Asymptotic analysis for a weakly damped wave equation with application to a problem arising in elasticity

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Abstract. The present work is devoted to the study of homogenization of the weakly damped wave equation

$$\begin{aligned} & \int_{\Omega} \rho^{\varepsilon} \frac{\partial^2 u_{\varepsilon}}{\partial t^2}(t) \cdot v dx + 2\varepsilon^2 \mu \int_{\Omega_f^{\varepsilon}} E_{ij} \left(\frac{\partial u_{\varepsilon}}{\partial t}(t) \right) E_{ij}(v) dx \\ & + \varepsilon^2 \lambda \int_{\Omega_f^{\varepsilon}} \operatorname{div} \left(\frac{\partial u_{\varepsilon}}{\partial t}(t) \right) \operatorname{div} v dx + \vartheta \int_{\Omega_f^{\varepsilon}} \operatorname{div}(u_{\varepsilon}(t)) \operatorname{div} v dx \\ & = \int_{\Omega} f(t) \cdot v dx \text{ for all } v = (v^1, v^2, v^3) \in V_{\varepsilon} \quad (0 < t < T), \end{aligned}$$

with initial conditions

$$u_{\varepsilon}(0) = \frac{\partial u_{\varepsilon}}{\partial t}(0) = \omega \text{ (the origin in } \mathbb{R}^3 \text{)}.$$

Convergence homogenization results are achieved using the two-scale convergence theory.

1. Introduction

Our main objective in the present work is to prove a homogenization theorem for the weakly damped wave equation

$$(1.1) \quad \int_{\Omega} \rho^{\varepsilon} \frac{\partial^2 u_{\varepsilon}}{\partial t^2}(t) \cdot v dx + \varepsilon^2 b^{\varepsilon} \left(\frac{\partial u_{\varepsilon}}{\partial t}(t), v \right) + a^{\varepsilon}(u_{\varepsilon}(t), v) = \int_{\Omega} f(t) \cdot v dx$$

for all $v = (v^1, v^2, v^3) \in V_{\varepsilon}$ ($0 < t < T$),

with initial conditions

$$(1.2) \quad u_{\varepsilon}(0) = \frac{\partial u_{\varepsilon}}{\partial t}(0) = \omega \text{ (the origin in } \mathbb{R}^3),$$

where Ω is a smooth bounded open set in the numerical space $\mathbb{R}_x^3$ of variables $x = (x_1, x_2, x_3)$, T is a fixed positive real, the dot denotes the usual Euclidean inner product in $\mathbb{R}^3$, $f = (f^1, f^2, f^3)$ is given in $L^2(0, T; \mathbb{L}^2(\Omega; \mathbb{R}))$ (with $\mathbb{L}^2(\Omega; \mathbb{R}) = L^2(\Omega; \mathbb{R}^3)$), dx denotes the Lebesgue measure on $\mathbb{R}_x^3$, the functions ρ^{ε} , a^{ε} , b^{ε} , and the set Ω_s^{ε} are defined farther on and

$$(1.3) \quad V_{\varepsilon} = \left\{ v \in \mathbb{H}_0^1(\Omega; \mathbb{R}) = H_0^1(\Omega; \mathbb{R})^3 : E_{ij}(v) = 0 \text{ in } \Omega_s^{\varepsilon} \text{ (} i, j = 1, 2, 3) \right\}$$

with $E_{ij}(v) = \frac{1}{2} \left(\frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i} \right) \quad i, j = 1, 2, 3.$

In what follows, all vector spaces F are assumed to be complex vector spaces, and $\mathbb{F}$ denotes the product $F \times F \times F$ (three times). If X denotes a locally compact space, we will write $\mathcal{C}(X) = \mathcal{C}(X; \mathbb{C})$ for the space of continuous mappings from X into $\mathbb{C}$ (complex numbers) and $\mathcal{B}(X) = \mathcal{B}(X; \mathbb{C})$ for that of bounded continuous mappings from X into $\mathbb{C}$. Likewise the usual spaces $L^2(X; \mathbb{C})$, $L_{loc}^2(X; \mathbb{C})$, $H^n(X; \mathbb{C})$ and $H_{loc}^n(X; \mathbb{C})$ (integer $n \geq 1$, X provided with a positive Radon measure β) will be denoted by $L^2(X)$, $L_{loc}^2(X)$, $H^n(X)$ and $H_{loc}^n(X)$. The corresponding subspaces of real functions will be denoted by $L^2(X; \mathbb{R})$, $L_{loc}^2(X; \mathbb{R})$, $H^n(X; \mathbb{R})$ and $H_{loc}^n(X; \mathbb{R})$. In the definitions of other function spaces, the functions are assumed to be measurable. If A is any Banach space, $\mathcal{C}(X; A)$ will denote the space of all continuous functions from X into A , and $L^2(X; A)$ will denote the space of all functions $u : X \rightarrow A$ with $\int_X \|u(x)\|_A^2 d\beta(x) < \infty$, $\|\cdot\|_A$ being the norm on A . Furthermore, we systematically use the convention of Einstein. For basic concepts and notations concerning integration theory we refer to [3, 4].

Let us now give a precise description of the problem. Let $\alpha, \lambda, \mu, \rho_s, \rho_f, \vartheta, c_0, b_{ijkh}$ be real numbers with

$$(1.4) \quad b_{ijkh} = \lambda \delta_{ij} \delta_{kh} + \mu (\delta_{ik} \delta_{jh} + \delta_{ih} \delta_{jk}) \text{ for } i, j, k, h = 1, 2, 3$$

(δ_{ij} being the Kronecker symbol), and

$$(1.5) \quad 0 < \alpha < 1, \mu > 0, \rho_s > 0, \rho_f > 0, \frac{\lambda}{\mu} \geq \left(-\frac{2}{3}\right) \alpha, \vartheta = c_0^2 \rho_f.$$

Then the symmetry condition

$$(1.6) \quad b_{ijkh} = b_{jikh} = b_{ijhk} = b_{khij}$$

and the ellipticity condition

$$(1.7) \quad b_{ijkh} \xi_{kh} \xi_{ij} \geq \delta \xi_{ij} \xi_{ij} \text{ (with } \delta = 2\mu(1 - \alpha) \text{) for all symmetric matrices of real numbers } (\xi_{ij})$$

follow immediately. Let Y_s be a smooth open set in $\mathbb{R}_y^3$ (the 3-dimensional numerical space of variables $y = (y_1, y_2, y_3)$) such that

$$(1.8) \quad \overline{Y}_s \subset Y = \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \quad (\overline{Y}_s \text{ the closure of } Y_s),$$

and let Y_f be the open set $Y \setminus \overline{Y}_s$. Let $\rho \in L^\infty(\mathbb{R}_y^3)$ be given by

$$(1.9) \quad \rho(y) = \rho_s \text{ in } \tilde{Y}_s \text{ and } \rho_f \text{ in } \tilde{Y}_f,$$

where, denoting by $\mathbb{Z}$ the set of integers, we denote

$$\tilde{Y}_s = \bigcup_{k \in \mathbb{Z}^3} (k + Y_s) \text{ and } \tilde{Y}_f = \mathbb{R}_y^3 \setminus \overline{\tilde{Y}_s} \quad (\overline{\tilde{Y}_s} \text{ the closure of } \tilde{Y}_s \text{ in } \mathbb{R}_y^3).$$

For fixed $\varepsilon > 0$, let ρ^ε be the function in $L^\infty(\mathbb{R}_x^3)$ defined by

$$\rho^\varepsilon(x) = \rho\left(\frac{x}{\varepsilon}\right) \quad (x \in \mathbb{R}^3).$$

Let Ω be as above. We endow $\mathbb{H}^1(\Omega; \mathbb{R}) = H^1(\Omega; \mathbb{R})^3$ with the norm

$$\|v\| = \left(\int_{\Omega} v^i v^i dx + \int_{\Omega} E_{ij}(v) E_{ij}(v) dx \right)^{\frac{1}{2}} \quad (v = (v^i) \in \mathbb{H}^1(\Omega; \mathbb{R})).$$

Thanks to Korn's inequality (see [14]), the norm $|||\cdot|||$ is equivalent to the usual $\mathbb{H}^1(\Omega)$ -norm; and the seminorm

$$|||v|||_0 = \left(\int_{\Omega} E_{ij}(v) E_{ij}(v) dx \right)^{\frac{1}{2}} \quad (v \in \mathbb{H}^1(\Omega; \mathbb{R}))$$

is a norm in $\mathbb{H}_0^1(\Omega; \mathbb{R})$ equivalent to the $\mathbb{H}^1(\Omega)$ -norm.

For $\varepsilon > 0$, we set

$$\Omega_s^\varepsilon = \Omega \cap \varepsilon \tilde{Y}_s \quad \text{and} \quad \Omega_f^\varepsilon = \Omega \cap \varepsilon \tilde{Y}_f,$$

which defines two open subsets of Ω . Let b^ε and a^ε be the two continuous bilinear forms on $\mathbb{H}^1(\Omega; \mathbb{R}) \times \mathbb{H}^1(\Omega; \mathbb{R})$ defined by

$$b^\varepsilon(u, v) = \int_{\Omega_f^\varepsilon} b_{ijkh} \frac{\partial u^k}{\partial x_h} \frac{\partial v^i}{\partial x_j} dx = \lambda \int_{\Omega_f^\varepsilon} \operatorname{div} u \operatorname{div} v dx + 2\mu \int_{\Omega_f^\varepsilon} E_{ij}(u) E_{ij}(v) dx$$

and

$$a^\varepsilon(u, v) = \vartheta \int_{\Omega_f^\varepsilon} \operatorname{div} u \operatorname{div} v \, dx, \text{ respectively,}$$

where $\operatorname{div} u = \frac{\partial u^1}{\partial x_1} + \frac{\partial u^2}{\partial x_2} + \frac{\partial u^3}{\partial x_3}$ for $u = (u^i)$. It is clear that the space V_ε in (1.3) is a closed vector subspace of $\mathbb{H}_0^1(\Omega; \mathbb{R})$, and the seminorm

$$|||v|||_\varepsilon = \left(\int_{\Omega_f^\varepsilon} E_{ij}(v) E_{ij}(v) dx \right)^{\frac{1}{2}} \quad (v \in \mathbb{H}_0^1(\Omega; \mathbb{R}))$$

is a norm on V_ε equivalent to $|||\cdot|||_0$.

By the Galerkin method one can prove the following existence and uniqueness result (see [2] or [13]) :

Theorem 1. *For each given $\varepsilon > 0$, there exists one and only one function u_ε from $[0, T]$ into V_ε defined by (1.1)-(1.2) and satisfying $u_\varepsilon \in \mathcal{C}([0, T]; V_\varepsilon)$, $\frac{\partial u_\varepsilon}{\partial t} \in \mathcal{C}([0, T]; \mathbb{L}^2(\Omega; \mathbb{R})) \cap L^2(0, T; V_\varepsilon)$, $\frac{\partial^2 u_\varepsilon}{\partial t^2} \in L^2(0, T; \mathbb{L}^2(\Omega; \mathbb{R}))$.*

The periodic homogenization of (1.1)-(1.2), together with many other hyperbolic problems, is studied in Sanchez-Palencia [14]. Here, the homogenization method is the pioneering approach using multiple-scale asymptotic expansions. We also refer to [11] for the homogenization, by means of two-scale convergence, of a mixture of solid and fluid. Another study of interest on this topic is presented in [8], where the homogenization of hyperbolic-parabolic equations in perforated domains is worked out.

The paper is organized as follows. Section 2 is devoted to some fundamentals of homogenization. In section 3 we study the homogenization problem for (1.2)-(1.2) in the periodic setting when $f \in \mathcal{C}^1([0, T]; \mathbb{L}^2(\Omega; \mathbb{R}))$ with $f(0) = \omega$. Section 4 is concerned with the study of the periodic homogenization problem for (1.1)-(1.2) in the general case $f \in L^2(0, T; \mathbb{L}^2(\Omega; \mathbb{R}))$.

2. Fundamentals of homogenization

The aim is to study the homogenization of (1.1)-(1.2) by using the two-scale convergence method (see [1, 8, 12]). We will need the notion of weak two-scale convergence in $L^2(Q)$. That is, given $(v_\varepsilon)_{\varepsilon>0} \subset L^2(Q)$ and $v_0 \in L^2(Q; L^2_{per}(Z))$, where

$$\begin{aligned} Q &= \Omega \times (0, T), \\ Z &= Y \times \Theta \text{ with } \Theta = \left(-\frac{1}{2}, \frac{1}{2}\right), \\ L^2_{per}(Z) &= \left\{v \in L^2_{loc}(\mathbb{R}^4) : v \text{ is } Z\text{-periodic}\right\}, \end{aligned}$$

we will say that $v_\varepsilon \rightarrow v_0$ in $L^2(Q)$ -weak two-scale if, as $\varepsilon \rightarrow 0$,

$$(2.1) \quad \int_Q v_\varepsilon \Phi^\varepsilon dx dt \rightarrow \iint_{Q \times Z} v_0 \Phi dx dt dy d\tau$$

for every $\Phi \in L^2(Q; \mathcal{C}_{per}(Z))$, $\mathcal{C}_{per}(Z) = \{v \in \mathcal{C}(\mathbb{R}^4) : v \text{ is } Z\text{-periodic}\}$, where $\Phi^\varepsilon(x, t) = \Phi(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon})$ in the sense of [12, Proposition 4.1], $(x, t) \in Q$, $\varepsilon > 0$. In particular (2.1) makes sense for $\Phi \in L^2(Q; \mathcal{C}_{per}(Y)) \otimes \mathcal{C}_{per}(\Theta)$, or for $\Phi \in \mathcal{C}(\bar{Q}; L^\infty_{per}(Z))$ [12, Proposition 4.5].

We recall the following fundamental results :

Lemma 1. *Let E be a fundamental sequence, i.e., E is an ordinary sequence of positive real numbers tending to 0; and suppose the sequence $(v_\varepsilon)_{\varepsilon \in E} \subset L^2(Q)$ is bounded in $L^2(Q)$. Then, a subsequence E' can be extracted from E such that the sequence $(v_\varepsilon)_{\varepsilon \in E'}$ is weakly two-scale convergent in $L^2(Q)$.*

Proof. See, e.g., [7]. □

Lemma 2. *Let $(v_\varepsilon)_{\varepsilon \in E}$ be a bounded sequence in $L^2(Q)$. Suppose that $(\varepsilon Dv_\varepsilon)_{\varepsilon \in E}$ is bounded in $\mathbb{L}^2(Q)$ (where $Dv_\varepsilon = \left(\frac{\partial v_\varepsilon}{\partial x_1}, \frac{\partial v_\varepsilon}{\partial x_2}, \frac{\partial v_\varepsilon}{\partial x_3}\right)$). Then, there exist a subsequence $E' \subset E$ and a function $v_0 \in L^2(Q; L^2_{per}(Z))$*

with $\frac{\partial v_0}{\partial y_i} \in L^2(Q; L^2_{per}(Z))$ such that, as $E' \ni \varepsilon \rightarrow 0$,

$$\int_Q v_\varepsilon \Phi^\varepsilon dx dt \rightarrow \iint_{Q \times Z} v_0 \Phi dx dt dy d\tau$$

and

$$\int_Q \varepsilon \frac{\partial v_\varepsilon}{\partial x_i} \Phi^\varepsilon dx dt \rightarrow \iint_{Q \times Z} \frac{\partial v_0}{\partial y_i} \Phi dx dt dy d\tau, \quad i = 1, 2, 3$$

for all $\Phi \in L^2(Q; \mathcal{C}_{per}(Z))$.

Proof. See, e.g., [1, 13]. $\square$

Lemma 3. Let $(v_\varepsilon)_{\varepsilon \in E}$ be a bounded sequence in $L^2(Q)$. Suppose that $(\frac{\partial v_\varepsilon}{\partial t})_{\varepsilon \in E}$ is bounded in $L^2(Q)$. Then, there exist a subsequence $E' \subset E$ and two functions $v_0 \in H^1(0, T; L^2(\Omega; L^2_{per}(Y)))$ and $v_1 \in L^2(Q; L^2_{per}(Y; H^1_\#(\Theta)))$ (with $H^1_\#(\Theta) = \{v \in H^1_{per}(\Theta) : \int_\Theta v d\tau = 0\}$, $H^1_{per}(\Theta) = \{v \in H^1_{loc}(\mathbb{R}) : v, \frac{dv}{d\tau} \text{ are } \Theta\text{-periodic}\}$, and $L^2_{per}(Y; H^1_\#(\Theta)) = \{v \in L^2_{loc}(\mathbb{R}^3; H^1_\#(\Theta)) : v \text{ is } Y\text{-periodic}\}$), such that as $E' \ni \varepsilon \rightarrow 0$,

$$(2.2) \quad \int_Q v_\varepsilon \Phi^\varepsilon dx dt \rightarrow \iint_{Q \times Z} v_0 \Phi dx dt dy d\tau,$$

and

$$(2.3) \quad \int_Q \frac{\partial v_\varepsilon}{\partial t} \Phi^\varepsilon dx dt \rightarrow \iint_{Q \times Z} \left(\frac{\partial v_0}{\partial t} + \frac{\partial v_1}{\partial \tau} \right) \Phi dx dt dy d\tau$$

for all $\Phi \in L^2(Q; \mathcal{C}_{per}(Z))$.

