Generalized composition operators from Bloch type spaces to $Q_K$ type spaces

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Abstract. This paper characterizes the boundedness and compactness of the generalized composition operator $(C^g_\varphi f)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi$ from Bloch type spaces to $Q_K$ type spaces.

1. Introduction

Let $\varphi$ be an analytic self-map of the unit disk $D$. For $g \in H(D)$, the class of all analytic functions on $D$, we define a linear operator as follows

$$(C^g_\varphi f)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi, \quad f \in H(D).$$

The operator $C^g_\varphi$ is called the generalized composition operator which is introduced in [5] for the first time. When $g = \varphi'$, we see that this operator is essentially composition operator $C_\varphi$ which is defined by $C_\varphi f = f \circ \varphi$. Therefore, $C^g_\varphi$ is a generalization of the composition operator $C_\varphi$. One of the critical problems on composition operators is to relate function theoretic properties of $\varphi$ to operator theoretic properties of the restriction of $C_\varphi$ to various Banach spaces of analytic functions. The composition
operators on $B^\alpha$, $Q_p$, $F(p,q,s)$ and $Q_k$ have been studied by some authors (see, for example, [3,6,7,10,12] and references therein). The purpose of this paper is to study the boundedness and compactness of the generalized composition operators from Bloch type spaces to $Q_k$ type spaces by the $K$–Carleson measure, which can be viewed as a development of the study on spaces $Q_k$ and $F(p,q,s)$. The corresponding problems for $Q_k$ type spaces were studied in [2] and [4]. For $a \in D$, the Green’s function with logarithmic singularity at $a$ is denoted by $g(z,a) = \log \frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{a - z}{1 - az}$.

Let $K : [0, \infty) \to [0, \infty)$ be a right-continuous function. For $0 < p < \infty$, $-2 < q < \infty$, we say $f \in Q_k(p,q)$ provided

$$\|f\|_{p,K,p,q} = \sup_{a \in D} \int_D |f'(z)|^p (1 - |z|^2)^q K(g(z,a)) dA(z) < \infty,$$

and $f \in Q_{K,0}(p,q)$ provided

$$\lim_{|a| \to 1} \int_D |f'(z)|^p (1 - |z|^2)^q K(g(z,a)) dA(z) = 0,$$

where $dA$ means the normalized Lebesgue area measure on $D$ such that $A(D) = 1$. $Q_{K,0}(p,q)$ is the subspace of $Q_k(p,q)$, and $Q_k(p,q)$ is a Banach space under the norm $\|f\|_{Q_k(p,q)} = |f(0)| + \|f\|_{K,p,q}$. Throughout the paper, we always assume that $K$ satisfies the following conditions:

(a) $K$ is nondecreasing;
(b) $K$ is two times differentiable on $(0, 1)$;
(c) $\int_0^{1/e} K(\log(1/r)) r dr < \infty$;
(d) $K(t) = K(1) > 0$, $t \geq 1$;
(e) $K(2t) \approx K(t)$, $t \geq 0$.

Also, we assume that $\int_0^1 (1 - r^2)^q K(\log \frac{1}{r}) r dr < \infty$. Otherwise, $Q_k(p,q)$ contains constant functions only (see [11]). In order to obtain the main results in this paper, we further assume that $\int_0^1 \varphi_k(s) \frac{ds}{s} < \infty$, where

$$\varphi_k(s) = \sup_{0 < t \leq 1} \frac{K(st)}{K(t)}, \ 0 < s \leq \infty.$$

If $p = 2, q = 0$, we have that $Q_K(p,q) = Q_k$. If $K(t) = t^s$, then $Q_k(p,q) = F(p,q,s)$. We know from [11] that $Q_k(p,q) \subset B^{\frac{2+q}{p-q}}$. For $0 < \alpha < \infty$, a function $f \in H(D)$ is said to belong to the Bloch type.
space $B^\alpha$ with the norm
\[ \|f\|_\alpha = \sup_{z \in D} |f'(z)|(1 - |z|^2)^\alpha. \]

Let $B^\alpha_0$ denote the subspace of $B^\alpha$ consisting of those $f \in B^\alpha$ such that
\[ \lim_{|z| \to 1} |f'(z)|(1 - |z|^2)^\alpha = 0. \]

$B^\alpha$ is also a Banach space under the norm $\|f\|_{B^{\alpha}} = |f(0)| + \|f\|_\alpha$.

