\leq C r^{\frac{p}{2}} \|\Omega\|^p_{L^\infty \times L^1} \left( \int_B |a(y)|dy \int_{8r}^\infty t^{-n-\varepsilon t^{-1}n-1} dt \right)^p . \|\epsilon^{(\varepsilon - \frac{1}{2})p} + (1-p) \|a\|^p_{L^2} \leq C.
\]
where we have used that \(|x - y| \sim |x|\).

As to the estimate of \(I_2\). Noting that if \(t \geq |x| + 2r\), then \(B \subset \{ y : |x - y| < t \}\). So by the cancellation condition (iii) of \(a\), we have

\[
\int_{|x-y| < t} a(y)dy = 0.
\]

From this and Minkowski's inequality for integrals, we obtain

\[
I_2 = \int_{(B^*)^c} \left( \int_{|x|+2r}^\infty \left( \int_{|x-y| \leq t} \left( \frac{\Omega(x, x - y)}{|x - y|^{n-1}} - \frac{\Omega(x, x)}{|x - y|^{n-1}} \right) \cdot a(y)\right)^2 |y| \frac{1}{t^3} dt \right)^{p/2} dx \\
\leq I_{21} + I_{22},
\]
where

\[
I_{21} = \int_{(B^*)^c} \left( \int_{|x|+2r}^\infty \left( \int_{|x-y| \leq t} \left| \frac{\Omega(x, x - y) - \Omega(x, x)}{|x - y|^{n-1}} \right| a(y)\right)^2 \frac{1}{t^3} dt \right)^{p/2} dx,
\]
and

\[
I_{22} = \int_{(B^*)^c} \left( \int_{|x|+2r}^\infty \left( \int_{|x-y| \leq t} \left| \frac{\Omega(x, x - y) - \Omega(x, x)}{|x - y|^{n-1}} \right| a(y)\right)^2 \frac{1}{t^3} dt \right)^{p/2} dx.
\]
Applying Minkowski inequality for integrals, Hölder inequality for integrals and Fubini theorem successively, we can obtain

\[
I_{21} \leq \int_{(B^*)^c} \left| \int_B \frac{|\Omega(x, x - y) - \Omega(x, x)|}{|x - y|^n} |a(y)| dy \right|^p dx
\]

\[
\leq \sum_{j=1}^{+\infty} \int_{2^j r \leq |x| < 2^{j+1} r} \left[ \int_B \frac{|\Omega(x, x - y) - \Omega(x, x)|}{|x - y|^n} |a(y)| dy \right]^p dx
\]

\[
\leq \sum_{j=1}^{+\infty} \left( \int_{2^j r \leq |x| < 2^{j+1} r} \left[ \int_B \frac{|\Omega(x, x - y) - \Omega(x, x)|}{|x - y|^n} |a(y)| dy dx \right]^p (2^j r)^{(1-p)} \right)
\]

\[
= \sum_{j=1}^{+\infty} \left( \int_B \left( \int_{2^j r \leq |x| < 2^{j+1} r} \frac{|\Omega(x, x - y) - \Omega(x, x)|}{|x - y|^n} dx \right)^p |a(y)| dy \right) (2^j r)^{(1-p)}
\]

To estimate the inner integral above, we note \(|y| < r\) and \(|x| > 2r\), which implies,

\[
\left| \frac{x - y}{|x - y|} - \frac{x}{|x|} \right| = \left| \frac{(|x| - |x - y|) x - |x| y}{|x||x - y|} \right|
\]

\[
\leq \frac{|x| - |x - y| + |y|}{|x - y|} \leq \frac{4|y|}{|x|}
\]

And thus

\[
\int_{\mathbb{S}^{n-1}} |\Omega(x, x - y) - \Omega(x, x)| d\sigma(x')
\]

\[
= \int_{\mathbb{S}^{n-1}} \Omega \left( x, \frac{x - y}{|x - y|} \right) - \Omega \left( x, \frac{x}{|x|} \right) d\sigma(x') \leq \varpi \left( \frac{4|y|}{|x|} \right).
\]