Proof. Based on Lemma 1, there exist a subsequence $E' \subset E$ and two functions $v_0, w_0 \in L^2(Q; L^2_{per}(Z))$ such that, as $E' \ni \varepsilon \rightarrow 0$,

$$(2.4) \quad \begin{aligned} \int_Q v_\varepsilon \Phi^\varepsilon dx dt &\rightarrow \iint_{Q \times Z} v_0 \Phi dx dt dy d\tau \text{ and} \\ \int_Q \frac{\partial v_\varepsilon}{\partial t} \Phi^\varepsilon dx dt &\rightarrow \iint_{Q \times Z} w_0 \Phi dx dt dy d\tau \end{aligned}$$

for every $\Phi \in L^2(Q; \mathcal{C}_{per}(Z))$.

Let us prove (2.2). This will be done if we show that v_0 in (2.4) does not depend on $\tau \in \mathbb{R}$. For $\varphi \in \mathcal{D}(Q)$, $\psi \in \mathcal{C}_{per}(Y)$ and $\theta \in \mathcal{C}^\infty_{per}(\Theta)$ arbitrarily fixed, we have

$$\int_Q \frac{\partial v_\varepsilon}{\partial t} \varphi \psi^\varepsilon \theta^\varepsilon dx dt = - \int_Q v_\varepsilon \frac{\partial \varphi}{\partial t} \psi^\varepsilon \theta^\varepsilon dx dt - \frac{1}{\varepsilon} \int_Q v_\varepsilon \varphi \psi^\varepsilon \left(\frac{d\theta}{d\tau} \right)^\varepsilon dx dt.$$

Taking into account the two limits in (2.4), it follows that

$$\int_Q v_\varepsilon \varphi \psi^\varepsilon \left(\frac{d\theta}{d\tau} \right)^\varepsilon dx dt \rightarrow 0 \text{ as } E' \ni \varepsilon \rightarrow 0;$$

that is,

$$\iint_{Q \times Z} v_0 \varphi \otimes \psi \otimes \frac{d\theta}{d\tau} dx dt dy d\tau = 0.$$

Then, for almost all $(x, t, y) \in Q \times Y$,

$$\int_\Theta v_0(x, t, y, \cdot) \frac{d\theta}{d\tau} d\tau = 0 \text{ for all } \theta \in \mathcal{C}_{per}^\infty(\Theta).$$

Therefore $\frac{\partial v_0(x, t, y, \cdot)}{\partial \tau} = 0$ (a.e. in $Q \times Y$). Hence v_0 does not depend to τ . In other words, $v_0 \in L^2(Q; L_{per}^2(Y))$ and we deduce (2.2).

Let us now prove (2.3). By taking in (2.4) $\Phi = \varphi \otimes \psi \otimes 1$, $\varphi \in \mathcal{D}(Q)$ and $\psi \in \mathcal{C}_{per}(Y)$, it follows that

$$\int_Q v_\varepsilon(x, t) \varphi(x, t) \psi\left(\frac{x}{\varepsilon}\right) dx dt \rightarrow \iint_{Q \times Y} v_0(x, t, y) \varphi(x, t) \psi(y) dx dt dy$$

and

$$\int_Q \frac{\partial v_\varepsilon}{\partial t}(x, t) \varphi(x, t) \psi\left(\frac{x}{\varepsilon}\right) dx dt \rightarrow \iint_{Q \times Z} w_0(x, t, y, \tau) \varphi(x, t) \psi(y) dx dt dy d\tau.$$

But we have

$$\int_Q \frac{\partial v_\varepsilon}{\partial t}(x, t) \varphi(x, t) \psi\left(\frac{x}{\varepsilon}\right) dx dt = - \int_Q v_\varepsilon(x, t) \frac{\partial \varphi}{\partial t}(x, t) \psi\left(\frac{x}{\varepsilon}\right) dx dt;$$

so that, letting $E' \ni \varepsilon \rightarrow 0$, the above limits drive us to

$$\iint_{Q \times Z} w_0(x, t, y, \tau) \varphi(x, t) \psi(y) dx dt dy d\tau = - \iint_{Q \times Y} v_0(x, t, y) \frac{\partial \varphi}{\partial t}(x, t) \psi(y) dx dt dy.$$

Thus, we have $\frac{\partial v_0}{\partial t} \in L^2(Q; L_{per}^2(Y))$ with

$$\int_\Theta \left(w_0(x, t, y, \tau) - \frac{\partial v_0}{\partial t}(x, t, y) \right) d\tau = 0$$

for almost all $(x, t, y) \in Q \times Y$. Thus, there exists a function u_1 from $Q \times Y$ into $H_\#^1(\Theta) = \{v \in H_{per}^1(\Theta) : \int_\Theta v d\tau = 0\}$ (with $H_{per}^1(\Theta) = \{v \in$

$H_{loc}^1(\mathbb{R}) : v, \frac{dv}{d\tau}$ are Θ -periodic}) such that

$$(2.5) \quad \begin{cases} -\frac{d^2 u_1(x, t, y, \cdot)}{d\tau^2} = w_0(x, t, y, \cdot) - \frac{\partial v_0}{\partial t}(x, t, y) \text{ in } \Theta = (-\frac{1}{2}, \frac{1}{2}), \\ \frac{du_1(x, t, y, -\frac{1}{2})}{d\tau} = \frac{du_1(x, t, y, \frac{1}{2})}{d\tau} = 0 \end{cases}$$

a.e. in $Q \times Y$. In fact, the above boundary value problem is an homogeneous Neumann problem, and the semi-norm $w \rightarrow \left(\int_{\Theta} \left| \frac{dw}{d\tau} \right|^2 \right)^{\frac{1}{2}}$ is equivalent to the $H^1(\Theta)$ -norm in $H_{\#}^1(\Theta)$; so that we can apply Lax-Milgram Lemma. By letting $-\frac{du_1(x, t, y, \cdot)}{d\tau} = v_1(x, t, y, \cdot)$, it follows from (2.5) that $w_0(x, t, y, \cdot) = \frac{\partial v_0}{\partial t}(x, t, y) + \frac{\partial v_1}{\partial \tau}(x, t, y, \cdot)$ a.e. $Q \times Y$; and (2.3) follows. $\square$

By successive iterations, we can generalize the above lemma as follows

Lemma 4. *Let $(v_{\varepsilon})_{\varepsilon \in E}$ be a bounded sequence in $L^2(Q)$. Suppose that for some integer $n \geq 1$, $(\frac{\partial^j v_{\varepsilon}}{\partial t^j})_{\varepsilon \in E}$ is bounded in $L^2(Q)$, $j = 1, \dots, n$. Then, there exist a subsequence $E' \subset E$ and two functions $v_0 \in H^n(0, T; L^2(\Omega; L_{per}^2(Y)))$ and $v_1 \in L^2(Q; L_{per}^2(Y; H_{\#}^n(\Theta)))$ with $H_{\#}^n(\Theta) = \{u \in H_{per}^n(\Theta) : \int_{\Theta} u d\tau = 0\}$, such that as $E' \ni \varepsilon \rightarrow 0$,*

$$\int_Q \frac{\partial^j v_{\varepsilon}}{\partial t^j} \Phi^{\varepsilon} dx dt \rightarrow \iint_{Q \times Z} \frac{\partial^j v_0}{\partial t^j} \Phi dx dt dy d\tau, \quad j = 0, \dots, n-1$$

and

$$\int_Q \frac{\partial^n v_{\varepsilon}}{\partial t^n} \Phi^{\varepsilon} dx dt \rightarrow \iint_{Q \times Z} \left(\frac{\partial^n v_0}{\partial t^n} + \frac{\partial^n v_1}{\partial \tau^n} \right) \Phi dx dt dy d\tau$$

for all $\Phi \in L^2(Q; C_{per}(Z))$.

3. The homogenization result in the case

$$f \in \mathcal{C}^1([0, T]; \mathbb{L}^2(\Omega; \mathbb{R})), \quad f(0) = \omega$$

The preliminary results below are based on the following general estimates, which will be easily proved by using a Gronwall lemma.

Lemma 5. *Let $(u_{\varepsilon})_{\varepsilon > 0}$ be the sequence of functions defined in Theorem 1. Then,*

$$\sup_{\varepsilon > 0} \left(\|u'_{\varepsilon}\|_{L^{\infty}(0, T; \mathbb{L}^2(\Omega))} + \|\varepsilon u'_{\varepsilon}\|_{L^2(0, T; \mathbb{H}_0^1(\Omega))} + \|\operatorname{div} u_{\varepsilon}\|_{L^{\infty}(0, T; L^2(\Omega))} \right) < +\infty$$

and

$$\sup_{\varepsilon > 0} \left(\|u_{\varepsilon}\|_{L^{\infty}(0, T; \mathbb{L}^2(\Omega))} + \|\varepsilon u_{\varepsilon}\|_{L^{\infty}(0, T; \mathbb{H}_0^1(\Omega))} \right) < +\infty.$$

Furthermore, if $f \in \mathcal{C}^1([0, T]; \mathbb{L}^2(\Omega))$ with $f(0) = \omega$, then

$$\sup_{\varepsilon > 0} \left(\|u''_\varepsilon\|_{L^\infty(0, T; \mathbb{L}^2(\Omega))} + \|\varepsilon u''_\varepsilon\|_{L^2(0, T; \mathbb{H}_0^1(\Omega))} + \|\operatorname{div} u'_\varepsilon\|_{L^\infty(0, T; L^2(\Omega))} \right) < +\infty.$$

Remark 1. The condition $f \in \mathcal{C}^1([0, T]; \mathbb{L}^2(\Omega; \mathbb{R}))$ with $f(0) = \omega$ allows us to prove that the solution u_ε of (1.2)-(1.2) is also a solution of the problem (see [13])

$$\int_{\Omega} \rho^\varepsilon \frac{\partial^3 u_\varepsilon^i}{\partial t^3}(t) v^i dx + \varepsilon^2 b^\varepsilon \left(\frac{\partial^2 u_\varepsilon}{\partial t^2}(t), v \right) + a^\varepsilon \left(\frac{\partial u_\varepsilon}{\partial t}(t), v \right) = \int_{\Omega} \frac{\partial f^i}{\partial t}(t) v^i dx$$

for all $v = (v^i) \in V_\varepsilon$ ($0 < t < T$),

$$u_\varepsilon(0) = \frac{\partial u_\varepsilon}{\partial t}(0) = \frac{\partial^2 u_\varepsilon}{\partial t^2}(0) = \omega.$$

Then, the last estimate becomes obvious. Further, the choice of f as an element of $\mathcal{C}^1([0, T]; \mathbb{L}^2(\Omega; \mathbb{R}))$ with $f(0) = \omega$, will permits us to study the weakly two-scale convergence of the sequence of second derivative $(u''_\varepsilon)_{\varepsilon > 0}$.

3.1 Preliminaries. In the sequel E denotes a fundamental sequence (see Lemma 1).

Lemma 6. *There exist a sequence $E' \subset E$ and two functions $p_0 \in L^2(Q; L^2_{\text{per}}(Y))$ with $\frac{\partial p_0}{\partial t} \in L^2(Q; L^2_{\text{per}}(Y))$, and $u_0 = (u_0^1, u_0^2, u_0^3) \in L^2(Q; \mathbb{H}^1_{\text{per}}(Y))$ with $\frac{\partial u_0}{\partial t} \in L^2(Q; \mathbb{H}^1_{\text{per}}(Y))$, $\frac{\partial^2 u_0}{\partial t^2} \in L^2(Q; \mathbb{L}^2_{\text{per}}(Y))$, such that, as $E' \ni \varepsilon \rightarrow 0$,*

$$(3.1) \quad u_\varepsilon^i \rightarrow u_0^i \text{ in } L^2(Q)\text{-weak two-scale,}$$

$$(3.2) \quad p_\varepsilon \rightarrow p_0 \text{ in } L^2(Q)\text{-weak two-scale,}$$

where $p_\varepsilon = -\vartheta \operatorname{div} u_\varepsilon$. Furthermore, we have

$$(3.3) \quad e_{ij}(u_0) = e_{ij} \left(\frac{\partial u_0}{\partial t} \right) = p_0 = 0 \text{ in } Q \times Y_s,$$

$$(3.4) \quad \operatorname{div}_y u_0 = \operatorname{div}_y \left(\frac{\partial u_0}{\partial t} \right) = 0 \text{ in } Q \times Y_f,$$

$$(3.5) \quad \int_{Y_f} p_0 dy + \vartheta \operatorname{div} \left(\int_Y u_0 dy \right) = 0 \text{ in } Q.$$

where $e_{ij}(u_0) = \frac{1}{2} \left(\frac{\partial u_0^i}{\partial y_j} + \frac{\partial u_0^j}{\partial y_i} \right)$, $i, j = 1, 2, 3$, $\operatorname{div}_y u_0 = e_{ii}(u_0) = \frac{\partial u_0^1}{\partial y_1} + \frac{\partial u_0^2}{\partial y_2} + \frac{\partial u_0^3}{\partial y_3}$ and $H_{per}^1(Y) = \{v \in H_{loc}^1(\mathbb{R}^3) : v, \frac{\partial v}{\partial y_i} \ (i = 1, 2, 3) \text{ are } Y\text{-periodic}\}$.

Proof. We proceed in three steps.

Step 1. Let us prove (3.1) and (3.2). Thanks to Lemma 5, we have $\sup_{\varepsilon \in E} \left(\|p_\varepsilon\|_{H^1(0,T;L^2(\Omega))} \right) < \infty$, and $\sup_{\varepsilon \in E} \left(\|u_\varepsilon^i\|_{H^2(0,T;L^2(\Omega))} \right) < \infty$ ($i = 1, 2, 3$). Hence, by Lemmas 1, 3 and 4, there exist a subsequence $E' \subset E$ and functions $p_0 \in H^1(0,T;L^2(\Omega;L_{per}^2(Y)))$, $p_1 \in L^2(Q;L_{per}^2(Y;H_{\#}^1(\Theta)))$, $u_0^i \in H^2(0,T;L^2(\Omega;L_{per}^2(Y)))$ and $u_1^i \in L^2(Q;L_{per}^2(Y;H_{\#}^2(\Theta)))$ such that, as $E' \ni \varepsilon \rightarrow 0$,

$$\begin{aligned}
 (3.6) \quad & \int_Q p_\varepsilon \Phi^\varepsilon dx dt \rightarrow \int \int_{Q \times Z} p_0 \Phi dx dt dy d\tau, \\
 & \int_Q \frac{\partial p_\varepsilon}{\partial t} \Phi^\varepsilon dx dt \rightarrow \int \int_{Q \times Z} \left(\frac{\partial p_0}{\partial t} + \frac{\partial p_1}{\partial \tau} \right) \Phi dx dt dy d\tau, \\
 & \int_Q u_\varepsilon^i \Phi^\varepsilon dx dt \rightarrow \int \int_{Q \times Z} u_0^i \Phi dx dt dy d\tau, \\
 & \int_Q \frac{\partial u_\varepsilon^i}{\partial t} \Phi^\varepsilon dx dt \rightarrow \int \int_{Q \times Z} \frac{\partial u_0^i}{\partial t} \Phi dx dt dy d\tau, \\
 & \int_Q \frac{\partial^2 u_\varepsilon^i}{\partial t^2} \Phi^\varepsilon dx dt \rightarrow \int \int_{Q \times Z} \left(\frac{\partial^2 u_0^i}{\partial t^2} + \frac{\partial^2 u_1^i}{\partial \tau^2} \right) \Phi dx dt dy d\tau
 \end{aligned}$$

($i = 1, 2, 3$) for all $\Phi \in L^2(Q;C_{per}(Z))$. Hence, (3.1) and (3.2) are the third line and the first line of (3.6) (respectively).

Step 2. Let us prove (3.3). By Lemma 5 we have

$$\sup_{\varepsilon \in E} \left(\left\| \varepsilon \frac{\partial u_\varepsilon^i}{\partial x_j} \right\|_{L^2(Q)} \right) < \infty, \text{ and } \sup_{\varepsilon \in E} \left(\left\| \varepsilon \frac{\partial^2 u_\varepsilon^i}{\partial x_j \partial t} \right\|_{L^2(Q)} \right) < \infty \ (i, j = 1, 2, 3).$$

Then according to (3.6), we conclude by Lemma 2 that

$$\begin{aligned}
 & \int_Q \varepsilon \frac{\partial u_\varepsilon^i}{\partial x_j} \varphi \psi^\varepsilon \theta^\varepsilon dx dt \rightarrow \int \int \int_{Q \times Y \times \Theta} \frac{\partial u_0^i}{\partial y_j} \varphi \otimes \psi \otimes \theta dx dt dy d\tau \\
 & \int_Q \varepsilon \frac{\partial^2 u_\varepsilon^i}{\partial x_j \partial t} \varphi \psi^\varepsilon \theta^\varepsilon dx dt \rightarrow \int \int \int_{Q \times Y \times \Theta} \frac{\partial^2 u_0^i}{\partial y_j \partial t} \varphi \otimes \psi \otimes \theta dx dt dy d\tau
 \end{aligned}$$

($i, j = 1, 2, 3$) for all $\varphi \in \mathcal{D}(Q)$, $\psi \in C_{per}^\infty(Y)$ and $\theta \in C_{per}(\Theta)$; hence $u_0, \frac{\partial u_0}{\partial t} \in L^2(Q; \mathbb{H}_{per}^1(Y))$, where $u_0 = (u_0^i) \equiv (u_0^i)_{i=1,2,3}$. Let us bear in mind

that the preceding two limits give

$$(3.7) \quad \begin{aligned} \int_Q \varepsilon E_{ij}(u_\varepsilon) \varphi \psi^\varepsilon \theta^\varepsilon dx dt &\rightarrow \iint_{Q \times Y} e_{ij}(u_0) \varphi \otimes \psi dx dt dy \\ \int_Q \varepsilon E_{ij} \left(\frac{\partial u_\varepsilon}{\partial t} \right) \varphi \psi^\varepsilon \theta^\varepsilon dx dt &\rightarrow \iint_{Q \times Y} e_{ij} \left(\frac{\partial u_0}{\partial t} \right) \varphi \otimes \psi dx dt dy \end{aligned}$$

($i, j = 1, 2, 3$) for all $\varphi \in \mathcal{D}(Q)$, $\psi \in \mathcal{C}_{per}^\infty(Y)$ and $\theta \in \mathcal{C}_{per}(\Theta)$ with $\int_\Theta \theta d\tau = 1$.