For a subarc $I \subset \partial D$, the boundary of $D$, let
\[ S(I) = \{ r\xi \in D : 1 - |I| < r < 1, \xi \in I \}, \]
where $|I|$ denotes the arc length of $I \subset \partial D$. If $|I| \geq 1$ then we set $S(I) = D$. A positive measure $\mu$ on $D$ is said to be a $K-$Carleson measure if
\[ \sup_{I \subset \partial D} \int_{S(I)} K\left(1 - \frac{|z|}{|I|}\right)d\mu(z) < \infty, \]

If
\[ \lim_{|I| \to 0} \int_{S(I)} K\left(1 - \frac{|z|}{|I|}\right)d\mu(z) = 0, \]
then we say $\mu$ is a vanishing $K-$Carleson measure. Clearly, if $K(t) = t^p, 0 < p < \infty$, then $\mu$ is a $K-$Carleson measure if and only if $(1 - |z|^2)^p d\mu(z)$ is a $p-$Carleson measure. Note that $p = 1$ give the classical Carleson measure.

In this paper, two quantities $A_1$ and $A_2$ are said to be equivalent if there exist two finite positive constants $C_1$ and $C_2$ such that $C_2 A_2 \leq A_1 \leq C_1 A_2$, written as $A_1 \approx A_2$. Throughout this paper, $C$ always denote positive constants and may be different at different occurrences.

\section{Preliminaries and lemmas}

\textbf{Lemma 2.1.} [7] Let $\alpha > 0$, there are two functions $f_1, f_2 \in B^\alpha$ such that
\[ |f'_1(z)| + |f'_2(z)| \geq \frac{1}{(1 - |z|^2)^\alpha} \]
for all $z \in D$.

\textbf{Lemma 2.2.} A positive measure $\mu$ on $D$ is a $K-$Carleson measure if and only if
\[ \sup_{a \in D} \int_D K(1 - |\varphi_a(z)|^2)d\mu(z) < \infty, \]
and a positive measure $\mu$ on $D$ is a vanishing $K$–Carleson measure if and only if
\[
\lim_{|a|\to 1} \int_D K(1 - |\varphi_a(z)|^2) d\mu(z) = 0.
\]

**Proof.** It can be obtained from the definitions of $K$–Carleson measure, compact $K$–Carleson measure and Theorem 2.1 in [9]. □

**Lemma 2.3.** Let $0 < p < \infty$, $-2 < q < \infty$. Then
\[
\|f\|_{K,p,q}^p \approx \sup_{a \in D} \int_D |f'(z)|^p (1 - |z|^2)^q K(1 - |\varphi_a(z)|^2) dA(z),
\]
and $f \in Q_{K,0}(p,q)$ if and only if
\[
\lim_{|a|\to 1} \int_D |f'(z)|^p (1 - |z|^2)^q K(1 - |\varphi_a(z)|^2) dA(z) = 0.
\]

**Proof.** It can be obtained from the proof of Theorem 2 in [11]. □

**Lemma 2.4.** Let $0 < p < \infty$, $-2 < q < \infty$. Then a closed set $K$ in $Q_{K,0}(p,q)$ is compact if and only if it is bounded and satisfies
\[
\lim \sup_{|a|\to 1} \int_{f \in K} |f'(z)|^p (1 - |z|^2)^q K(1 - |\varphi_a(z)|^2) dA(z) = 0.
\]

**Proof.** It can be proved similar to Lemma 1 in [8]. □

**Lemma 2.5.** [5] Let $0 < \alpha < \infty$, $g \in H(D)$ and $\varphi$ be an analytic self-map of $D$. Suppose that $X$ is a Banach space. Then $C_\varphi^g : B_0^\alpha \to X$ is compact if and only if $C_\varphi^g : B_0^\alpha \to X$ is weakly compact.