This and a direct computation give

\[
\int_{2^j r \leq |x| < 2^{j+1} r} \frac{|\Omega(x, x - y) - \Omega(x, x)|}{|x - y|^n} dx
\]

\[
\leq C \int_{2^j r}^{2^{j+1} r} t^{-1} \varpi \left( \frac{4|y|}{t} \right) dt
\]

\[
= C \int_{\frac{4|y|}{2^j r}}^{\frac{4|y|}{2^{j+1} r}} \varpi(\delta) \frac{\delta d\delta}{\delta}
\]

Now using the condition \(\int_0^1 \frac{\varpi(\delta)}{\delta^p} d\delta < \infty\), and \(1 > p > \frac{n}{n+\alpha}\), we can get
\[ I_{21} \leq C \sum_{j=3}^{+\infty} \left( \int_B |a(y)| \left( \int_{\frac{|y|}{2^{j+3}r}}^{\frac{|y|}{2^{j-1}r}} \frac{\omega(\delta)}{\delta^{1+\alpha}} \delta^\alpha d\delta dy \right)^p \right) (2^j r)^{n(1-p)} \]

\[ \leq C \sum_{j=3}^{+\infty} (2^j r)^{n(1-p)} (2^j)^{-\rho_\alpha} \left( \int_B |a(y)| dy \right)^p \]

\[ \leq C r^{n(1-p)} \left( \int_B |a(y)| dy \right)^p \leq C. \]

At last, we give the estimate of \( I_{22} \). Obviously, mean value theorem gives

\[ \frac{1}{|x-y|^{n-1}} - \frac{1}{|x|^{n-1}} \leq C \frac{|y|}{|x|^n}. \]

Applying this inequality, Minkowski's inequality for integrals, and the fact \( \frac{n}{n+1} < p < 1 \), we deduce that

\[ I_{22} \leq \int_{|x|>8r} \left( \int_{|x|+2r}^{+\infty} \left( \int_{|x-y| \leq t} \frac{|\Omega(x,x)||y|}{|x|^{n}} |a(y)| dy \right)^2 \frac{dt}{t^3} \right)^{p/2} dx \]

\[ \leq C \int_{|x|>8r} \left( \int_B \frac{|\Omega(x,x)||y|}{|x|^{n+1}} |a(y)| dy \right)^p dx \]

\[ \leq C r^p \left( \int_{|x|>8r} |\Omega(x,x)|^p |x|^{(-n-1)p} dx \right) \left( \int_B |a(y)| dy \right)^p \]

\[ \leq C r^p \left( \int_{|x|>8r} |\Omega(x,x)|^p |x|^{(-n-1)p} dx \right) \left( \int_B |a(y)| dy \right)^p \]

\[ \leq C. \]

Combining (2.3), and the estimates of \( I_1, I_{21} \) and \( I_{22} \), we get (2.2) and then complete the proof of Theorem 1.1.

### 3. Proof of theorem 1.4

In order to prove theorem 1.4, we need the following decomposition theorem for distributions in \( H^{p,\infty}(\mathbb{R}^n) \).

**Lemma 3.1.** [10] Given a distribution \( f \in H^{p,\infty}(\mathbb{R}^n) \), there exits a bounded function sequence \( \{f_k\}_{k=-\infty}^{+\infty} \) which has the following properties:

(a) \( f = \sum_k f_k \) in the sense of distributions;
(b) Each \( f_k \) may be further decomposed in \( L^p \) as \( f_k = \sum_i h_i^k \), where \( \{h_i^k\} \) satisfies the following three conditions:

(i) \( \text{supp}(h_i^k) \subset B_i^k := B(x_i^k, r_i^k) \), where \( B(x, r) \) denotes the ball in \( \mathbb{R}^n \) with the center at \( x \) and radius \( r \). Moreover, \( \sum_i |B_i^k| \leq C_1 2^{-kp} \), where \( C_1 \sim \|f\|_{H^p, \infty} \) and \( \sum_i \chi_{B_i^k}(x) \leq C \);

(ii) \( \|h_i^k\|_p \leq C2^k \);

(iii) \( \int h_i^k(x)dx = 0 \).