Since $E_{ij}(u_\varepsilon) = E_{ij}(\frac{\partial u_\varepsilon}{\partial t}) = 0$ ($i, j = 1, 2, 3$) and $p_\varepsilon = \frac{\partial p_\varepsilon}{\partial t} = 0$ in $\Omega_s^\varepsilon \times (0, T)$ (see the definition of V_ε), we take in the first line in (3.6) and in (3.7) $\psi = \chi_s w$ where $w \in \mathcal{C}_{per}(Y)$ and χ_s is the characteristic function of $\tilde{Y}_s$ defined on $\mathbb{R}_y^3$ [so that, as in (2.1), $\varphi \otimes \psi \otimes \theta \in \mathcal{C}(\overline{Q}; L_{per}^\infty(Z))$]. It follows that

$$\begin{aligned} \int \int_{Q \times Y} \chi_s e_{ij}(u_0) \varphi \otimes w dx dt dy &= 0, \\ \int \int_{Q \times Y} \chi_s e_{ij} \left(\frac{\partial u_0}{\partial t} \right) \varphi \otimes w dx dt dy &= 0, \\ \int \int_{Q \times Y} \chi_s p_0 \varphi \otimes w dx dt dy &= 0 \end{aligned}$$

for all $\varphi \in \mathcal{D}(Q)$ and all $w \in \mathcal{C}_{per}(Y)$; and (3.3) is deduced.

Step 3. Let us prove (3.4) and (3.5). The first and second lines in (3.6) combining with (3.7) yield

$$\begin{aligned} \int \int_{Q \times Y} e_{ii}(u_0) \varphi \otimes w dx dt dy &= 0, \\ \int \int_{Q \times Y} e_{ii} \left(\frac{\partial u_0}{\partial t} \right) \varphi \otimes w dx dt dy &= 0 \end{aligned}$$

for all $\varphi \in \mathcal{D}(Q)$, $w \in \mathcal{C}_{per}^\infty(Y)$; and (3.4) follows. Furthermore, it follows to the first and third lines in (3.6) combining with (3.3), that

$$\int_Q \left(\int_{Y_f} p_0 dy \right) \varphi dx dt = \vartheta \int_Q \left(\int_Y u_0^i dy \right) \frac{\partial \varphi}{\partial x_i} dx dt$$

for all $\varphi \in \mathcal{D}(Q)$, and (3.5) is proved. $\square$

Let us verify that u_0 satisfies the homogeneous initial conditions. Precisely we have the following lemma.

Lemma 7. $u_0, \frac{\partial u_0}{\partial t} \in \mathcal{C}([0, T]; L^2(\Omega; \mathbb{L}_{per}^2(Y)))$ with $u_0(0) = \frac{\partial u_0}{\partial t}(0) = \omega$.

Proof. By Lemma 6, u_0 and $\frac{\partial u_0}{\partial t}$ belong to $\mathcal{C}([0, T]; L^2(\Omega; \mathbb{L}_{per}^2(Y)))$, so that $u_0(0)$ and $\frac{\partial u_0}{\partial t}(0)$ are well defined. Now, denoting by $[\cdot, \cdot]$ the

scalar product in $L^2(\Omega; \mathbb{L}_{per}^2(Y))$, if $v \in \mathcal{C}_T^1([0, T]; L^2(\Omega; \mathbb{L}_{per}^2(Y))) = \{w \in \mathcal{C}^1([0, T]; L^2(\Omega; \mathbb{L}_{per}^2(Y))) : w(T) = \omega\}$ then we have

$$(3.8) \quad \int_0^T [u_0(t), v'(t)] dt = - \int_0^T [u_0'(t), v(t)] dt - [u_0(0), v(0)].$$

On the other hand, recalling that u_ε and $\frac{\partial u_\varepsilon}{\partial t}$ lie in $L^2(0, T; \mathbb{H}_0^1(\Omega))$ with $u_\varepsilon(0) = \omega$, if $\beta \in \mathcal{C}_T^1([0, T]) = \{\beta \in \mathcal{C}^1([0, T]) : \beta(T) = 0\}$ and $\psi \in L^2(\Omega; \mathcal{C}_{per}(Y)^3)$ it follows that

$$\int_Q \frac{\partial u_\varepsilon}{\partial t} \psi^\varepsilon \beta dx dt = - \int_Q u_\varepsilon \psi^\varepsilon \beta' dx dt.$$

The passage to the limit, as $E' \ni \varepsilon \rightarrow 0$, permits to deduce

$$\int_0^T [u_0'(t), \psi] \beta(t) dt = - \int_0^T [u_0(t), \psi] \beta'(t) dt.$$

Using the density of $L^2(\Omega; \mathcal{C}_{per}(Y)^3)$ in $L^2(\Omega; \mathbb{L}_{per}^2(Y))$, together with (3.8) we conclude that $[u_0(0), \varphi] \beta(0) = 0$ for all $\varphi \in L^2(\Omega; \mathbb{L}_{per}^2(Y))$ and all $\beta \in \mathcal{C}_T^1([0, T])$. Therefore, we have $u_0(0) = \omega$. The same arguments give $\frac{\partial u_0}{\partial t}(0) = \omega$. $\square$

The following lemma will permit to achieve one part of the proof of Lemma 9.

Lemma 8. *Let $S = (S^i) \in \mathcal{D}'(Q)^3$ ($\mathcal{D}'(Q)$ being the space of distributions on Q). Assume that*

$$(3.9) \quad \left(\int_{Y_f} w^i dy \right) S^i \in L^2(Q) \text{ for all } w = (w^i) \in \mathcal{C}_{per}^\infty(Y_f)^3, \operatorname{div}_y w = 0.$$

Then

$$(3.10) \quad S = (S^i) \in \mathbb{L}^2(Q),$$

where $\mathcal{C}_{per}^\infty(Y_f)$ denotes the space of functions in $\mathcal{C}_{per}^\infty(Y)$ with support contained in $\tilde{Y}_f$ (see [11]).

Proof. We proceed in two steps.

Step 1. Let us prove the following: if $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ with $\xi \neq \omega$, then there is $w = (w^i) \in \mathcal{C}_{per}^\infty(Y_f)^3$, $\operatorname{div}_y w = 0$ such that $\xi_i \int_{Y_f} w^i dy \neq 0$,

or equivalently if $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ such that

$$(3.11) \quad \xi_i \int_{Y_f} w^i dy = 0 \text{ for all } w = (w^i) \in \mathcal{C}_{per}^\infty(Y_f)^3, \operatorname{div}_y w = 0,$$

then $\xi = \omega$.

Let us consider a point $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ having property (3.11). There is $\psi \in H_{per}^1(Y_f)$ such that $\frac{\partial \psi}{\partial y_i} = \xi_i$ ($i = 1, 2, 3$). Let χ be a polynomial function on $\tilde{Y}_f$ defined by $\chi(y) = \xi_1 y_1 + \xi_2 y_2 + \xi_3 y_3$ for all $y = (y_1, y_2, y_3) \in \tilde{Y}_f$. There exists a constant $c \in \mathbb{R}$ such that $\psi = \chi + c$ in $\tilde{Y}_f$. By virtue of periodicity of ψ , such equality is possible only if $\xi_1 = \xi_2 = \xi_3 = 0$.

Step 2. Now, let us prove (3.10). In what follows we assume there is $S = (S^i) \in \mathcal{D}'(Q)^3$ satisfying (3.9).

Let $\xi = (1, 0, 0)$. According to the above step, there is $w_1 = (w_1^i) \in \mathcal{C}_{per}^\infty(Y_f)^3$, $\operatorname{div}_y w_1 = 0$ such that

$$\int_{Y_f} w_1^1 dy \neq 0.$$

We put

$$a_i = \int_{Y_f} w_1^i dy \quad (i = 1, 2, 3).$$

By virtue of (3.9), there is $q_1 \in L^2(Q)$ such that

$$(3.12) \quad a_1 S^1 + a_2 S^2 = q_1 - a_3 S^3.$$

Next let $\xi = (-a_2, a_1, 0)$. We have $a_1 \neq 0$, then $\xi \neq \omega$, so that there is $w_2 = (w_2^i) \in \mathcal{C}_{per}^\infty(Y_f)^3$, $\operatorname{div}_y w_2 = 0$ such that

$$-a_2 \int_{Y_f} w_2^1 dy + a_1 \int_{Y_f} w_2^2 dy \neq 0.$$

By letting

$$b_i = \int_{Y_f} w_2^i dy \quad (i = 1, 2, 3),$$

we have $a_1 b_2 - a_2 b_1 \neq 0$ and, according to (3.9), there is $q_2 \in L^2(Q)$ such that

$$(3.13) \quad b_1 S^1 + b_2 S^2 = q_2 - b_3 S^3.$$

Hence, we are led to consider the system of linear equations (3.12)-(3.13). Since $a_1b_2 - a_2b_1 \neq 0$, then S^1 and S^2 are uniquely determined by

$$(3.14) \quad S^1 = \lambda_1 q_1 + \lambda_2 q_2 + \lambda_3 S^3 \text{ and } S^2 = \mu_1 q_1 + \mu_2 q_2 + \mu_3 S^3$$

λ_i, μ_i ($i = 1, 2, 3$) being well defined.

Finally let $\xi = (\lambda_3, \mu_3, 1)$. There is $w_3 = (w_3^i) \in \mathcal{C}_{per}^\infty(Y_f)^3$, $\operatorname{div}_y w_3 = 0$, such that

$$\lambda_3 \int_{Y_f} w_3^1 dy + \mu_3 \int_{Y_f} w_3^2 dy + \int_{Y_f} w_3^3 dy \neq 0.$$

We put

$$c_i = \int_{Y_f} w_3^i dy \quad (i = 1, 2, 3).$$

Then,

$$(3.15) \quad \lambda_3 c_1 + \mu_3 c_2 + c_3 \neq 0.$$

According to (3.9), there is $q_3 \in L^2(Q)$ such that

$$c_1 S^1 + c_2 S^2 + c_3 S^3 = q_3.$$

By substituting (3.14), we have

$$(\lambda_3 c_1 + \mu_3 c_2 + c_3) S^3 = q_3 - (\lambda_2 c_1 + \mu_2 c_2) q_2 - (\lambda_1 c_1 + \mu_1 c_2) q_1.$$

Thanks to (3.15), S^3 is determined as a linear combination of $q_1, q_2, q_3 \in L^2(Q)$, that is $S^3 \in L^2(Q)$. Therefore $S^1, S^2 \in L^2(Q)$ by virtue of (3.13). $\square$

This being so, we will need the following lemma to derive the homogenized problem.

Lemma 9. *By letting*

$$(3.16) \quad p(x, t) = -\frac{\vartheta}{|Y_f|} \operatorname{div} \left(\int_Y u_0(x, t, y) dy \right) \quad ((x, t) \in Q),$$

where $|Y_f|$ is the Lebesgue measure of Y_f , for almost all $(x, t) \in Q$ we have

$$(3.17) \quad p_0(x, t, \cdot) = 0 \text{ in } Y_s \text{ and } p(x, t) \text{ in } Y_f.$$

Furthermore

$$(3.18) \quad \frac{\partial p}{\partial x_i} \in L^2(0, T; L^2(\Omega)), \quad i = 1, 2, 3;$$

and the function $u_1 = (u_1^i)$ in (3.6) verifies

$$(3.19) \quad \int_{Y_f} u_1^i(x, t, y, \cdot) dy = \int_{Y_s} u_1^i(x, t, y, \cdot) dy = 0, \quad i = 1, 2, 3.$$

Proof. Let us prove (3.17). We take in (1.2) test functions of the form $v = \varepsilon \varphi w^\varepsilon \phi(t) \theta^\varepsilon(t)$ for $0 < t < T$, $\varepsilon \in E'$, with $\varphi \in \mathcal{D}(\Omega)$, $\phi \in \mathcal{D}((0, T))$, $\theta \in \mathcal{C}_{per}(\Theta)$, $w = (w_1, w_2, w_3)$ each w^i belonging to $\mathcal{C}_{per}^\infty(Y_f)$ [11]; next we integrate from 0 to T and it follows that

$$\begin{aligned} & \varepsilon \int_Q \rho^\varepsilon \frac{\partial^2 u_\varepsilon}{\partial t^2} \varphi w^\varepsilon \phi \theta^\varepsilon dx dt + \varepsilon \int_Q b_{ijkh} E_{kh} \left(\varepsilon \frac{\partial u_\varepsilon}{\partial t} \right) (e_{ij}(w))^\varepsilon \varphi \phi \theta^\varepsilon dx dt \\ & - \varepsilon \int_Q p_\varepsilon (\operatorname{div} \varphi w)^\varepsilon \phi \theta^\varepsilon dx dt + \varepsilon^2 \int_Q b_{ijkh} E_{kh} \left(\varepsilon \frac{\partial u_\varepsilon}{\partial t} \right) (E_{ij}(\varphi w))^\varepsilon \phi \theta^\varepsilon dx dt \\ & - \varepsilon \int_Q f \varphi \phi w^\varepsilon \theta^\varepsilon dx dt = \int_Q p_\varepsilon \varphi (\operatorname{div}_y w)^\varepsilon \phi \theta^\varepsilon dx dt. \end{aligned}$$

Passing to the limit, as $E' \ni \varepsilon \rightarrow 0$, it follows by (3.6)-(3.7) that

$$\iint_{Q \times Y_f} p_0 \operatorname{div}_y w \varphi \phi dx dt dy = 0.$$

This yields

$$\int_{Y_f} p_0(x, t, \cdot) \frac{\partial \psi}{\partial y_i} dy = 0 \text{ a.e. in } Q \text{ for all } \psi \in \mathcal{C}_{per}^\infty(Y_f) \quad (1 \leq i \leq 3).$$

Thus $p_0(x, t, \cdot)$ is constant in Y_f . Taking into account (3.3) and (3.5), we have derived (3.17).

Now, let us prove (3.18) and (3.19). We select in (1.2) test functions of the form $v = \varphi w^\varepsilon \phi(t) \theta^\varepsilon(t)$, $t \in (0, T)$, with $\varphi \in \mathcal{D}(\Omega)$, $\phi \in \mathcal{D}((0, T))$, $w \in \mathcal{C}_{per}^\infty(Y_f)^3$, $\operatorname{div}_y w = 0$ and $\theta \in \mathcal{C}_{per}(\Theta)$; next we integrate from 0 to T and obtain

$$\begin{aligned} & \int_Q \rho^\varepsilon \frac{\partial^2 u_\varepsilon}{\partial t^2} \varphi w^\varepsilon \phi \theta^\varepsilon dx dt + \int_Q b_{ijkh} E_{kh} \left(\varepsilon \frac{\partial u_\varepsilon}{\partial t} \right) (e_{ij}(w))^\varepsilon \varphi \phi \theta^\varepsilon dx dt \\ & - \int_Q p_\varepsilon (\operatorname{div}(\varphi w))^\varepsilon \phi \theta^\varepsilon dx dt + \varepsilon \int_Q b_{ijkh} E_{kh} \left(\varepsilon \frac{\partial u_\varepsilon}{\partial t} \right) (E_{ij}(\varphi w))^\varepsilon \phi \theta^\varepsilon dx dt \\ & = \int_Q f \varphi w^\varepsilon \phi \theta^\varepsilon dx dt. \end{aligned}$$

Further we pass to the limit, as $E' \ni \varepsilon \rightarrow 0$. Using Lemma 6 and property (3.17), we are led to

$$\begin{aligned} & \left(\int_{\Theta} \theta d\tau \right) \int_Q \left[\int_{Y_f} \left(\rho_f \frac{\partial^2 u_0}{\partial t^2} \cdot w + \mu_{ijkh} e_{kh} \left(\frac{\partial u_0}{\partial t} \right) e_{ij}(w) - f \cdot w \right) dy \right] \varphi \phi dx dt \\ & + \int \int \int_{Q \times Y_f \times \Theta} \rho_f \frac{\partial^2 u_1}{\partial \tau^2} \cdot w \varphi \phi \theta dx dt dy d\tau \\ & = \left(\int_{\Theta} \theta d\tau \right) \left(\int_{Y_f} w^i dy \right) \int_Q p(x, t) \frac{\partial \varphi}{\partial x_i}(x) \phi(t) dx dt \end{aligned}$$

for all $\varphi \in \mathcal{D}(\Omega)$, $\phi \in \mathcal{D}((0, T))$, $w \in \mathcal{C}_{per}^\infty(Y_f)^3$, $\operatorname{div}_y w = 0$ and $\theta \in \mathcal{C}_{per}(\Theta)$, where $\mu_{ijkh} = \mu(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk})$. Taking $\theta = 1$, we are drived to

$$\left(\int_{Y_f} w^i dy \right) \frac{\partial p}{\partial x_i} \in L^2(Q) \quad \text{for all } w \in \mathcal{C}_{per}^\infty(Y_f)^3, \operatorname{div}_y w = 0$$

with

$$\begin{aligned} & \left(\int_{Y_f} w^i dy \right) \frac{\partial p}{\partial x_i} = \int_{Y_f} \left(\rho_f \frac{\partial^2 u_0}{\partial t^2} \cdot w + \mu_{ijkh} e_{kh} \left(\frac{\partial u_0}{\partial t} \right) e_{ij}(w) - f \cdot w \right) dy \\ & (i = 1, 2, 3). \end{aligned}$$

Hence, taking into account Lemma 8, we have $\frac{\partial p}{\partial x_i} \in L^2(Q)$, $i = 1, 2, 3$; and (3.18) follows.