**Lemma 2.6.** Let $\alpha > 0$, $0 < p < \infty$, $-2 < q < \infty$, $g \in H(D)$ and $\varphi$ be an analytic self-map of $D$. Then the operator $C_\varphi^g : B^\alpha \to Q_K(p,q)$ is compact if and only if for any bounded sequence $\{f_n\}$ in $B^\alpha$ which converges to zero uniformly on compact subsets of $D$ as $n \to \infty$, we have $\|C_\varphi^g f_n\|_{Q_K(p,q)} \to 0$ as $n \to \infty$.

**Proof.** It can be proved similar to Proposition 3.11 in [1]. □

**Lemma 2.7.** Let $\alpha > 0$, $0 < p < \infty$, $-2 < q < \infty$, $g \in H(D)$ and $\varphi$ be an analytic self-map of $D$. Suppose that $C_\varphi^g : B^\alpha \to Q_K(p,q)$ is compact, then for every $a \in D$,
\[
\lim_{r \to 1} \int_{|z| > r} |g(z)|^p (1 - |z|^2)^q K(g(z,a)) dA(z) = 0.
\]
Let $f_n(z) = n^{a-1}z^n$, we can check that $\limsup_{n \to \infty} |f_n'(z)| (1 - |z|^2)^\alpha = (2\alpha)\alpha e^{-\alpha}$, and thus $\{f_n\}$ is norm bounded in $B^\alpha$ and converges to zero uniformly on compact subsets of $D$. In view of Lemma 2.6 it follows that $\|C^\alpha_{f_n}\|_{K(p,q)} \to 0$, as $n \to \infty$. Therefore, for given $\varepsilon > 0$ and each $a \in D$, there is an $N > 1$ such that if $n \geq N$, then
\[
\int_D |f_n'(\varphi(z))|^p |g(z)|^p (1 - |z|^2)^q K(g(z), a) dA(z) < \varepsilon.
\]
Given $r \in (0, 1)$, we have
\[
n^{apr(n-1)} \int_{|\varphi| > r} |g(z)|^p (1 - |z|^2)^q K(g(z), a) dA(z) < \varepsilon.
\]
Choosing $r$ so that $n^{apr(n-1)} = 1$, we complete the proof. \hfill \Box

**Lemma 2.8.** Let $\alpha > 0, 0 < p < \infty, -2 < q < \infty, g \in H(D)$ and $\varphi$ be an analytic self-map of $D$. Suppose $C^\alpha_{\varphi} : B^\alpha \to Q_{K(p,q)}$ is compact, then for every $a \in D$,
\[
\lim_{r \to 1} \sup_{f \in B^\alpha} \int_{|\varphi| > r} |f'(\varphi(z))|^p |g(z)|^p (1 - |z|^2)^q K(g(z), a) dA(z) = 0.
\]