We now give the proof of Theorem 1.4. For any \( f \in H^{p, \infty}(\mathbb{R}^n) \) and \( \beta > 0 \), we choose \( k_0 \) satisfying \( 2^{k_0} \leq \beta < 2^{k_0+1} \). Applying Lemma 3.1, we can write

\[
f = \sum_{k=-\infty}^{k_0} f_k + \sum_{k=k_0+1}^{+\infty} f_k = F_1 + F_2, \quad f_k = \sum_i h_i^k,
\]

where \( h_i^k \) satisfies the properties (i), (ii) and (iii) of Lemma 3.1.

Let \( A_k = \text{supp}(f_k) \), then \( A_k = \bigcup_i B_i^k \) and \( |A_k| \leq \sum_i |B_i^k| \leq C2^{-kp}\|f\|_{H^p, \infty} \), also \( \|f_k\|_\infty \leq C2^k \), so we can get

\[
\|F_1\|_2 \leq \sum_{k=-\infty}^{k_0} \|f_k\|_2 \leq C \sum_{k=-\infty}^{k_0} 2^k |A_k|^{\frac{2}{p}} \leq C \sum_{k=-\infty}^{k_0} 2^{k(1-\frac{2}{p})}\|f\|_{H^p, \infty}^\frac{2}{p} \leq C\|f\|_{H^p, \infty}^{\frac{2}{p}} \beta^{1-\frac{2}{p}}.
\]

By the \( L^2 \)-boundedness of \( \mu_{\Omega} \), it follows

\[
\text{(3.1)} \quad \{|x: \mu_{\Omega}(F_1)(x) > \beta| \leq \|\mu_{\Omega}(F_1)\| \leq C \frac{\|F_1\|_2^2}{\beta^2} \leq C \|f\|_{H^p, \infty}^p \beta^{-p}.
\]

On the other hand, we denote \( B_i^k = B(x_i^k, 2R_i^k) \) and \( R_i^k = \left( \frac{4}{3} \right)^{k-k_0} r_i^k \), and let \( B_{k_0} = \bigcup_{k=k_0+1}^{+\infty} \bigcup_i B_i^k \), we can get

\[
|B_{k_0}| \leq C \sum_{k=k_0+1}^{+\infty} \sum_i |B_i^k| \leq C \sum_{k=k_0+1}^{+\infty} \sum_i 2^n (\frac{4}{3})^{k-k_0} |B_i^k|
\]

\[
\text{(3.2)} \quad \leq C \sum_{k=k_0+1}^{+\infty} \sum_i (\frac{4}{3})^{k-k_0} |B_i^k| \leq C \sum_{k=k_0+1}^{+\infty} (\frac{4}{3})^{k-k_0} 2^{-kp} \|f\|_{H^p, \infty} \leq C \frac{\|f\|_{H^p, \infty}^p}{\beta^p},
\]

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where the last inequality holds owing to \( p > \frac{n}{n+\frac{3}{2}} \). Therefore, in order to prove Theorem 1.4, it suffices to show

\[
(3.3) \quad |\{ x \in (B_{y_0})^c : \mu_\Omega(F_2)(x) > \beta \}| \leq C \frac{\| f \|_{\mathcal{H}^{p,\infty}}}{\beta p}.
\]