Consequently $\int \int \int_{Q \times Y_f \times \Theta} \rho_f \frac{\partial^2 u_1}{\partial \tau^2} \cdot w \varphi \phi \theta dx dt dy d\tau = 0$ for all $\varphi \in \mathcal{D}(\Omega)$, $\phi \in \mathcal{D}((0, T))$, $w \in \mathcal{C}_{per}^\infty(Y_f)^3$, $\operatorname{div}_y w = 0$ and $\theta \in \mathcal{C}_{per}(\Theta)$. Therefore $\frac{\partial^2}{\partial \tau^2} \int_{Y_f} u_1^i dy = 0$ in $Q \times \Theta$ ($i = 1, 2, 3$); that is for almost all $(x, t) \in Q$ the function $\frac{\partial}{\partial \tau} \int_{Y_f} u_1^i(x, t, y, \cdot) dy$ is constant with respect to τ . By periodicity with respect to τ , $\int_{Y_f} u_1^i(x, t, y, \cdot) dy = \lambda^i$ ($\lambda^i \in \mathbb{R}$). But $\int_{\Theta} u_1^i d\tau = 0$, then we have $\int_{Y_f} u_1^i(x, t, y, \cdot) dy = 0$, $i = 1, 2, 3$.

Finally, we select in (1.2) test functions of the form $v = \varphi w^\varepsilon \phi(t) \theta^\varepsilon(t)$, $t \in (0, T)$, with $\varphi \in \mathcal{D}(\Omega)$, $\phi \in \mathcal{D}((0, T))$, w constant in Y and $\theta \in \mathcal{C}_{per}(\Theta)$; and we obtain, at the limit when $E' \ni \varepsilon \rightarrow 0$

$$\int \int_{Q \times Y \times \Theta} \rho \left(\frac{\partial^2 u_0^i}{\partial t^2} + \frac{\partial^2 u_1^i}{\partial \tau^2} \right) \varphi \phi \theta dx dt dy d\tau = \int \int_{Q \times \Theta} \left(f \varphi + p \frac{\partial \varphi}{\partial x_i} \right) \phi \theta dx dt d\tau$$

$i = 1, 2, 3$. Since

$$\int \int_{Q \times Y} \rho \frac{\partial^2 u_0^i}{\partial t^2} \varphi \phi dx dt dy = \int_Q \left(f \varphi + p \frac{\partial \varphi}{\partial x_i} \right) \phi dx dt, \quad i = 1, 2, 3$$

then $\int \int_{Q \times Y \times \Theta} \rho \frac{\partial^2 u_0^i}{\partial \tau^2} \varphi \phi \theta dx dt dy d\tau = 0$. Therefore $\int_{Y_s} u_1^i(x, t, y, \cdot) dy = 0$, $i = 1, 2, 3$. $\square$

3.2 A suitable variational equation. Let us first prove a general result.

Lemma 10. *Let Ω_0 and Ω_1 be two smooth open bounded sets in $\mathbb{R}^N$ ($N \geq 1$) with the property*

$$(3.20) \quad \overline{\Omega}_0 \subset \Omega_1 \quad (\overline{\Omega}_0 \text{ being the closure of } \Omega_0 \text{ in } \mathbb{R}^N),$$

and let $\gamma_\circ \in \mathcal{L}(H^1(\Omega_0); H^{\frac{1}{2}}(\partial\Omega_0))$, the usual trace operator on Ω_0 . Then there exist two continuous linear mappings $\Gamma_\circ \in \mathcal{L}(H_0^1(\Omega_1); H^{\frac{1}{2}}(\partial\Omega_0))$ and $R_\circ \in \mathcal{L}(H^{\frac{1}{2}}(\partial\Omega_0); H_0^1(\Omega_1))$ ($H_0^1(\Omega_1)$ endowed with the $H^1(\Omega_1)$ -norm) such that

$$(3.21) \quad \begin{aligned} (\Gamma_\circ \circ R_\circ)g &= g \quad \text{for all } g \in H^{\frac{1}{2}}(\partial\Omega_0), \text{ and} \\ (R_\circ \circ \Gamma_\circ)v &= v \quad \text{for all } v \in (\ker \Gamma_\circ)^\perp \end{aligned}$$

where $(\ker \Gamma_\circ)^\perp$ is the orthogonal in $H_0^1(\Omega_1)$ of the kernel of $\Gamma_\circ$, $\circ$ denotes usual composition, $\mathcal{L}(H_0^1(\Omega_1); H^{\frac{1}{2}}(\partial\Omega_0))$ being the vector space of continuous linear mappings from $H_0^1(\Omega_1)$ into $H^{\frac{1}{2}}(\partial\Omega_0)$, and $\mathcal{L}(H^{\frac{1}{2}}(\partial\Omega_0); H_0^1(\Omega_1))$ being that of continuous linear mappings from $H^{\frac{1}{2}}(\partial\Omega_0)$ into $H_0^1(\Omega_1)$.

Proof. By noting $v|_{\Omega_0}$ the restriction of v on Ω_0 , we construct a surjective mapping $\Gamma_\circ$ by $\Gamma_\circ v = \gamma_\circ(v|_{\Omega_0})$ ($v \in H_0^1(\Omega_1)$). It is an obvious matter to verify that $\Gamma_\circ$ belongs to $\mathcal{L}(H_0^1(\Omega_1); H^{\frac{1}{2}}(\partial\Omega_0))$, where $H_0^1(\Omega_1)$ and $H^{\frac{1}{2}}(\partial\Omega_0)$ are endowed with the usual $H^1(\Omega_1)$ -norm and $H^{\frac{1}{2}}(\partial\Omega_0)$ -norm respectively. Using (3.20) and the construction of $\Gamma_\circ$, we prove easily the equality $H^{\frac{1}{2}}(\partial\Omega_0) = \{g \in L^2(\partial\Omega_0) : \exists v \in H_0^1(\Omega_1) \text{ with } \Gamma_\circ v = g\}$. This being so, we endow $H^{\frac{1}{2}}(\partial\Omega_0)$ with the norm $\|g\| = \inf\{\|v\|_{H^1(\Omega_1)} : v \in H_0^1(\Omega_1) \text{ with } \Gamma_\circ v = g\}$ ($g \in H^{\frac{1}{2}}(\partial\Omega_0)$) which makes it a Banach space. By a classical result of minimization of functionals, for fixed $g \in H^{\frac{1}{2}}(\partial\Omega_0)$ there exists a unique couple $(v_\circ, w_\circ) \in H_0^1(\Omega_1) \times H^1(\Omega_0)$ such that $g = \Gamma_\circ v_\circ = \gamma_\circ w_\circ$, $\|g\| = \|v_\circ\|_{H^1(\Omega_1)}$ and $\|g\|_{H^{\frac{1}{2}}(\partial\Omega_0)} = \|w_\circ\|_{H^1(\Omega_0)}$. Since $\|g\|_{H^{\frac{1}{2}}(\partial\Omega_0)} \leq \|(v_\circ|_{\Omega_0})\|_{H^1(\Omega_0)}$, then we have $\|g\|_{H^{\frac{1}{2}}(\partial\Omega_0)} \leq \|g\|$; and by virtue of the open mapping theorem (see, e.g., [5]), it follows that both the

norms $\|\cdot\|_{H^{\frac{1}{2}}(\partial\Omega_0)}$ and $\|\cdot\|$ are equivalent on $H^{\frac{1}{2}}(\partial\Omega_0)$. Finally, we apply the usual canonical decomposition of $\Gamma_\circ$ to define $R_\circ$ as a linear mapping from $H^{\frac{1}{2}}(\partial\Omega_0)$ into $H_0^1(\Omega_1)$ with (3.21) and there are two constants $c_\circ, c_1$ such that $\|R_\circ g\|_{H^1(\Omega_1)} \leq c_\circ \|g\| \leq c_\circ c_1 \|g\|_{H^{\frac{1}{2}}(\partial\Omega_0)}$ for all $g \in H^{\frac{1}{2}}(\partial\Omega_0)$. $\square$

Lemma 10 leads us to the following

Lemma 11. *Let Ω_0 and Ω_1 be two smooth open bounded sets in $\mathbb{R}^N$ ($N \geq 1$) with the property (3.20), let $\Phi_\circ$ be a linear form on $H_0^1(\Omega_1)$, and let V be a fixed vector subspace of $H_0^1(\Omega_1)$. If*

$$\Phi_\circ = 0 \text{ in } V \cap \ker \Gamma_\circ,$$

then

$$\Phi_\circ = \Phi_\circ \circ R_\circ \circ \Gamma_\circ \text{ in } V.$$

Proof. Under the above assumptions, for any $v \in V$, $v = v_\circ + v_1$, $v_\circ \in (\ker \Gamma_\circ)^\perp$ and $v_1 \in \ker \Gamma_\circ$, we have $(\Phi_\circ \circ R_\circ \circ \Gamma_\circ)v = (\Phi_\circ \circ R_\circ \circ \Gamma_\circ)v_\circ = \Phi_\circ v_\circ = \Phi_\circ v$ (see (3.21) for the second equality). $\square$

We are now in position to prove that the solution u_ε of (1.1)-(1.2) can be inserted in an appropriate variational formulation. Indeed, it is not easy to take test functions in (1.1) of the form $v(x, t) = \varphi(x, t) w(\frac{x}{\varepsilon}) \theta(\frac{t}{\varepsilon})$, $\varphi \in \mathcal{D}(Q)$, $w \in \mathbb{H}_{per}^1(Y)$, $\theta \in H_{per}^1(\Theta)$, which belong to V_ε . The following proposition is then fundamental in the rest of this section.

Proposition 1. *There is $0 < \varepsilon_0 \leq 1$ such that for $0 < \varepsilon \leq \varepsilon_0$, there exists $z_\varepsilon \in L^\infty(0, T; \mathbb{H}^1(\Omega_s^\varepsilon; \mathbb{R}))$ verifying*

$$(3.22) \quad \int_Q \rho^\varepsilon \frac{\partial^2 u_\varepsilon}{\partial t^2} \cdot v dx dt + \varepsilon^2 \int_Q b_{ijkh} E_{kh} \left(\frac{\partial u_\varepsilon}{\partial t} \right) E_{ij}(v) dx dt + \vartheta \int_Q \operatorname{div} u_\varepsilon \operatorname{div} v dx dt = \int_{Q_s^\varepsilon} E_{ij}(z_\varepsilon) E_{ij}(v) dx dt + \int_Q f \cdot v dx dt$$

for all $v \in L^2(0, T; \mathbb{H}_0^1(\Omega; \mathbb{R}))$, where u_ε is the solution of (1.1)-(1.2). Furthermore,

$$(3.23) \quad \sup_{0 < \varepsilon \leq \varepsilon_0} \left(\|E_{ij}(z_\varepsilon)\|_{L^2(0, T; L^2(\Omega_s^\varepsilon))} \right) < \infty \quad (1 \leq i, j \leq 3).$$

Proof. We will proceed in five steps.

Step 1. Let us introduce $\Omega_1 = \{x \in \mathbb{R}^3 : d(x, \overline{\Omega}) < 1\}$, where d designates the Euclidean metric and $\overline{\Omega}$ the closure of Ω in $\mathbb{R}_x^3$. For

$0 < \varepsilon \leq 1$, let $t_\varepsilon = \{k \in \mathbb{Z}^3 : \varepsilon(k + Y_s) \subset \Omega_1\}$. Clearly there exists $0 < \varepsilon_0 \leq 1$ such that t_ε is nonempty for $0 < \varepsilon \leq \varepsilon_0$. By letting

$$\Omega_0^\varepsilon = \bigcup_{k \in t_\varepsilon} \varepsilon(k + Y_s) \text{ and } \partial\Omega_0^\varepsilon = \bigcup_{k \in t_\varepsilon} \varepsilon(k + \partial Y_s), \quad 0 < \varepsilon \leq \varepsilon_0,$$

the bounded sets Ω_1 and Ω_0^ε are opened and at least of class $\mathcal{C}^1$ in $\mathbb{R}_x^3$ with $\overline{\Omega_0^\varepsilon} \subset \Omega_1$. Let $0 < \varepsilon \leq \varepsilon_0$ be fixed, and let u_ε be the function defined in Theorem 1. For each $0 \leq t \leq T$, the linear form $\Phi(\varepsilon, t)$ on $\mathbb{H}_0^1(\Omega_1; \mathbb{R})$ defined by

$$\begin{aligned} \Phi(\varepsilon, t)(v) &= \int_{\Omega} \rho^\varepsilon u_\varepsilon''(t) \cdot v dx + \varepsilon^2 \int_{\Omega} b_{ijkh} E_{kh}(u_\varepsilon'(t)) E_{ij}(v) dx \\ (3.24) \quad &+ \vartheta \int_{\Omega} \operatorname{div} u_\varepsilon(t) \operatorname{div} v dx - \int_{\Omega} f(t) \cdot v dx \quad (v \in \mathbb{H}_0^1(\Omega_1; \mathbb{R})) \end{aligned}$$

is continuous.

Step 2. We want to decompose $\Phi(\varepsilon, t)$, further we will apply Lemma 11. Let $\gamma_\circ^\varepsilon$ be the trace operator on $\partial\Omega_0^\varepsilon$. Let $\Gamma_\circ^\varepsilon$ be the operator on $\mathbb{H}_0^1(\Omega_1; \mathbb{R})$ defined by $\Gamma_\circ^\varepsilon v = \gamma_\circ^\varepsilon(v|_{\Omega_0^\varepsilon}) = (\gamma_\circ^\varepsilon(v^i|_{\Omega_0^\varepsilon}))$ ($v = (v^i) \in \mathbb{H}_0^1(\Omega_1; \mathbb{R})$), and let $R_\circ^\varepsilon$ be defined in the sense of Lemma 10. We put $\Phi(\varepsilon, t) = \Phi_\circ(\varepsilon, t) + \Phi_1(\varepsilon, t)$ and $\Upsilon(\varepsilon, t) = \Phi_\circ(\varepsilon, t) \circ R_\circ^\varepsilon$ (usual composition), where $\Phi_\circ(\varepsilon, t)$ and $\Phi_1(\varepsilon, t)$ are the continuous linear forms on $\mathbb{H}_0^1(\Omega_1; \mathbb{R})$ defined by

$$\begin{aligned} \Phi_\circ(\varepsilon, t)(v) &= \int_{\Omega_f^\varepsilon} \rho_f u_\varepsilon''(t) \cdot v dx + \varepsilon^2 \int_{\Omega} b_{ijkh} E_{kh}(u_\varepsilon'(t)) E_{ij}(v) dx \\ &+ \vartheta \int_{\Omega} \operatorname{div} u_\varepsilon(t) \operatorname{div} v dx - \int_{\Omega_f^\varepsilon} f(t) \cdot v dx \end{aligned}$$

and

$$\Phi_1(\varepsilon, t)(v) = \int_{\Omega_s^\varepsilon} \rho_s u_\varepsilon''(t) \cdot v dx - \int_{\Omega_s^\varepsilon} f(t) \cdot v dx$$

for all $v \in \mathbb{H}_0^1(\Omega_1; \mathbb{R})$. It follows immediately that $\Upsilon(\varepsilon, t)$ is a continuous linear form on $\mathbb{H}_0^{\frac{1}{2}}(\partial\Omega_0^\varepsilon)$. By letting

$$\begin{aligned} \widetilde{\mathbb{H}_0^1(\Omega)} &= \{v \in \mathbb{H}_0^1(\Omega_1; \mathbb{R}) : v = \omega \text{ in } \Omega_1 \setminus \Omega\} \text{ and} \\ \widetilde{V}_\varepsilon &= \{v \in \widetilde{\mathbb{H}_0^1(\Omega)} : E_{ij}(v) = 0 \text{ in } \Omega_0^\varepsilon \text{ (} i, j = 1, 2, 3 \text{)}\}, \end{aligned}$$