**Proof.** Let $B_\alpha$ denote the unit ball of $B_\alpha$ and $f_t(z) = f(tz), t \in (0, 1)$. Suppose $f \in B_\alpha$, then $f_t \to f$ uniformly on compact subsets of $D$ as $t \to 1$ and $\{f_t\}$ is bounded on $B^\alpha$, thus for any $\varepsilon > 0$ and $a \in D$ there is a $t \in (0, 1)$ such that
\[
\int_D |f_t'(\varphi(z)) - f'(\varphi(z))|^p |g(z)|^p (1 - |z|^2)^q K(g(z), a) dA(z) < \frac{\varepsilon}{2^p}.
\]
On the other hand, by Lemma 2.7, for above $\varepsilon$ there is a $\delta = \delta(f, \varepsilon), \delta \in (0, 1)$ such that as $r \in [\delta, 1)$
\[
\int_{|\varphi| > r} |g(z)|^p (1 - |z|^2)^q K(g(z), a) dA(z) < \frac{\varepsilon}{2^p \sup_{t} \|f_t'\|_\infty}.
\]
Therefore, by the triangle inequality, we obtain that
\[
\int_{|\varphi| > r} |f'(\varphi(z))|^p |g(z)|^p (1 - |z|^2)^q K(g(z), a) dA(z)
\]
\[
= \int_{|\varphi| > r} |f'(\varphi(z)) - f_t'(\varphi(z)) + f_t'(\varphi(z))|^p |g(z)|^p (1 - |z|^2)^q K(g(z), a) dA(z)
\]
\[
< \varepsilon.
\]
Next we will show that above $\delta = \delta(f, \varepsilon)$ in fact is independent of $f$. Since $C_\varphi(B_\alpha)$ is relatively compact in $Q_K(p, q)$. It means that there are $f_1, f_2, \ldots, f_m \in B_{B_\alpha}$ such that for any $\varepsilon > 0$ and each $f \in B_{B_\alpha}$ there is a $k, k = 1, 2, \ldots, m$ such that

$$\int_D |f'(\varphi(z)) - f'_k(\varphi(z))|^p |g(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \varepsilon.$$ 

If $\max_{1 \leq k \leq m} \delta_k(f_k, \varepsilon) = \delta < r < 1$, from above proof we have for all $k = 1, 2, \ldots, m$,

$$\int_{|\varphi| > r} |f'_k(\varphi(z))|^p |g(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \varepsilon.$$ 

Hence for any $f \in B_{B_\alpha}$, by the triangle inequality, we obtain that

$$\int_{|\varphi| > r} |f'(\varphi(z))|^p |g(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < 2^p \varepsilon.$$ 

Thus the proof is completed. \(\Box\)

3. Main results

**Theorem 3.1.** Let $\alpha > 0, 0 < p < \infty, -2 < q < \infty, g \in H(D)$ and $\varphi$ be an analytic self-map of $D$. Then following statements are equivalent:

1. $C_\varphi^g : B^\alpha \to Q_K(p, q)$ is bounded;
2. $C_\varphi^g : B_0^\alpha \to Q_K(p, q)$ is bounded;
3. $\sup_{\alpha \in D} \int_D \frac{|g(z)|^p (1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{pa}} K(g(z, a)) dA(z) < \infty$;
4. $\frac{|g(z)|^p (1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{pa}} dA(z)$ is a $K-Carleson$ measure.

**Proof.** (1) $\Rightarrow$ (2) It is obvious.

(2) $\Rightarrow$ (3) For $f \in B^\alpha$ if we set $f_s(t) = f(st)$ for $0 < s < 1$, then $f_s \in B_0^\alpha$ and $\|f_s\|_{B^\alpha} \leq \|f\|_{B^\alpha}$. Thus, by the condition (2) for all $f \in B^\alpha$, we have

$$\|C_\varphi^g(f_s)\|_{Q_K(p, q)} \leq \|C_\varphi^g\| \|f_s\|_{B^\alpha} \leq C \|f\|_{B^\alpha}.$$
From Lemma 2.1, there exist \( f_1 \) and \( f_2 \in B^\alpha \) such that

\[
\sup_{a \in D} \int_D \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}} K(g(z),a) dA(z)
\]

\[
\leq 2^{p-1} \sup_{a \in D} \int_D |f_1(s \varphi(z))|^p|g(z)|^p(1-|z|^2)^q K(g(z),a) dA(z)
\]

\[+ 2^{p-1} \sup_{a \in D} \int_D |f_2(s \varphi(z))|^p|g(z)|^p(1-|z|^2)^q K(g(z),a) dA(z)
\]

\[
\leq C \|C^\alpha_2(f_1)\|_{Q_K(p,q)} + C \|C^\alpha_2(f_2)\|_{Q_K(p,q)}
\]

\[
\leq C \|C^\alpha_2(f_1)\|_{B^\alpha} + C \|C^\alpha_2(f_2)\|_{B^\alpha} < \infty.
\]