Firstly, a similar argument as the one used in (2.1) and Minkowski’s inequality for series give

\[
(3.4) \quad \int_{(B_{y_0})^c} |\mu_\Omega(F_2)(x)|^p dx \leq \int_{(B_{y_0})^c} \sum_{k=k_0+1}^{+\infty} \sum_{i} |\mu_\Omega(h^k_i)(x)|^p dx \\
\leq C \sum_{k=k_0+1}^{+\infty} \sum_{i} (J_1 + J_2),
\]

where

\[
J_1 = \int_{(B_{y_0})^c} \left| \int_{|x-y| \leq t} \frac{\Omega(x, x-y)}{|x-y|^{n-1}} h^k_i(y) dy \right|^2 \frac{dt}{t^3} dx,
\]

and

\[
J_2 = \int_{(B_{y_0})^c} \left| \int_{|x-y| \leq t} \frac{\Omega(x, x-y)}{|x-y|^{n-1}} h^k_i(y) dy \right|^2 \frac{dt}{t^3} dx.
\]

It is obvious that when \( y \in B^k_i, x \in B^c_{y_0} \), we will have \( |x-x^k_i| \sim |x-y| \sim |x-x^k_i| + 2r_i^k \) and

\[
\left| \frac{1}{(|x-x^k_i| + 2r_i^k)^2} - \frac{1}{|x-y|^2} \right| \leq C \frac{r_i^k}{|x-y|^3}.
\]

Because of \( \frac{n}{n+\frac{3}{2}} < p < 1 \), one can choose proper \( \varepsilon \) such that \( 0 < \varepsilon < n + \frac{1}{2} - \frac{3}{p} \). Combining the above inequality and Minkowski’s inequality for integrals, the size condition of \( h^k_i \) and Hölder inequality for integrals, we deduce that

\[
J_1 \leq \int_{(B_{y_0})^c} \left| \int_{B^k_i} \frac{|\Omega(x, x-y)|}{|x-y|^{n-1}} \left| h^k_i(y) \right| \left| \frac{(r_i^k)^{\frac{3}{2}}}{|x-y|^\frac{3}{2}} \right| dy \right|^p dx \\
\leq C 2^{kp} (r_i^k)^{\frac{3}{2}} \int_{(B_{y_0})^c} \left| \int_{B^k_i} \frac{|\Omega(x, x-y)|}{|x-y|^{n+\varepsilon}} \left| x-x^k_i \right|^{-\frac{3}{2} - \frac{3}{2}} dy \right|^p dx
\]
\[ = C 2^{kp} (r_i^k)^\frac{2}{3} \int_{(B_k^c)^c} \left| \int_{B_k^c} \left| \frac{\Omega(x, x - y)}{|x - y|} \right| dy \right|^p \left| x - x_i^k \right|^{-\frac{1}{2}} p dx \]
\[ \leq C 2^{kp} (r_i^k)^\frac{2}{3} \left( \int_{(B_k^c)^c} \left| \frac{\Omega(x, x - y)}{|x - y|^{n + \epsilon}} \right| dy dx \right)^{\frac{p}{2}} \left( \int_{(B_k^c)^c} \left| x - x_i^k \right|^{\frac{2}{p} + 1 + \frac{1}{p}} dx \right)^{1 - p} \]
\[ = C 2^{kp} (r_i^k)^\frac{2}{3} \left( \int_{B_k^c} \int_{(B_k^c)^c} \left| \frac{\Omega(x, x - y)}{|x - y|^{n + \epsilon}} \right| dy dx \right)^{\frac{p}{2}} \left( \int_{2R_i^k}^{+\infty} r^{\frac{(1 - p) p + n - 1}{p}} dr \right)^{1 - p} \]
\[ \leq C 2^{kp} (r_i^k)^\frac{2}{3} \left| B_i^k \right|^p \Omega \left( \int_{2R_i^k}^{+\infty} r^{\frac{1 - \epsilon - 1}{p}} dr \right)^{\frac{p}{2}} \left( 2 R_i^k \right)^{n(1 - p) - \frac{2}{3}} \leq C 2^{kp} \left| B_i^k \right|^p \left( \frac{3}{4} \right)^{(k - k_0)(n p + \frac{2}{3} - n)} \]