$\widetilde{\mathbb{H}_0^1(\Omega)}$ is a closed vector subspace of $\mathbb{H}_0^1(\Omega_1)$ and $\widetilde{V}_\varepsilon$ is that of $\widetilde{\mathbb{H}_0^1(\Omega)}$. Since $\Phi_\circ(\varepsilon, t) = 0$ in $\widetilde{\mathbb{H}_0^1(\Omega)} \cap \ker \Gamma_\circ^\varepsilon$, Lemma 11 leads us to

$$(3.25) \quad \Phi(\varepsilon, t) = \Upsilon(\varepsilon, t) \circ \Gamma_\circ^\varepsilon + \Phi_1(\varepsilon, t) \text{ in } \widetilde{\mathbb{H}_0^1(\Omega)}$$

with

$$(3.26) \quad \Upsilon(\varepsilon, t) \circ \Gamma_o^\varepsilon + \Phi_1(\varepsilon, t) = 0 \quad \text{in } \widetilde{V}_\varepsilon.$$

Step 3. We want to solve the variational problem (for fixed $t \in [0, T]$ and $0 < \varepsilon \leq \varepsilon_0$) :

(3.27)

Find $z_\varepsilon(t) \in \mathbb{H}^1(\Omega_s^\varepsilon; \mathbb{R})$ such that

$$\int_{\Omega_s^\varepsilon} E_{ij}(z_\varepsilon(t)) E_{ij}(v) dx = \Upsilon(\varepsilon, t)(\Gamma_o^\varepsilon v) + \Phi_1(\varepsilon, t)(v) \text{ for } v \in \widetilde{\mathbb{H}_0^1(\Omega)}.$$

We begin by noting that the mapping $v \rightarrow \left(\int_{\Omega_s^\varepsilon} E_{ij}(v) E_{ij}(v) dx \right)^{\frac{1}{2}}$ is a norm in the closed vector subspace $\widetilde{\mathbb{H}^1(\Omega_s^\varepsilon)} = \{v \in \mathbb{H}^1(\Omega_0^\varepsilon; \mathbb{R}) : v = \omega \text{ in } \Omega_0^\varepsilon \setminus \Omega_s^\varepsilon\}$ of $\mathbb{H}^1(\Omega_0^\varepsilon; \mathbb{R})$, equivalent to the usual $\mathbb{H}^1(\Omega_0^\varepsilon)$ -norm. Further, we decompose the space $\widetilde{\mathbb{H}^1(\Omega_s^\varepsilon)}$ as $\widetilde{\mathbb{H}^1(\Omega_s^\varepsilon)} = W_\varepsilon \oplus (\widetilde{\mathbb{H}^1(\Omega_s^\varepsilon)}/W_\varepsilon)$, where $W_\varepsilon = \{v \in \widetilde{\mathbb{H}^1(\Omega_s^\varepsilon)} : E_{ij}(v) = 0 \text{ (} i, j = 1, 2, 3 \text{)}\}$ and $\widetilde{\mathbb{H}^1(\Omega_s^\varepsilon)}/W_\varepsilon = \{v \in \widetilde{\mathbb{H}^1(\Omega_s^\varepsilon)} : \int_{\Omega_s^\varepsilon} uv dx = 0 \text{ for all } u \in W_\varepsilon\}$. Let $v \in W_\varepsilon$ and $\tilde{v}$ its extension on Ω_1 such that $\tilde{v} = \omega$ on $\Omega_1 \setminus \Omega$. Then $\tilde{v} \in \widetilde{V}_\varepsilon$ and (3.26) implies

$$(3.28) \quad \Upsilon(\varepsilon, t)(\gamma_o^\varepsilon v) + \int_{\Omega_s^\varepsilon} u_\varepsilon''(t) \cdot v dx - \int_{\Omega_s^\varepsilon} f(t) \cdot v dx = 0 \quad (v \in W_\varepsilon)$$

with $\gamma_o^\varepsilon v = \Gamma_o^\varepsilon \tilde{v}$. Since $\|\gamma_o^\varepsilon v\|_{\mathbb{H}^1(\partial\Omega_0^\varepsilon)} \leq \|\tilde{v}\|_{\mathbb{H}^1(\Omega_0^\varepsilon)}$, the linear form

$$v \rightarrow \Upsilon(\varepsilon, t)(\gamma_o^\varepsilon v) + \int_{\Omega_s^\varepsilon} u_\varepsilon''(t) \cdot v dx - \int_{\Omega_s^\varepsilon} f(t) \cdot v dx$$

is continuous on $\widetilde{\mathbb{H}^1(\Omega_s^\varepsilon)}$. By virtue of (3.28), there exists a unique function $z_\varepsilon(t) \in \widetilde{\mathbb{H}^1(\Omega_s^\varepsilon)}/W_\varepsilon$ such that

$$(3.29) \quad \int_{\Omega_s^\varepsilon} E_{ij}(z_\varepsilon(t)) E_{ij}(v) dx = \Upsilon(\varepsilon, t)(\gamma_o^\varepsilon v) + \int_{\Omega_s^\varepsilon} u_\varepsilon''(t) \cdot v dx - \int_{\Omega_s^\varepsilon} f(t) \cdot v dx$$

for all $v \in \widetilde{\mathbb{H}^1(\Omega_s^\varepsilon)}/W_\varepsilon$. But, according to the definition of W_ε , we may replace the space of test functions $\widetilde{\mathbb{H}^1(\Omega_s^\varepsilon)}/W_\varepsilon$ by $\widetilde{\mathbb{H}^1(\Omega_s^\varepsilon)}$, next $\widetilde{\mathbb{H}^1(\Omega_s^\varepsilon)}$ is replaced by $\widetilde{\mathbb{H}_0^1(\Omega)}$ to obtain (3.27).

Step 4. Let us prove that the function $z_\varepsilon : t \rightarrow z_\varepsilon(t)$ from $[0, T]$ into $\mathbb{H}^1(\Omega_s^\varepsilon)$ belongs to $L^\infty(0, T; \mathbb{H}^1(\Omega_s^\varepsilon))$ with (3.23) : Let $t \in (0, T)$ and $h \in \mathbb{R}$

such that $t + h \in (0, T)$. According to (3.29) we have

$$\begin{aligned} \int_{\Omega_s^\varepsilon} E_{ij}(z_\varepsilon(t+h) - z_\varepsilon(t)) E_{ij}(v) dx &= (\Upsilon(\varepsilon, t+h) - \Upsilon(\varepsilon, t)) (\gamma_o^\varepsilon v) \\ &+ \int_{\Omega_s^\varepsilon} (u_\varepsilon''(t+h) - u_\varepsilon''(t)) \cdot v dx \\ &- \int_{\Omega_s^\varepsilon} (f(t+h) - f(t)) \cdot v dx. \end{aligned}$$

We take $v = z_\varepsilon(t+h) - z_\varepsilon(t)$, and it follows

$$\begin{aligned} c \|z_\varepsilon(t+h) - z_\varepsilon(t)\|_{\widetilde{\mathbb{H}^1(\Omega_s^\varepsilon)}} &\leq \|\Upsilon(\varepsilon, t+h) - \Upsilon(\varepsilon, t)\|_{\mathbb{H}^{-\frac{1}{2}}(\partial\Omega_0^\varepsilon)} \\ &+ \|u_\varepsilon''(t+h) - u_\varepsilon''(t)\|_{\mathbb{L}^2(\Omega_s^\varepsilon)} \\ &+ \|f(t+h) - f(t)\|_{\mathbb{L}^2(\Omega_s^\varepsilon)} \end{aligned}$$

where c does not depend to t , h and $0 < \varepsilon \leq \varepsilon_0$. But $\Upsilon(\varepsilon, t) = \Phi_o(\varepsilon, t) \circ R_o^\varepsilon$, then there exists a constant c independent to t , h and $0 < \varepsilon \leq \varepsilon_0$ such that

$$\begin{aligned} c \|\Upsilon(\varepsilon, t+h) - \Upsilon(\varepsilon, t)\|_{\mathbb{H}^{-\frac{1}{2}}(\partial\Omega_0^\varepsilon)} &\leq \|u_\varepsilon''(t+h) - u_\varepsilon''(t)\|_{\mathbb{L}^2(\Omega_f^\varepsilon)} + \varepsilon \|u_\varepsilon'(t+h) - u_\varepsilon'(t)\|_{\mathbb{H}^1(\Omega)} \\ &+ \|\operatorname{div} u_\varepsilon(t+h) - \operatorname{div} u_\varepsilon(t)\|_{\mathbb{L}^2(\Omega)} + \|f(t+h) - f(t)\|_{\mathbb{L}^2(\Omega_f^\varepsilon)}. \end{aligned}$$

Thus, there is a constant $c > 0$ independent to t , h and $0 < \varepsilon \leq \varepsilon_0$ such that

$$\begin{aligned} c \|z_\varepsilon(t+h) - z_\varepsilon(t)\|_{\widetilde{\mathbb{H}^1(\Omega_s^\varepsilon)}} &\leq \|u_\varepsilon''(t+h) - u_\varepsilon''(t)\|_{\mathbb{L}^2(\Omega)} + \varepsilon \|u_\varepsilon'(t+h) - u_\varepsilon'(t)\|_{\mathbb{H}^1(\Omega)} \\ &+ \|\operatorname{div} u_\varepsilon(t+h) - \operatorname{div} u_\varepsilon(t)\|_{\mathbb{L}^2(\Omega)} + \|f(t+h) - f(t)\|_{\mathbb{L}^2(\Omega)}. \end{aligned}$$

Since f , u_ε , $\operatorname{div} u_\varepsilon$, u_ε' and u_ε'' are measurable, then the function $z_\varepsilon : t \rightarrow z_\varepsilon(t)$ from $[0, T]$ into $\mathbb{H}^1(\Omega_s^\varepsilon)$ is measurable. Furthermore, we are quickly drived to $\|z_\varepsilon(t)\|_{\widetilde{\mathbb{H}^1(\Omega_s^\varepsilon)}} \leq c$ for almost all $t \in (0, T)$ and all $0 < \varepsilon \leq \varepsilon_0$, which proves that z_ε belongs to $L^\infty(0, T; \mathbb{H}^1(\Omega_s^\varepsilon))$ with (3.23).

Step 5. Let us derive (3.22). By combining (3.24), (3.25) and (3.27), we deduce (3.22). $\square$

We end this subsection by proving a crucial lemma for the choice of test functions. We denote by R_{per}^s , the space of $w \in \mathbb{H}_{per}^1(Y_s; \mathbb{R})$, $e_{ij}(w) = 0$,

$i, j = 1, 2, 3$ and by $\mathbb{H}_{per}^1(Y_s; \mathbb{R})/R_{per}^s$, the space of $u \in \mathbb{H}_{per}^1(Y_s; \mathbb{R})$, $\int_{Y_s} u \cdot w dy = 0$ for all $w \in R_{per}^s$.

We have $\mathbb{H}_{per}^1(Y_s; \mathbb{R}) = R_{per}^s \oplus (\mathbb{H}_{per}^1(Y_s; \mathbb{R})/R_{per}^s)$, and the seminorm

$$u \rightarrow e_s(u) = \left(\int_{Y_s} e_{ij}(u) e_{ij}(u) dy \right)^{\frac{1}{2}}$$

is a norm on $\mathbb{H}_{per}^1(Y_s; \mathbb{R})/R_{per}^s$ equivalent to the usual $\mathbb{H}^1(Y_s)$ -norm. We endow $L^2(Q; \mathbb{H}_{per}^1(Y_s; \mathbb{R})/R_{per}^s)$ with the hilbertian norm

$$\|u\|_{Q, Y_s} = \left(\int_Q [e_s(u(x))]^2 dx \right)^{\frac{1}{2}} \quad (u \in L^2(Q; \mathbb{H}_{per}^1(Y_s; \mathbb{R})/R_{per}^s)).$$

Lemma 12. *Given $\Psi_0 \in H^1(Q; \mathbb{H}_{per}^1(Y; \mathbb{R}))$, $e_{ij}(\Psi_0) = 0$ in $Q \times Y_s$ ($i, j = 1, 2, 3$), there exists $\Psi_1 \in L^2(Q; \mathbb{H}_{per}^1(Y_s; \mathbb{R}))$ such that*

$$(3.30) \quad \iint_{Q \times Y_s} e_{ij}(\Psi_1) e_{ij}(v) dx dt dy = - \iint_{Q \times Y_s} E_{ij}(\Psi_0) e_{ij}(v) dx dt dy$$

for all $v \in L^2(Q; \mathbb{H}_{per}^1(Y; \mathbb{R}))$.

Proof. There exists exactly one function $u_1 \in L^2(Q; \mathbb{H}^1(Y_s; \mathbb{R})/R^s)$ such that

$$\iint_{Q \times Y_s} e_{ij}(u_1) e_{ij}(v) dx dt dy = - \iint_{Q \times Y_s} E_{ij}(\Psi_0) e_{ij}(v) dx dt dy$$

for all $v \in L^2(Q; \mathbb{H}^1(Y_s; \mathbb{R})/R^s)$, where $R^s = \{w \in \mathbb{H}^1(Y_s; \mathbb{R}) : e_{ij}(w) = 0, i, j = 1, 2, 3\}$. Extending by periodicity, we obtain a unique function (still denoted u_1) $u_1 \in L^2(Q; \mathbb{H}_{per}^1(Y_s; \mathbb{R})/R_{per}^s)$ such that

$$\iint_{Q \times Y_s} e_{ij}(u_1) e_{ij}(v) dx dt dy = - \iint_{Q \times Y_s} E_{ij}(\Psi_0) e_{ij}(v) dx dt dy$$

for all $v \in L^2(Q; \mathbb{H}_{per}^1(Y_s; \mathbb{R})/R_{per}^s)$. Hence, by extension, there exists a function $\Psi_1 \in L^2(Q; \mathbb{H}_{per}^1(Y_s; \mathbb{R}))$ such that

$$\iint_{Q \times Y_s} e_{ij}(\Psi_1) e_{ij}(v) dx dt dy = - \iint_{Q \times Y_s} E_{ij}(\Psi_0) e_{ij}(v) dx dt dy$$

for all $v \in L^2(Q; \mathbb{H}_{per}^1(Y; \mathbb{R}))$. □

Remark 2. We have

$$\int_{Y_s} e_{ij}(\Psi_1(x, t)) dy = - \int_{Y_s} E_{ij}(\Psi_0(x, t)) dy \text{ a.e. in } Q \text{ for } i, j = 1, 2, 3.$$

Remark 3. Let $\mathcal{C}_{per}^1(Y_s) = \left\{ v \in \mathcal{C}^1(\mathbb{R}^3) : v, \frac{\partial v}{\partial y_i} (i = 1, 2, 3) \text{ are } Y\text{-periodic} \right\}$. Since $\mathcal{D}((0, T)) \otimes \mathcal{D}(\Omega) \otimes \mathcal{C}_{per}^1(Y_s)^3$ is dense in $L^2(Q; H_{per}^1(Y_s))$, we will choose in (3.22) test functions of the form $v = \Psi_\varepsilon \theta^\varepsilon$ with

$$(3.31) \quad \Psi_\varepsilon = \Psi_0^\varepsilon + \varepsilon \Psi_{1n}^\varepsilon \text{ and } \theta = 1 \text{ in } \mathbb{R}$$

$n \in \mathbb{N}$, where

$$(3.32) \quad \begin{cases} \Psi_0 \in \mathcal{D}((0, T)) \otimes \mathcal{D}(\Omega) \otimes \mathcal{W}, \\ \Psi_{1n} \in \mathcal{D}((0, T)) \otimes \mathcal{D}(\Omega) \otimes \mathcal{C}_{per}^1(Y)^3, n \in \mathbb{N}, \\ \text{the restriction } \Psi_{1n}|_{Q \times \tilde{Y}_s} \text{ of } \Psi_{1n} \text{ on } Q \times \tilde{Y}_s \text{ satisfying :} \\ \Psi_{1n}|_{Q \times \tilde{Y}_s} \rightarrow \Psi_1 \text{ in } L^2(Q; \mathbb{H}_{per}^1(Y_s)) \text{ as } n \rightarrow \infty \\ \text{such that } \Psi_1 \text{ is the solution of (3.30) for } \Psi_0 \text{ above} \end{cases}$$

with

$$\mathcal{W} = \{w \in \mathcal{C}_{per}^1(Y; \mathbb{R})^3 : \operatorname{div}_y w = 0 \text{ in } Y_f \text{ and } e_{ij}(w) = 0 \text{ in } Y_s (i, j = 1, 2, 3)\},$$

$$\text{and } \Psi_0^\varepsilon(x, t) = \Psi_0(x, t, \frac{x}{\varepsilon}), \quad \Psi_{1n}^\varepsilon(x, t) = \Psi_{1n}(x, t, \frac{x}{\varepsilon}) \quad (x, t) \in Q.$$

3.3 Convergence theorem for the homogenization problem.

According to Lemma 6, we need to define some vector spaces. It is well-known that $E(\Omega) = \{u = (u^i) \in \mathbb{L}^2(\Omega) : \operatorname{div} u \in L^2(\Omega)\}$ is a Hilbert space for the norm $\|u\|_{E(\Omega)} = \left(\|u\|_{\mathbb{L}^2(\Omega)}^2 + \|\operatorname{div} u\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}$, $u \in E(\Omega)$, further $E_0(\Omega) = \{u \in E(\Omega; \mathbb{R}) : \gamma_n u = 0\}$ is one of its closed vector subspace (cf. [15]), where γ_n is the continuous linear operator from $E(\Omega)$ into $H^{-\frac{1}{2}}(\partial\Omega)$ such that $\gamma_n u(x) = u^i(x) n^i(x)$ for all $x \in \partial\Omega$ and all $u = (u^i) \in \mathcal{D}(\overline{\Omega})^3$, $n = (n^i)$ being the outer unit normal to $\partial\Omega$.