Applying Fatou’s lemma to the above inequality, we get

\[
\sup_{a \in D} \int_D \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}} K(g(z),a) dA(z) < \infty.
\]

(3) \( \Rightarrow \) (4) From properties of \( K \) and the condition (3), we obtain

\[
\sup_{a \in D} \int_D \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}} K(1-|\varphi_a(z)|^2) dA(z)
\]

\[
\leq \sup_{a \in D} \int_D \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}} K(2g(z),a) dA(z)
\]

\[
\approx \sup_{a \in D} \int_D \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}} K(g(z),a) dA(z) < \infty.
\]

Thus, by Lemma 2.2, \( \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}} dA(z) \) is a \( K \)-Carleson measure.

(4) \( \Rightarrow \) (1) For any \( f \in B^\alpha \), we have

\[
\sup_{a \in D} \int_D |f'(\varphi(z))|^p|g(z)|^p(1-|z|^2)^q K(1-|\varphi_a(z)|^2) dA(z)
\]

\[
\leq \|f\|_{B^\alpha} \sup_{a \in D} \int_D \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}} K(1-|\varphi_a(z)|^2) dA(z).
\]

In addition, that \( (C^\alpha_\varphi f)(0) = 0 \). From Lemma 2.2 and Lemma 2.3, we have that \( C^\alpha_\varphi : B^\alpha \to Q_K(p,q) \) is bounded.

\[\square\]

**Theorem 3.2.** Let \( \alpha > 0, 0 < p < \infty, -2 < q < \infty, g \in H(D) \) and \( \varphi \) be an analytic self-map of \( D \). Then following statements are equivalent:

1. \( C^\alpha_\varphi : B^\alpha \to Q_K(p,q) \) is bounded;
(2) \( C^\varphi_g : B^\alpha \to Q_{K,0}(p, q) \) is compact;
(3) \( C^\varphi_g : B^\alpha_0 \to Q_{K,0}(p, q) \) is weakly compact;
(4) \( C^\varphi_g : B^\alpha_0 \to Q_{K,0}(p, q) \) is compact;
(5) \( \lim_{|\alpha| \to 1} \int_D \frac{|g(z)|^p(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{p\alpha}} K(g(z, a))dA(z) = 0; \)
(6) \( \frac{|g(z)|^p(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{p\alpha}}dA(z) \) is a vanishing \( K \)-Carleson measure.

**Proof.** \( (1) \Leftrightarrow (3) \) Since \( (C^\varphi_g)^**((B^\alpha_0)^**) = C^\varphi_g(B^\alpha) \subset Q_{K,0}(p, q) \), it follows from Gantmacher’s theorem.

(3) \( \Leftrightarrow (4) \) It follows from Lemma 2.5.

(4) \( \Rightarrow (5) \) Assume that \( C^\varphi_g : B^\alpha \to Q_{K,0}(p, q) \) is compact, then \( C^\varphi_g : B^\alpha_0 \to Q_{K,0}(p, q) \) is weakly compact. From above proofs we have that \( C^\varphi_g : B^\alpha \to Q_{K,0}(p, q) \) is bounded. Hence, as in the proof of Theorem 3.1, there exist \( f_1, f_2 \in B^\alpha \) such that

\[
\lim_{|\alpha| \to 1} \int_D \frac{|g(z)|^p(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{p\alpha}} K(g(z, a))dA(z) \\
\leq 2^{p-1} \lim_{|\alpha| \to 1} \int_D |f'_1(\varphi(z))|^p |g(z)|^p(1 - |z|^2)^q K(g(z, a))dA(z) \\
+ 2^{p-1} \lim_{|\alpha| \to 1} \int_D |f'_2(\varphi(z))|^p |g(z)|^p(1 - |z|^2)^q K(g(z, a))dA(z) \\
= 0.
\]