It follows

\[ \sum_{k = k_0 + 1}^{+\infty} \sum_{t} J_1 \leq C \sum_{k = k_0 + 1}^{+\infty} \left( \frac{3}{4} \right)^{(k - k_0)(n p + \frac{2}{3} - n)} \| f \|^p_{L^{\infty}} \leq C \| f \|^p_{H^{p, \infty}} \]

Next we estimate \( J_2 \). One notes that \( B_k^c \subset \{ y; |x - y| < t \} \) since \( x \in B_k^c \) and \( t > |x - x_i^k| + 2 r_i^k \). By the cancellation of \( h_i^k \) and Minkowski’s inequality for integrals, we can get

\[ J_2 = \int_{(B_k^c)^c} \int_{|x - y| \leq t} \left| \frac{\Omega(x, x - y)}{|x - y|^{n - 1}} h_i^k(y) \right| dy \left| \frac{dt}{t^3} \right| dx \]
\[ = \int_{(B_k^c)^c} \int_{|x - y| \leq t} \left( \frac{\Omega(x, x - y)}{|x - y|^{n - 1}} - \frac{\Omega(x, x - x_i^k)}{|x - x_i^k|^{n - 1}} \right) h_i^k(y) dy \left| \frac{dt}{t^3} \right| dx \]
\[ \leq C 2^{kp} \int_{(B_k^c)^c} \left| \frac{1}{|x - x_i^k|} \right| \int_{B_k^c} \left| \frac{\Omega(x, x - y)}{|x - y|^{n - 1}} - \frac{\Omega(x, x - x_i^k)}{|x - x_i^k|^{n - 1}} \right| dy \left| \frac{dt}{t^3} \right| dx \]
\[ + C 2^{kp} \int_{(B_k^c)^c} \left| \frac{1}{|x - x_i^k|} \right| \int_{B_k^c} \left| \frac{\Omega(x, x - y)}{|x - y|^{n - 1}} \right| dy \left| \frac{dt}{t^3} \right| dx \]
\[ = C 2^{kp} (J_{2, 1} + J_{2, 2}). \]
Decompose $J_{2,1}$ as following

\[
J_{2,1} = \sum_{j=1}^{+\infty} \int_{2^{-j} R_i^k \leq |x-x_i^k| \leq 2^{j+1} R_i^k} \left| \int_{B_i^k} \frac{\Omega(x, x-y)}{|x-x_i^k|^n} - \frac{\Omega(x, x-x_i^k)}{|x-x_i^k|^n} \right| dx
\]

\[
\leq \sum_{j=1}^{+\infty} (2^j R_i^k)^{n(1-p)} \left( \int_{2^{-j} R_i^k \leq |x-x_i^k| \leq 2^{j+1} R_i^k} \left| \int_{B_i^k} \frac{\Omega(x, x-y)}{|x-x_i^k|^n} \right| dx \right)^p
\]

Observe that $|y-x_i^k| < r_i^k$ and $|x-x_i^k| > R_i^k \geq 2r_i^k$. Using a similar argument as in inequalities (2.4) and (2.5), we can deduce that

\[
\left| \frac{x-y}{|x-y|} - \frac{x-x_i^k}{|x-x_i^k|} \right| \leq 4 \left| \frac{y-x_i^k}{|x-x_i^k|} \right|
\]

and then

\[
\int_{2^{-j} R_i^k \leq |x-x_i^k| \leq 2^{j+1} R_i^k} \left| \frac{\Omega(x, x-y)}{|x-x_i^k|^n} - \frac{\Omega(x, x-x_i^k)}{|x-x_i^k|^n} \right| dx
\]

\[
= \int_{2^{j} R_i^k}^{2^{j+1} R_i^k} \int_{2^{-j} R_i^k \leq |x-x_i^k| \leq 2^{j+1} R_i^k} \left| \Omega(rx' + x_i^k, \frac{rx' + x_i^k - y}{|rx' + x_i^k - y|}) - \Omega(rx', x_i^k) \right| d\sigma(x') r^{-1} dr
\]