Now, the set

$$W = \{w \in \mathbb{H}_{per}^1(Y; \mathbb{R}) : \operatorname{div}_y w = 0 \text{ in } Y_f \text{ and } e_{ij}(w) = 0 \text{ in } Y_s (i, j = 1, 2, 3)\}$$

is a closed vector subspace of $\mathbb{H}_{per}^1(Y; \mathbb{R})$, and it is obvious that the mapping

$$w \rightarrow \|w\|_W = \left(\|w\|_{\mathbb{L}^2(Y)}^2 + \int_{Y_f} e_{ij}(w) e_{ij}(w) dy \right)^{\frac{1}{2}}$$

from W into $\mathbb{R}$ is a norm in W equivalent to the usual norm of $\mathbb{H}_{per}^1(Y)$. Furthermore, the space $\mathcal{W}$ (of Remark 3) is dense in W . Let

$$E_0(\Omega; W) = \{u = (u^i) \in L^2(\Omega; W) : \tilde{u} \in E_0(\Omega)\},$$

where $\tilde{u} = (\tilde{u}^i)$, $\tilde{u}^i$ being the function on Ω defined by $\tilde{u}^i(x) = \int_Y u^i(x, y) dy$ ($x \in \Omega$). Clearly, $E_0(\Omega; W)$ is a vector subspace of $L^2(\Omega; W)$. Furthermore, $E_0(\Omega; W)$ is a Hilbert space for the norm

$$\|u\|_{E_0(\Omega; W)} = \left(\|u\|_{L^2(\Omega; W)}^2 + \|\operatorname{div} \tilde{u}\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \quad (u \in E_0(\Omega; W)).$$

Theorem 2. *Let $(u_\varepsilon)_{\varepsilon>0}$ be the sequence defined by Theorem 1. Then, as $\varepsilon \rightarrow 0$, we have*

$$(3.33) \quad u_\varepsilon \rightarrow \tilde{u}_0 \text{ in } L^\infty(0, T; E_0(\Omega))\text{-weak}^*,$$

where $\tilde{u}_0$ is the function from Q into $\mathbb{R}^3$ defined by $\tilde{u}_0^i(x, t) = \int_Y u_0^i(x, t, y) dy$ ($(x, t) \in Q$), $u_0 = (u_0^i)$ being the unique function from $[0, T]$ into $L^2(\Omega; W)$ satisfying

$$(3.34) \quad u_0 \in L^2(0, T; E_0(\Omega; W)),$$

$$(3.35) \quad \frac{\partial u_0}{\partial t} \in L^2(0, T; E_0(\Omega; W)) \cap L^\infty(0, T; L^2(\Omega; \mathbb{H}_{per}^1(Y))),$$

$$(3.36) \quad \frac{\partial^2 u_0}{\partial t^2} \in L^2(0, T; L^2(\Omega; \mathbb{L}_{per}^2(Y))),$$

and the variational wave equation

$$(3.37) \quad \iint_{\Omega \times Y} \rho \frac{\partial^2 u_0^i}{\partial t^2}(t) v_0^i dx dy + \iint_{\Omega \times Y_f} \mu_{ijkh} e_{kh} \left(\frac{\partial u_0}{\partial t}(t) \right) e_{ij}(v_0) dx dy + \frac{\vartheta}{|Y_f|} \int_{\Omega} \operatorname{div} \tilde{u}_0(t) \operatorname{div} \tilde{v}_0 dx = \int_{\Omega} f^i(t) \tilde{v}_0^i dx \quad \text{for all } v_0 \in E_0(\Omega; W),$$

$0 < t < T$, with initial conditions

$$(3.38) \quad u_0(0) = \frac{\partial u_0}{\partial t}(0) = \omega,$$

where $\mu_{ijkh} = \mu(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk})$.

Proof. **Step 1.** Let us apply the weak two-scale convergence on a subsequence of $(z_\varepsilon)_{0 < \varepsilon \leq \varepsilon_0}$. According to (3.23), we can extend each z_ε

such that

$$z_\varepsilon \in L^2(0, T; \widetilde{\mathbb{H}_0^1(\Omega)}), \quad 0 < \varepsilon \leq \varepsilon_0, \quad \text{and} \quad \sup_{0 < \varepsilon \leq \varepsilon_0} \left(\|z_\varepsilon\|_{L^2(0, T; \mathbb{H}_0^1(\Omega))} \right) < \infty.$$

Then, there exists a fundamental sequence extracted from $E' \cap (0, \varepsilon_0]$, still denoted E' for simplicity, such that we have [10, Theorem 3], as $E' \ni \varepsilon \rightarrow 0$,

$$(3.39) \quad \int_Q E_{ij}(z_\varepsilon) \Phi^\varepsilon dx dt \rightarrow \iint_{Q \times Z} (E_{ij}(r) + e_{ij}(r_1)) \Phi dx dt dy d\tau$$

for all $\Phi \in L^2(Q; \mathcal{C}_{per}(Z))$ and for $i, j = 1, 2, 3$, where $r \in L^2(0, T; L_{per}^2(\Theta; \mathbb{H}^1(\Omega)))$, $r_1 \in L^2(Q; L_{per}^2(\Theta; \mathbb{H}_\#^1(Y)))$.

Step 2. Here, we will derive (3.33) for a subsequence, further (3.34), (3.35), (3.36), (3.38). Thanks to Lemmas 5-9, it follows that $u_\varepsilon \in L^\infty(0, T; E_0(\Omega))$ with (3.33) as $E' \ni \varepsilon \rightarrow 0$, and properties (3.34), (3.35), (3.36), (3.38). Now, we want to obtain (3.37).

Step 3. We will pass to the limit in (3.22) for a suitable test function. We take in (3.22) test functions of the form $v = (\Psi_0^\varepsilon + \varepsilon \Psi_{1n}^\varepsilon) \theta^\varepsilon$, $n \in \mathbb{N}$, where Ψ_0 and Ψ_{1n} satisfy (3.32), $\theta \in \mathcal{C}_{per}(\Theta)$, $\theta = 1$. We obtain

$$\begin{aligned} & \int_Q \rho^\varepsilon \frac{\partial^2 u_\varepsilon}{\partial t^2} \cdot \Psi_0^\varepsilon \theta^\varepsilon dx dt + \int_{Q_f^\varepsilon} \varepsilon b_{ijkh} E_{kh} \left(\frac{\partial u_\varepsilon}{\partial t} \right) (e_{ij}(\Psi_0))^\varepsilon \theta^\varepsilon dx dt \\ & - \int_{Q_f^\varepsilon} p_\varepsilon (\operatorname{div} \Psi_0)^\varepsilon \theta^\varepsilon dx dt - \int_{Q_f^\varepsilon} p_\varepsilon (\operatorname{div}_y \Psi_{1n})^\varepsilon \theta^\varepsilon dx dt - \int_Q f \cdot \Psi_0^\varepsilon \theta^\varepsilon dx dt \\ & - \int_Q E_{ij}(z_\varepsilon) \chi_s^\varepsilon (E_{ij}(\Psi_0))^\varepsilon \theta^\varepsilon dx dt - \int_Q E_{ij}(z_\varepsilon) \chi_s^\varepsilon (e_{ij}(\Psi_{1n}))^\varepsilon \theta^\varepsilon dx dt \\ & = \varepsilon \int_Q f \cdot \Psi_{1n}^\varepsilon \theta^\varepsilon dx dt + \varepsilon \int_{Q_f^\varepsilon} \varepsilon b_{ijkh} E_{kh} \left(\frac{\partial u_\varepsilon}{\partial t} \right) (E_{ij}(\Psi_0))^\varepsilon \theta^\varepsilon dx dt \\ & + \varepsilon \int_{Q_f^\varepsilon} p_\varepsilon (\operatorname{div} \Psi_{1n})^\varepsilon \theta^\varepsilon dx dt + \varepsilon \int_{Q_s^\varepsilon} E_{ij}(z_\varepsilon) (E_{ij}(\Psi_{1n}))^\varepsilon \theta^\varepsilon dx dt \\ & - \varepsilon \int_Q \rho^\varepsilon \frac{\partial^2 u_\varepsilon}{\partial t^2} \cdot \Psi_{1n}^\varepsilon \theta^\varepsilon dx dt + \varepsilon \int_{Q_f^\varepsilon} \varepsilon b_{ijkh} E_{kh} \left(\frac{\partial u_\varepsilon}{\partial t} \right) (e_{ij}(\Psi_{1n}))^\varepsilon \theta^\varepsilon dx dt \\ & - \varepsilon^2 \int_{Q_f^\varepsilon} \varepsilon b_{ijkh} E_{kh} \left(\frac{\partial u_\varepsilon}{\partial t} \right) (E_{ij}(\Psi_{1n}))^\varepsilon \theta^\varepsilon dx dt + \frac{1}{\varepsilon} \int_{Q_f^\varepsilon} p_\varepsilon (\operatorname{div}_y \Psi_0)^\varepsilon \theta^\varepsilon dx dt \\ & + \frac{1}{\varepsilon} \int_{Q_s^\varepsilon} E_{ij}(z_\varepsilon) (e_{ij}(\Psi_0))^\varepsilon \theta^\varepsilon dx dt, \end{aligned}$$

where χ_s is the characteristic function of $\tilde{Y}_s$ defined on $\mathbb{R}_y^3$. But now, since $\operatorname{div}_y \Psi_0 = 0$ in $Q \times Y_f$ and $e_{ij}(\Psi_0) = 0$ in $Q \times Y_s$ ($i, j = 1, 2, 3$), the

last two terms at the right hand side are equal to zero. Next, we observe that $\chi_s E_{ij}(\Psi_0)\theta$ and $\chi_s e_{ij}(\Psi_{1n})\theta$ belong to $\mathcal{C}(\overline{Q}; L_{per}^\infty(Z))$. Thus, by using (3.39), Lemmas 6 and 9, at the limit (when $E' \ni \varepsilon \rightarrow 0$), we conclude that the right-hand side converges to zero, and the left-hand side converges to

$$\begin{aligned} & \iint_{Q \times Y} \rho \frac{\partial^2 u_0^i}{\partial t^2} \Psi_0^i dx dt dy + \iint_{Q \times Y_f} \mu_{ijkh} e_{kh} \left(\frac{\partial u_0}{\partial t} \right) e_{ij}(\Psi_0) dx dt dy \\ & - \int_Q p \left(\int_{Y_f} \operatorname{div}_y \Psi_{1n} dy \right) dx dt - \iint_{Q \times Y_f} p \operatorname{div} \Psi_0 dx dt dy \\ & - \iiint_{Q \times Y_s \times \Theta} (E_{ij}(r) + e_{ij}(r_1)) (E_{ij}(\Psi_0) + e_{ij}(\Psi_{1n})) dx dt dy d\tau \\ & - \iint_{Q \times Y} f^i \Psi_0^i dx dt dy \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

Letting $z = \int_\Theta r d\tau$ and $z_1 = \int_\Theta r_1 d\tau$, we have $z \in L^2(0, T; \mathbb{H}^1(\Omega))$, $z_1 \in L^2(Q; \mathbb{H}_{per}^1(Y))$ with $E_{ij}(z) = \int_\Theta E_{ij}(r) d\tau$ and $e_{ij}(z_1) = \int_\Theta e_{ij}(r_1) d\tau$. Further it follows that for all $n \in \mathbb{N}$

$$\begin{aligned} & \iint_{Q \times Y} \rho \frac{\partial^2 u_0^i}{\partial t^2} \Psi_0^i dx dt dy + \iint_{Q \times Y_f} \mu_{ijkh} e_{kh} \left(\frac{\partial u_0}{\partial t} \right) e_{ij}(\Psi_0) dx dt dy \\ & - \int_Q p \left(\int_{Y_f} \operatorname{div}_y \Psi_{1n} dy \right) dx dt - \iint_{Q \times Y_f} p \operatorname{div} \Psi_0 dx dt dy \\ & = \iint_{Q \times Y} f^i \Psi_0^i dx dt dy \\ & + \iint_{Q \times Y_s} (E_{ij}(z) + e_{ij}(z_1)) (E_{ij}(\Psi_0) + e_{ij}(\Psi_{1n})) dx dt dy. \end{aligned}$$

Step 4. Derivation of (3.37). Let us consider the function $\xi = (\xi^i)$ from $\tilde{Y}_s$ into $\mathbb{R}^3$ defined by : for $k = (k_i) \in \mathbb{Z}^3$ and $y \in Y_s$, $\xi^i(y + k) = k_i$, $i = 1, 2, 3$. Then the function $y \rightarrow y - \xi(y)$ from $\tilde{Y}_s$ into $\mathbb{R}^3$ belongs to $\mathbb{H}_{per}^1(Y_s)$.

Since $-\int_{Y_f} \operatorname{div}_y \Psi_{1n} dy = \int_{Y_s} \operatorname{div}_y \Psi_{1n} dy$ in Q , we have

$$\begin{aligned} - \iint_{Q \times Y_f} p \operatorname{div}_y \Psi_{1n} dx dt dy &= \iint_{Q \times Y_s} p \operatorname{div}_y \Psi_{1n} dx dt dy \\ &= \iint_{Q \times Y_s} e_{ij}(\Psi_{1n}) e_{ij}(p(y - \xi(y))) dx dt dy. \end{aligned}$$

According to (3.30) and (3.32) it follows that, as $n \rightarrow \infty$,

$$\begin{aligned} - \iint_{Q \times Y_f} p \operatorname{div}_y \Psi_{1n} dx dt dy &\rightarrow - \iint_{Q \times Y_s} E_{ij}(\Psi_0) e_{ij}(p(y - \xi(y))) dx dt dy \\ &= - \iint_{Q \times Y_s} p \operatorname{div} \Psi_0 dx dt dy. \end{aligned}$$

On the other hand, according to the convergence in (3.32), it follows that, up to a subsequence, still denoted $(\Psi_{1n})_n$,

$$\begin{aligned} \iint_{Q \times Y_s} E_{ij}(z) (E_{ij}(\Psi_0) + e_{ij}(\Psi_{1n})) dx dt dy &\rightarrow \\ \iint_{Q \times Y_s} E_{ij}(z) (E_{ij}(\Psi_0) + e_{ij}(\Psi_1)) dx dt dy & \end{aligned}$$

as $n \rightarrow \infty$. Therefore

$$\begin{aligned} \iint_{Q \times Y_s} (E_{ij}(z) + e_{ij}(z_1)) (E_{ij}(\Psi_0) + e_{ij}(\Psi_{1n})) dx dt dy &\rightarrow \\ \iint_{Q \times Y_s} (E_{ij}(z) + e_{ij}(z_1)) (E_{ij}(\Psi_0) + e_{ij}(\Psi_1)) dx dt dy. & \end{aligned}$$

But we have

$$\begin{aligned} &\iint_{Q \times Y_s} E_{ij}(z) (E_{ij}(\Psi_0) + e_{ij}(\Psi_1)) dx dt dy \\ &= \int_Q E_{ij}(z(x, t)) \left(\int_{Y_s} (E_{ij}(\Psi_0(x, t)) + e_{ij}(\Psi_1(x, t))) dy \right) dx dt = 0 \quad (\text{Remark 2}) \end{aligned}$$

and

$$\iint_{Q \times Y_s} e_{ij}(z_1) (E_{ij}(\Psi_0) + e_{ij}(\Psi_1)) dx dt dy = 0 \quad (\text{Lemma 12}).$$

Consequently

$$\begin{aligned} &\iint_{Q \times Y} \rho \frac{\partial^2 u_0^i}{\partial t^2} \Psi_0^i dx dt dy + \iint_{Q \times Y_f} \mu_{ijkh} e_{kh} \left(\frac{\partial u_0}{\partial t} \right) e_{ij}(\Psi_0) dx dt dy \\ &- \iint_{Q \times Y} \frac{\partial p}{\partial x_i} \Psi_0^i dx dt dy = \iint_{Q \times Y} f^i \Psi_0^i dx dt dy. \end{aligned}$$

Clearly (3.16) implies

$$\begin{aligned} & \iint_{Q \times Y} \rho \frac{\partial^2 u_0^i}{\partial t^2} \Psi_0^i dx dt dy + \iint_{Q \times Y_f} \mu_{ijkh} e_{kh} \left(\frac{\partial u_0}{\partial t} \right) e_{ij}(\Psi_0) dx dt dy \\ & + \frac{\vartheta}{|Y_f|} \int_Q \operatorname{div} \tilde{u}_0 \operatorname{div} \left(\int_Y \Psi_0 dy \right) dx dt = \int_Q f^i \left(\int_Y \Psi_0^i dy \right) dx dt, \end{aligned}$$

for all $\Psi_0 \in \mathcal{D}(\Omega) \otimes \mathcal{W}$. This yields (3.37) thanks to the density of $\mathcal{D}(\Omega) \otimes \mathcal{W}$ into $E_0(\Omega; W)$, where

$$\tilde{v}_0(x, t) = \int_Y v_0(x, t, y) dy = \left(\int_Y v_0^i(x, t, y) dy \right).$$

Step 5. It remains to prove that u_0 is the unique function satisfying (3.33)-(3.38). Let us assume that $f = \omega$ and $v_0 = \frac{\partial u_0}{\partial t}(t)$ in (3.37). Then, we have

$$\begin{aligned} & \frac{d}{ds} \left(\iint_{\Omega \times Y} \rho \frac{\partial u_0^i}{\partial t}(s) \frac{\partial u_0^i}{\partial t}(s) dx dy + \frac{\vartheta}{|Y_f|} \int_{\Omega} \operatorname{div} \tilde{u}_0(s) \operatorname{div} \tilde{u}_0(s) dx \right) \\ & + 2 \iint_{\Omega \times Y_f} \mu_{ijkh} e_{kh} \left(\frac{\partial u_0}{\partial t}(s) \right) e_{ij} \left(\frac{\partial u_0}{\partial t}(s) \right) dx dy = 0 \quad (0 < s < t < T). \end{aligned}$$

Integrating from 0 to t , and using the initial conditions (see Lemma 7), we obtain

$$\begin{aligned} & \iint_{\Omega \times Y} \rho \frac{\partial u_0^i}{\partial t}(t) \frac{\partial u_0^i}{\partial t}(t) dx dy + \int_{\Omega} \operatorname{div} \tilde{u}_0(t) \operatorname{div} \tilde{u}_0(t) dx \\ & + \int_0^t \iint_{\Omega \times Y_f} \mu_{ijkh} e_{kh} \left(\frac{\partial u_0}{\partial t}(s) \right) e_{ij} \left(\frac{\partial u_0}{\partial t}(s) \right) dx dy ds \leq 0, \quad 0 < t < T. \end{aligned}$$

Therefore $\|u_0\|_{L^\infty(0, T; E_0(\Omega; W))} = 0$, and u_0 of Lemma 6 is the unique function from $[0, T]$ into $E_0(\Omega; W)$ satisfying (3.33)-(3.38). $\square$

3.4 Physical description in brief of the local problem. Let u_0 be the function defined in Theorem 2. There exists a negligible set $\mathcal{N}_0 \subset \Omega$ such that for $x \in \Omega \setminus \mathcal{N}_0$ the function $t \rightarrow u_0(x, t, \cdot)$ from $[0, T]$ into $\mathbb{L}_{per}^2(Y)$, denoted by $u_0(x)$, belongs to $L^2(0, T; \mathbb{L}_{per}^2(Y))$. Thanks to the isometric isomorphism between the spaces $L^2(0, T; L^2(\Omega; \mathbb{L}_{per}^2(Y)))$ and $L^2(\Omega; L^2(0, T; \mathbb{L}_{per}^2(Y)))$, we show easily that $u_0(x)$, $\frac{\partial}{\partial t} u_0(x) \in L^2(0, T; \mathbb{H}_{per}^1(Y))$ and $\frac{\partial^2}{\partial t^2} u_0(x) \in L^2(0, T; \mathbb{L}_{per}^2(Y))$ with $\frac{\partial}{\partial t} u_0(x) = \frac{\partial u_0}{\partial t}(x, \cdot, \cdot)$ and $\frac{\partial^2}{\partial t^2} u_0(x) = \frac{\partial^2 u_0}{\partial t^2}(x, \cdot, \cdot)$ for all $x \in \Omega \setminus \mathcal{N}$ ($\mathcal{N}$ being a negligible set such that $\mathcal{N}_0 \subset \mathcal{N} \subset \Omega$).

Theorem 3. *Let u_0 be the function in Theorem 2. For almost all $x \in \Omega$, $u_0(x)$ is uniquely determined by $u_0(x) \in \mathcal{C}([0, T]; W)$, $\frac{\partial}{\partial t}u_0(x) \in L^2(0, T; W) \cap \mathcal{C}([0, T]; \mathbb{L}_{per}^2(Y))$, $\frac{\partial^2}{\partial t^2}u_0(x) \in L^2(0, T; \mathbb{L}_{per}^2(Y))$ and the variational problem*

$$(3.40) \quad \int_Y \rho \frac{\partial^2}{\partial t^2} u_0^i(x)(t) w^i dy + \int_{Y_f} \mu_{ijkh} e_{kh} \left(\frac{\partial}{\partial t} u_0(x)(t) \right) e_{ij}(w) dy = \left(f^i(x, t) - \frac{\partial p}{\partial x_i}(x, t) \right) \int_Y w^i dy, \quad w \in W, \quad 0 < t < T,$$

with initial conditions

$$(3.41) \quad u_0(x)(0) = \frac{\partial}{\partial t} u_0(x)(0) = \omega.$$

Proof. It suffices to take in (3.37) test functions of the form $v_0 = \varphi \otimes w$, $\varphi \in \mathcal{D}(\Omega)$, $w \in W$, to obtain (3.40). Let us prove (3.41) : Let $v \in \mathcal{C}_T^1([0, T]; \mathbb{L}_{per}^2(Y)) = \{v \in \mathcal{C}^1([0, T]; \mathbb{L}_{per}^2(Y)) : v(T) = 0\}$. We have

$$\iint_{[0, T] \times Y} u_0(x) v' dt dy = - \int_Y u_0(x)(0) v(0) dy - \iint_{[0, T] \times Y} \frac{\partial}{\partial t} u_0(x) v dt dy.$$

We multiply both the members by $\varphi(x)$, $x \in \Omega \setminus \mathcal{N}$, $\varphi \in \mathcal{D}(\Omega)$, and we integrate on $\Omega \setminus \mathcal{N}$. It results that

$$\begin{aligned} \iint_{Q \times Y} u_0 \varphi \frac{\partial v}{\partial t} dx dt dy = \\ - \iint_{\Omega \times Y} u_0(x)(0) \varphi(x) v(0) dx dy - \iint_{Q \times Y} \frac{\partial u_0}{\partial t} \varphi v dx dt dy \end{aligned}$$

for all $\varphi \in \mathcal{D}(\Omega)$ and all $v \in \mathcal{C}_T^1([0, T]; \mathbb{L}_{per}^2(Y))$. Since $\frac{\partial u_0}{\partial t}(0) = \omega$ (Lemma 7), it follows that $\int_Y u_0(x)(0) w dy = 0$ for all $w \in \mathbb{L}_{per}^2(Y)$. Therefore $u_0(x)(0) = \omega$ for all $x \in \Omega \setminus \mathcal{N}$. We proceed as above, where $u_0(x)$ is replaced by $\frac{\partial}{\partial t} u_0(x)$, to obtain $\frac{\partial}{\partial t} u_0(x)(0) = \omega$.

We use the same arguments as in theorem 1 to end the proof. $\square$

The local problem (3.40)-(3.41) describes the motion of an incompressible viscous fluid with solid rigid particles in suspension. It results to (3.40) that

$$\begin{aligned} \int_{Y_f} \rho_f \frac{\partial^2}{\partial t^2} u_0^i(x)(t) w^i dy + \int_{Y_f} \mu_{ijkh} e_{kh} \left(\frac{\partial}{\partial t} u_0(x)(t) \right) e_{ij}(w) dy \\ - \left(f^i(x, t) - \frac{\partial p}{\partial x_i}(x, t) \right) \int_{Y_f} w^i dy = 0 \end{aligned}$$

for all $w \in \mathbb{H}_{per}^1(Y_f)$, $w = \omega$ on $\partial\widetilde{Y}_s$ and $\operatorname{div}_y w = 0$. Then, for almost all $(x, t) \in Q$ there exists $p_1(x, t) \in L_{per}^2(Y_f)/\mathbb{R} = \{v \in L_{per}^2(Y_f) : \int_{Y_f} v dy = 0\}$ such that (see, e.g., [9])

$$\begin{aligned} & \rho_f \frac{\partial^2}{\partial t^2} u_0^i(x)(t) - \frac{\partial}{\partial y_j} \left(\mu_{ijkh} e_{kh} \left(\frac{\partial}{\partial t} u_0(x)(t) \right) \right) + \frac{\partial p}{\partial x_i}(x, t) \\ & + \frac{p_1(x, t)}{\partial y_i} = f^i(x, t) \quad \text{in } Y_f \quad (i = 1, 2, 3). \end{aligned}$$

But the distribution $\rho_f \frac{\partial^2}{\partial t^2} u_0^i(x) - \frac{\partial}{\partial y_j} (\mu_{ijkh} e_{kh} (\frac{\partial}{\partial t} u_0(x))) + \frac{\partial p}{\partial x_i}(x) - f^i(x)$ belongs to $L^2(0, T; H_{per}^{-1}(Y_f))$ [$H_{per}^{-1}(Y_f)$ denoting the dual of the space $\{w \in H_{per}^1(Y_f) : w = 0 \text{ on } \partial Y_s\}$]; so that $\frac{p_1(x, t)}{\partial y_i} = \frac{p_1(x)}{\partial y_i}(t)$. On letting

$$\sigma_{ij}^0(x) = -\delta_{ij} p_1(x) + \mu_{ijkh} e_{kh} \left(\frac{\partial}{\partial t} u_0(x) \right), \quad i = 1, 2, 3,$$

we have $\frac{\partial \sigma_{ij}^0(x)}{\partial y_j} \in L^2(0, T; L_{per}^2(Y_f))$, $i, j = 1, 2, 3$, and the partial differential equations which govern this motion are (for almost all $x \in \Omega$) :

$$\begin{aligned} (3.42) \quad & \rho_f \frac{\partial^2}{\partial t^2} u_0(x) = \operatorname{div}_y \sigma^0(x) + f(x) - Dp(x) \quad \text{in } Y_f \times (0, T) \\ & \operatorname{div}_y u_0(x) = 0 \quad \text{in } Y_f \times (0, T) \\ & e_{ij}(u_0(x)) = 0 \quad \text{in } Y_s \times (0, T) \quad (i, j = 1, 2, 3) \\ & u_0(x) \text{ is } Y\text{-periodic} \end{aligned}$$

where $Dp = \left(\frac{\partial p}{\partial x_i} \right)_{i=1,2,3}$ and $\sigma^0(x) = (\sigma_{ij}^0(x))_{i,j=1,2,3}$. Otherwise, since (3.40) is equivalent to

$$\begin{aligned} & \int_{Y_f} \rho_f \frac{\partial^2}{\partial t^2} u_0^i(x) w^i dy + \int_{Y_f} \mu_{ijkh} e_{kh} \left(\frac{\partial}{\partial t} u_0(x) \right) e_{ij}(w) dy \\ & - \left(f^i(x) - \frac{\partial p}{\partial x_i}(x) \right) \int_{Y_f} w^i dy = - \int_{Y_s} \left(\rho_s \frac{\partial^2}{\partial t^2} u_0^i(x) - f^i(x) + \frac{\partial p}{\partial x_i}(x) \right) w^i dy \end{aligned}$$

for all $w \in W$, we add $-\int_{Y_f} p_1(x) \operatorname{div}_y w dy$ to both sides and, by using Green formula combining with the first equation in (3.42), we obtain

$$- \int_{Y_s} \left(\rho_s \frac{\partial^2}{\partial t^2} u_0^i(x) - f^i(x) + \frac{\partial p}{\partial x_i}(x) \right) w^i dy + [\sigma_{ij}^0(x) n^j, \gamma_\circ^s w^i]_s = 0 \quad \text{for all } w \in W$$

where $\gamma_\circ^s$ is the trace operator on ∂Y_s , $n = (n^i)$ is the outer unit normal to ∂Y_s with respect to Y_s , $[\cdot, \cdot]_s$ denotes the scalar product in the duality between $H^{-\frac{1}{2}}(\partial Y_s)$ and $H^{\frac{1}{2}}(\partial Y_s)$. Since $w(y) = a + b \wedge y$ in Y_s ($a, b \in \mathbb{R}^3$),

it follows that with $\mathcal{I} = (1, 1, 1)$

$$(3.43) \quad \begin{aligned} \int_{Y_s} \left(\rho_s \frac{\partial^2}{\partial t^2} u_0^i(x) - f^i(x) + \frac{\partial p}{\partial x_i}(x) \right) dy + [\sigma_{ij}^0(x) n^j, 1]_s &= 0 \\ \int_{Y_s} y \wedge \left(\rho_s \frac{\partial^2}{\partial t^2} u_0(x) - f(x) + Dp(x) \right) dy + [y \wedge \sigma^0(x) n, \mathcal{I}]_s &= 0. \end{aligned}$$

Remark 4. If $\sigma_{ij}^0(x) n^j \in L^2(\partial Y_s)$, $i = 1, 2, 3$, then we have $[\sigma_{ij}^0(x) n^j, 1]_s = \int_{\partial Y_s} \sigma_{ij}^0(x) n^j d\sigma$ and $[y \wedge \sigma^0(x) n, \mathcal{I}]_s = \int_{\partial Y_s} y \wedge (\sigma^0(x) n) d\sigma$, where $d\sigma$ is the superficial measure on ∂Y_s .

4. The homogenization result in the case

$$f \in L^2(0, T; \mathbb{L}^2(\Omega))$$

4.1 Preliminaries. In the sequel, we assume that f is fixed in $L^2(0, T; \mathbb{L}^2(\Omega))$. Then, it results just both the first and second estimates of Lemma 5. Therefore we have the following lemma, where we recall that $(u_\varepsilon)_{\varepsilon>0}$ is the sequence obtained in Theorem 1. In what follows, E designates a fundamental sequence.

Lemma 13. *A subsequence E' can be extracted from E such that, as $E' \ni \varepsilon \rightarrow 0$, we have*

$$(4.1) \quad u_\varepsilon^i \rightarrow u_0^i \text{ in } L^2(Q)\text{-weak two-scale, } i = 1, 2, 3,$$

$$(4.2) \quad u_\varepsilon \rightarrow \tilde{u}_0 \text{ in } L^\infty(0, T; E_0(\Omega))\text{-weak } *,$$

where $\tilde{u}_0 = (\tilde{u}_0^i) = (\int_Y u_0^i(\cdot, \cdot, y) dy)$, $u_0 = (u_0^i)$ being a function from $[0, T]$ into W satisfying

$$u_0 \in L^2(0, T; E_0(\Omega; W)), \quad \frac{\partial u_0}{\partial t} \in L^2(0, T; L^2(\Omega; W)), \text{ and } u_0(0) = \omega.$$

Proof. As in the proof of Lemmas 6 and 7, all those properties follow to first and second estimates of Lemma 5. $\square$

Now, let $(f_n)_{n \in \mathbb{N}^*}$ be a sequence in $\mathcal{C}^1([0, T]; \mathbb{L}^2(\Omega))$ such that $f_n(0) = \omega$ for all $n \in \mathbb{N}^*$ and

$$(4.3) \quad \|f_n - f\|_{L^2(0, T; \mathbb{L}^2(\Omega))} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

According to Theorem 1, there exists exactly one sequence $(u_{n\varepsilon})_{n \in \mathbb{N}^*, \varepsilon > 0}$ such that, given $n \in \mathbb{N}^*$ and $\varepsilon > 0$, $u_{n\varepsilon}$ is the unique function from $[0, T]$ into V_ε defined by :

$$u_{n\varepsilon} \in \mathcal{C}([0, T]; V_\varepsilon),$$

$$\frac{\partial u_{n\varepsilon}}{\partial t} \in \mathcal{C}([0, T]; \mathbb{L}^2(\Omega)) \cap L^2(0, T; V_\varepsilon), \quad \frac{\partial^2 u_{n\varepsilon}}{\partial t^2} \in L^2(0, T; \mathbb{L}^2(\Omega))$$

and

$$(4.4) \quad \begin{aligned} & \int_{\Omega} \rho^\varepsilon \frac{\partial^2 u_{n\varepsilon}}{\partial t^2}(t) \cdot v dx + \varepsilon^2 b^\varepsilon \left(\frac{\partial u_{n\varepsilon}}{\partial t}(t), v \right) + a^\varepsilon(u_{n\varepsilon}(t), v) \\ &= \int_{\Omega} f_n(t) \cdot v dx \quad \text{for all } v \in V_\varepsilon \quad (0 < t < T), \\ & u_{n\varepsilon}(0) = \frac{\partial u_{n\varepsilon}}{\partial t}(0) = \omega. \end{aligned}$$

Lemma 14. *Let u_0 and E' be defined in Lemma 13. Then we have*

$$\begin{aligned} \lim_{E' \ni \varepsilon \rightarrow 0} \left(\lim_{n \rightarrow \infty} \int_Q u_{n\varepsilon}^i \Psi^\varepsilon dx dt \right) &= \iint_{Q \times Y} u_0^i \Psi dx dt dy \\ &= \lim_{n \rightarrow \infty} \left(\lim_{E' \ni \varepsilon \rightarrow 0} \int_Q u_{n\varepsilon}^i \Psi^\varepsilon dx dt \right), \end{aligned}$$

for all $\Psi \in L^2(Q; \mathcal{C}_{per}(Y))$, $i = 1, 2, 3$.