\[
\leq C \int_{2^{j} R_i^k}^{2^{j+1} R_i^k} \varpi \left( \frac{|y-x_i^k|}{r} \right)^{-1} r^{-1} dr
\]

\[
= C \int_{2^{-j} R_i^k}^{2^{j+1} R_i^k} \varpi \left( \frac{|y-x_i^k|}{\delta} \right)^{-1} \varpi(\delta) \frac{|y-x_i^k|}{\delta^2} d\delta
\]

\[
= C \int_{2^{-j} R_i^k}^{2^{j+1} R_i^k} \varpi(\delta) \frac{|y-x_i^k|}{\delta^{1+\alpha}} d\delta \leq C \left( \frac{r_i^k}{2^j R_i^k} \right)^{\alpha}.
\]

This follows

\[
J_{2,1} \leq C \sum_{j=1}^{+\infty} (2^j R_i^k)^{n(1-p)} \left( \frac{x_i^k}{2^j R_i^k} \right)^{\alpha p} |B_i^k|^p
\]

\[
\leq C |B_i^k| \left( \frac{3}{4} \right)^{k \cdot \frac{\alpha}{np} (np-n+\alpha)} \cdot \sum_{j=1}^{+\infty} 2^{j(n-np-\alpha p)} \leq C |B_i^k| \left( \frac{3}{4} \right)^{k \cdot \frac{\alpha}{np} (np-n+\alpha)}.
\]
To deal with $J_{2,2}$, we use the mean value theorem, Hölder inequality for integrals, and the condition $\frac{n}{n+\frac{2}{p}} < p < 1$ to give

$$J_{2,2} = \int_{(B_{0})^c} \left( \frac{1}{|x-x_i^k|} \int_{B_{i}^k} \frac{\Omega(x, x-y) - \Omega(x, x-y)}{|x-y|^{n-1}} dy \right)^p dx$$

$$\leq C \int_{(B_{0})^c} \left( \frac{1}{|x-x_i^k|} \int_{B_{i}^k} \frac{\Omega(x, x-y)}{|x-x_i^k|^{n+\frac{2}{p}}} dy \right)^p dx$$

$$= C(r_i^k)^p \int_{(B_{0})^c} |x-x_i^k|^{-\frac{2}{p}} \left( \int_{B_{i}^k} \frac{\Omega(x, x-y)}{|x-x_i^k|^{n+\frac{2}{p}}} dy \right)^p dx$$

$$\leq (r_i^k)^p \left( \int_{(B_{0})^c} \int_{B_{i}^k} |\Omega(x, x-y)| dx \right)^p \left( \int_{(B_{0})^c} |x-x_i^k|^{-\frac{2}{p}+n(1-p)} dx \right)$$

$$\leq C(r_i^k)^p |B_{i}^k|^p \left( \frac{4}{3} \right)^{-\frac{k}{n}} r_i^k \frac{k}{n} - \frac{2}{n} + n(1-p) = C |B_i^k| \left( \frac{3}{4} \right)^{\frac{k}{n}}.$$ 

Combining the estimates of $J_{2,1}$ and $J_{2,2}$, we obtain that

$$\sum_{k=k_0+1}^{+\infty} \sum_i J_2 \leq C \sum_{k=k_0+1}^{+\infty} \sum_i 2^{kp} (J_{2,1} + J_{2,2})$$

$$\leq C \left( \sum_{k=k_0+1}^{+\infty} \left( \left( \frac{3}{4} \right)^{\frac{k}{n}} \right)^{\frac{k}{n}} \right) \|f\|_{L^p}^p$$

$$\leq C \|f\|_{L^p}^p.$$

Obviously, the inequalities (3.5), (3.6) and (3.4) yield the inequality (3.3) and the result follows. 

\[\square\]

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