Proof. **Step 1.** Let us prove the first equality. Let $\varepsilon \in E'$ be fixed. We show easily that $(u_{n\varepsilon})_{n \in \mathbb{N}^*}$ is a Cauchy sequence in $L^2(0, T; \mathbb{L}^2(\Omega))$, and

$$\lim_{n \rightarrow \infty} \int_Q u_{n\varepsilon}^i \Psi^\varepsilon dx dt = \int_Q u_\varepsilon^i \Psi^\varepsilon dx dt \quad \text{for all } \Psi \in L^2(Q; \mathcal{C}_{per}(Y)), i = 1, 2, 3,$$

where $u_\varepsilon = (u_\varepsilon^i)$ is a term of the sequence $(u_\varepsilon)_{\varepsilon \in E'}$ of Lemma 13. The first equality of Lemma 14 result to (4.1).

Step 2. Now, let us prove the second equality. It follows by Theorem 2 that, for each $n \in \mathbb{N}^*$,

$$(4.5) \quad \lim_{E' \ni \varepsilon \rightarrow 0} \int_Q u_{n\varepsilon}^i \Psi^\varepsilon dx dt = \iint_{Q \times Y} u_{n0}^i \Psi dx dt dy$$

for all $\Psi \in L^2(Q; \mathcal{C}_{per}(Y))$, $i = 1, 2, 3$, where $u_{n0} = (u_{n0}^i)$ is the unique function from $[0, T]$ into W satisfying

$$\begin{aligned} & u_{n0} \in L^2(0, T; E_0(\Omega; W)), \\ & \frac{\partial u_{n0}}{\partial t} \in L^2(0, T; E_0(\Omega; W)) \cap L^\infty(0, T; L^2(\Omega; \mathbb{H}_{per}^1(Y))) \end{aligned}$$

and

$$\frac{\partial^2 u_{n0}}{\partial t^2} \in L^2(Q; \mathbb{L}_{per}^2(Y))$$

and the variational problem

$$(4.6) \quad \iint_{\Omega \times Y} \rho \frac{\partial^2 u_{n0}^i}{\partial t^2}(t) v_0^i dx dy + \iint_{\Omega \times Y_f} \mu_{ijkh} e_{kh} \left(\frac{\partial u_{n0}}{\partial t}(t) \right) e_{ij}(v_0) dx dy \\ + \frac{\vartheta}{|Y_f|} \int_{\Omega} \operatorname{div} \tilde{u}_{n0}(t) \operatorname{div} \tilde{v}_0 dx = \int_{\Omega} f_n^i(t) \tilde{v}_0^i dx \quad \text{for all } v_0 \in E_0(\Omega; W),$$

$0 < t < T$, with initial conditions

$$u_{n0}(0) = \frac{\partial u_{n0}}{\partial t}(0) = \omega.$$

Since n is arbitrarily fixed, we obtain the a priori estimate

$$(4.7) \quad \left\| \frac{\partial u_{n0}}{\partial t} - \frac{\partial u_{m0}}{\partial t} \right\|_{L^\infty(0,T;L^2(\Omega;\mathbb{L}^2(Y)))}^2 + \sum_{i,j=1}^3 \left\| e_{ij} \left(\frac{\partial u_{n0}}{\partial t} - \frac{\partial u_{m0}}{\partial t} \right) \right\|_{L^2(Q \times Y_f)}^2 \\ + \|\operatorname{div}(\tilde{u}_{n0} - \tilde{u}_{m0})\|_{L^\infty(0,T;L^2(\Omega))}^2 \leq c \|f_n - f_m\|_{L^2(0,T;\mathbb{L}^2(\Omega))}^2,$$

for all $m, n \in \mathbb{N}^*$, $m \neq n$, where the constant c does not depend on m and n . Hence $(u_{n0})_{n \in \mathbb{N}^*}$ is a Cauchy sequence in $L^2(Q; \mathbb{L}_{per}^2(Y))$. There exists $u_\infty \in L^2(Q; \mathbb{L}_{per}^2(Y))$ such that $u_{n0} \rightarrow u_\infty$ in $L^2(Q; \mathbb{L}_{per}^2(Y))$ as $n \rightarrow \infty$. This strong limit implies

$$\lim_{n \rightarrow \infty} \iint_{Q \times Y} u_{n0}^i \Psi dx dt dy = \iint_{Q \times Y} u_\infty^i \Psi dx dt dy \quad (\Psi \in L^2(Q; \mathcal{C}_{per}(Y))),$$

for $i = 1, 2, 3$. Combining with (4.5), it follows that

$$\lim_{n \rightarrow \infty} \left(\lim_{E' \ni \varepsilon \rightarrow 0} \int_Q u_{n\varepsilon}^i \Psi^\varepsilon dx dt \right) = \iint_{Q \times Y} u_\infty^i \Psi dx dt dy \quad (\Psi \in L^2(Q; \mathcal{C}_{per}(Y))),$$

for $i = 1, 2, 3$.

Let us prove that $u_0 = u_\infty$ in $L^2(Q; \mathbb{L}^2(Y))$. We fix $\delta > 0$. Let $\Psi \in L^2(Q; \mathcal{C}_{per}(Y))$ be arbitrary fixed, further put

$$a_{n\varepsilon}^i = \int_Q u_{n\varepsilon}^i \Psi^\varepsilon dx dt, \quad a_\varepsilon^i = \int_Q u_\varepsilon^i \Psi^\varepsilon dx dt, \quad a_n^i = \iint_{Q \times Y} u_{n0}^i \Psi dx dt dy \\ a_\infty^i = \iint_{Q \times Y} u_\infty^i \Psi dx dt dy, \quad \text{and } a_0^i = \iint_{Q \times Y} u_0^i \Psi dx dt dy.$$

There are $n_1 \in \mathbb{N}^*$ and $\varepsilon_1 \in E'$ such that

$$|a_{n_1}^i - a_\infty^i| < \frac{\delta}{4}, \quad |a_{n_1\varepsilon_1}^i - a_{\varepsilon_1}^i| < \frac{\delta}{4}, \quad |a_{\varepsilon_1}^i - a_0^i| < \frac{\delta}{4}, \quad \text{and} \quad |a_{n_1\varepsilon_1}^i - a_{n_1}^i| < \frac{\delta}{4}$$

for $i = 1, 2, 3$. Therefore

$$|a_\infty^i - a_0^i| \leq |a_\infty^i - a_{n_1}^i| + |a_{n_1}^i - a_{n_1\varepsilon_1}^i| + |a_{n_1\varepsilon_1}^i - a_{\varepsilon_1}^i| + |a_{\varepsilon_1}^i - a_0^i| < \delta.$$

Since $\delta > 0$ is arbitrarily fixed, we obtain the desired equality.

Thanks to the density of $L^2(Q; \mathcal{C}_{per}(Y))$ in $L^2(Q; L_{per}^2(Y))$, we deduce that $\|u_\infty^i - u_0^i\|_{L^2(Q \times Y)} = 0$ for each $i = 1, 2, 3$. $\square$

Remark 5. It results to (4.7) that $(\frac{\partial u_{n0}}{\partial t})_{n \in \mathbb{N}^*}$ is a Cauchy sequence in $L^2(Q; W)$, so that there is a function $v_\infty \in L^2(Q; W)$ such that

$$\frac{\partial u_{n0}}{\partial t} \rightarrow v_\infty \text{ in } L^2(Q; W) \text{ as } n \rightarrow \infty.$$

According to the continuity of the derivation operator in the distributional space $\mathcal{D}'((0, T); L^2(\Omega; W))$, one proves that $v_\infty = \frac{\partial u_\infty}{\partial t} = \frac{\partial u_0}{\partial t}$.

4.2 Convergence theorem for the homogenization problem. We are now capable to prove the following

Theorem 4. *Let $(u_\varepsilon)_{\varepsilon > 0}$ be the sequence defined by Theorem 1. When $\varepsilon \rightarrow 0$,*

$$(4.8) \quad u_\varepsilon \rightarrow \tilde{u}_0 \text{ in } L^\infty(0, T; E_0(\Omega))\text{-weak}^*$$

where u_0 is the unique function from $[0, T]$ into $L^2(\Omega; W)$ satisfying

$$(4.9) \quad u_0 \in L^\infty(0, T; E_0(\Omega; W)),$$

$$(4.10) \quad \frac{\partial u_0}{\partial t} \in L^2(0, T; L^2(\Omega; W)),$$

and the variational equation

$$(4.11) \quad \begin{aligned} & \frac{d}{dt} \iint_{\Omega \times Y} \rho \frac{\partial u_0}{\partial t}(t) \cdot v_0 dx dy + \iint_{\Omega \times Y_f} \mu_{ijkh} e_{kh} \left(\frac{\partial u_0}{\partial t}(t) \right) e_{ij}(v_0) dx dy \\ & + \frac{\vartheta}{|Y_f|} \int_{\Omega} \operatorname{div} \tilde{u}_0(t) \operatorname{div} \tilde{v}_0 dx = \int_{\Omega} f(t) \cdot \tilde{v}_0 dx \end{aligned}$$

for all $v_0 \in E_0(\Omega; W)$, $0 < t < T$ combined with initial conditions

$$(4.12) \quad u_0(0) = \frac{\partial u_0}{\partial t}(0) = \omega.$$

Proof. By virtue of Lemma 13, it suffices to prove that the weak two-scale limit (4.1) satisfies (4.11)-(4.12), further it is the unique solution of this variational problem. We will proceed in three steps.

Step 1. Let us derive (4.11). We multiply (4.6) by $\phi(t)$, $0 < t < T$ fixed, $\phi \in \mathcal{D}((0, T))$ and integrate by parts from 0 to T to obtain

$$\begin{aligned} & - \iint_{Q \times Y} \rho \frac{\partial u_{n0}^i}{\partial t}(t) v_0^i \phi'(t) dx dy dt \\ & + \iint_{Q \times Y_f} \mu_{ijkh} e_{kh} \left(\frac{\partial u_{n0}}{\partial t}(t) \right) e_{ij}(v_0 \phi(t)) dx dy dt \\ & + \frac{\vartheta}{|Y_f|} \int_Q \operatorname{div} \tilde{u}_{n0}(t) \operatorname{div} \tilde{v}_0 \phi(t) dx dt = \int_Q f_n^i(t) \tilde{v}_0^i \phi(t) dx dt \quad (v_0 \in E_0(\Omega; W)). \end{aligned}$$

We pass to the limit, as $n \rightarrow \infty$, and it follows by (4.3) combined with Lemma 14 and Remark 5 that

$$\begin{aligned} & - \iint_{Q \times Y} \rho \frac{\partial u_0^i}{\partial t}(t) v_0^i \phi'(t) dx dy dt \\ & + \iint_{Q \times Y_f} \mu_{ijkh} e_{kh} \left(\frac{\partial u_0}{\partial t}(t) \right) e_{ij}(v_0 \phi(t)) dx dy dt \\ (4.13) \quad & + \frac{\vartheta}{|Y_f|} \int_Q \operatorname{div} \tilde{u}_0(t) \operatorname{div} \tilde{v}_0 \phi(t) dx dt = \int_Q f^i(t) \tilde{v}_0^i \phi(t) dx dt. \end{aligned}$$

This leads to the variational equation (4.11).

Step 2. In (4.11)-(4.12), it remains to prove that $\frac{\partial u_0}{\partial t}(0) = \omega$. Let $H(\Omega; W)$ be the closure of $L^2(\Omega; W)$ in $L^2(\Omega; \mathbb{L}_{per}^2(Y))$. We endow $H(\Omega; W)$ with the hilbertian norm

$$\|u\|_{H(\Omega; W)} = \left(\iint_{\Omega \times Y} \rho u^i u^i dx dy \right)^{\frac{1}{2}} \quad (u = (u^i) \in H(\Omega; W)).$$

The associated scalar product will be still denoted $[\cdot, \cdot]$. We will assume that $H(\Omega; W)$ is identified to its dual. Since $\mathcal{D}(\Omega) \otimes W$ is dense in $L^2(\Omega; W)$, one has the following continuous canonical injections (with density)

$$E_0(\Omega; W) \hookrightarrow L^2(\Omega; W) \hookrightarrow H(\Omega; W) \hookrightarrow L^2(\Omega; W') \hookrightarrow E_0(\Omega; W)',$$

where W' and $E_0(\Omega; W)'$ are the duals of W and $E_0(\Omega; W)$ respectively. Then, we will also denote by $[\cdot, \cdot]$ the duality pairing between $E_0(\Omega; W)'$ and $E_0(\Omega; W)$. Let A be a continuous linear operator from $E_0(\Omega; W)$ into $E_0(\Omega; W)'$ and B a continuous linear operator from $L^2(\Omega; W)$ into $E_0(\Omega; W)'$ defined by

$$\begin{aligned} [Av, w] &= \frac{\vartheta}{|Y_f|} \int_{\Omega} \operatorname{div} \tilde{v} \operatorname{div} \tilde{w} dx \quad (v, w \in E_0(\Omega; W)), \\ [Bv, w] &= \iint_{\Omega \times Y_f} \mu_{ijkh} e_{kh}(v) e_{ij}(w) dx dy \quad (v \in L^2(\Omega; W), w \in E_0(\Omega; W)). \end{aligned}$$

Let L be the unique function of $L^2(0, T; H(\Omega; W))$ such that for almost all $t \in [0, T]$,

$$\int_{\Omega} f^i(t) \tilde{v}^i dx = [L(t), v] \text{ for all } v \in H(\Omega; W).$$

Then (4.11) becomes

$$(4.14) \quad \frac{d}{dt} \left[\frac{\partial u_0}{\partial t}(t), v_0 \right] = \left[L(t) - B \frac{\partial u_0}{\partial t}(t) - Au_0(t), v_0 \right]$$

for all $v_0 \in E_0(\Omega; W)$, $0 < t < T$. Since $\frac{\partial u_0}{\partial t}$ and $L - B \frac{\partial u_0}{\partial t} - Au_0$ are elements of $L^2(0, T; E_0(\Omega; W)')$, then $\frac{d}{dt} \left(\frac{\partial u_0}{\partial t} \right)$ belongs to $L^2(0, T; E_0(\Omega; W)')$ with $\frac{d}{dt} \left(\frac{\partial u_0}{\partial t} \right) + B \frac{\partial u_0}{\partial t} + Au_0 = L$. Therefore we have $\frac{\partial u_0}{\partial t} \in \mathcal{C}([0, T]; E_0(\Omega; W)')$, so that $\frac{\partial u_0}{\partial t}(0)$ is well defined. Multiplying (4.11) by $\phi(t)$, $t \in (0, T)$, $\phi \in \mathcal{C}_T^1([0, T])$, next integrating by parts from 0 to T yields

$$\begin{aligned} & - \left[\frac{\partial u_0}{\partial t}(0), v_0 \right] \phi(0) - \iint_{Q \times Y} \rho \frac{\partial u_0^i}{\partial t}(t) v_0^i \phi'(t) dx dy dt \\ & + \iint_{Q \times Y_f} \mu_{ijkh} e_{kh} \left(\frac{\partial u_0}{\partial t}(t) \right) e_{ij}(v_0 \phi(t)) dx dy dt \\ (4.15) \quad & + \frac{\vartheta}{|Y_f|} \int_Q \operatorname{div} \tilde{u}_0(t) \operatorname{div} \tilde{v}_0 \phi(t) dx dt \end{aligned}$$

$$(4.16) \quad = \int_Q f^i(t) \tilde{v}_0^i \phi(t) dx dt.$$

On the other hand, multiplying (4.6) by $\phi(t)$, $t \in (0, T)$, $\phi \in \mathcal{C}_T^1([0, T])$, then integration by parts from 0 to T yields

$$\begin{aligned}
& - \iint_{Q \times Y} \rho \frac{\partial u_{n0}^i}{\partial t}(t) v_0^i \phi'(t) dx dy dt \\
& + \iint_{Q \times Y_f} \mu_{ijkh} e_{kh} \left(\frac{\partial u_{n0}}{\partial t}(t) \right) e_{ij}(v_0 \phi(t)) dx dy dt \\
& \frac{\vartheta}{|Y_f|} \int_Q \operatorname{div} \tilde{u}_{n0}(t) \operatorname{div} \tilde{v}_0 \phi(t) dx dt = \int_Q f_n^i(t) \tilde{v}_0^i \phi(t) dx dt, \quad n \in \mathbb{N}^*
\end{aligned}$$

(condition $\frac{\partial u_{n0}}{\partial t}(0) = \omega$ being taking into account). When $n \rightarrow \infty$, one obtains equation (4.13). Combining with (4.16), we are driven to

$$\left[\frac{\partial u_0}{\partial t}(0), v_0 \right] = 0 \text{ for all } v_0 \in E_0(\Omega; W),$$

that is $\frac{\partial u_0}{\partial t}(0) = \omega$ (in $E_0(\Omega; W)'$).

Step 3. Let us derive (4.8). The uniqueness of the solution u_0 in the problem (4.11)-(4.12) is proved in the above section, by assuming that $f = \omega$ (then $f \in \mathcal{C}^1([0, T]; \mathbb{L}^2(\Omega))$ with $f(0) = \omega$). Consequently, we can replace the fundamental sequence E' in Lemma 13 by all the sequence $0 < \varepsilon \leq 1$. The proof follows. $\square$

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