(5) \( \Rightarrow (6) \) From properties of \( K \) and the condition (5), we obtain

\[
\lim_{|\alpha| \to 1} \int_D \frac{|g(z)|^p(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{p\alpha}} K(1 - |\varphi_a(z)|^2) dA(z) \\
\leq \lim_{|\alpha| \to 1} \int_D \frac{|g(z)|^p(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{p\alpha}} K(2g(z, a)) dA(z) \\
\approx \lim_{|\alpha| \to 1} \int_D \frac{|g(z)|^p(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{p\alpha}} K(g(z, a)) dA(z) \\
= 0.
\]

Thus, by Lemma 2.2, \( \frac{|g(z)|^p(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{p\alpha}}dA(z) \) is a compact \( K \)-Carleson measure.

(6) \( \Rightarrow (2) \) It follows from Lemma 2.4 that \( C^\varphi_g \) is compact if and only if

\[
\lim_{|\alpha| \to 1} \sup_{\|f\|_{B^\alpha_0} \leq 1} \int_D |f'(\varphi(z))|^p |g(z)|^p(1 - |z|^2)^q K(g(z, a))dA(z) = 0.
\]
For \( \|f\|_{B^\alpha} \leq 1 \), by Lemma 2.3, we have
\[
\int_D |f'(\varphi(z))|^p |g(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z)
\]
\[
\approx \int_D |f'(\varphi(z))|^p |g(z)|^p (1 - |z|^2)^q K(1 - |\varphi_a(z)|^2) dA(z)
\]
\[
= \int_D |f'(\varphi(z))|^p (1 - |\varphi(z)|^2)^{p\alpha} \frac{|g(z)|^p (1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{p\alpha}} K(1 - |\varphi_a(z)|^2) dA(z).
\]
Since \( \sup_{\|f\|_{B^\alpha} \leq 1} |f'(w)|(1 - |w|^2)^\alpha = 1 \) for each \( w \in D \), from Lemma 2.2 we get the desired result.

(2) \( \implies \) (1) It is obvious. \( \square \)

**Theorem 3.3.** Let \( \alpha > 0, 0 < p < \infty, -2 < q < \infty, g \in H(D) \) and \( \varphi \) be an analytic self-map of \( D \). Then following statements are equivalent:

1. \( C^\alpha : B^\alpha_0 \to Q_{K,0}(p,q) \) is bounded;

(2)
\[
\lim_{|a| \to 1} \int_D |g(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) = 0
\]
and
\[
\sup_{a \in D} \int_D |g(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \infty.
\]

**Proof.** (2) \( \implies \) (1) Suppose that condition (2) holds and let \( f \in B^\alpha_0 \). Then for any \( \varepsilon > 0 \) there is a \( r \in (0,1) \) such that as \( r < |w| < 1, |f'(w)|^p (1 - |w|^2)^{p\alpha} < \varepsilon \). Thus
\[
\sup_{a \in D} \int_{|\varphi| > r} |f'(\varphi(z))|^p |g(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z)
\]
\[
\leq \varepsilon \sup_{a \in D} \int_{|\varphi| > r} \frac{|g(z)|^p (1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{p\alpha}} K(g(z, a)) dA(z)
\]
\[
\leq C \varepsilon.
\]
On the other hand,
\[
\lim_{|a| \to 1} \int_{|\varphi| \leq r} |f'(\varphi(z))|^p |g(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z)
\]
\[
\leq \frac{\|f\|_{B^\alpha}^p}{(1 - r^2)^{p\alpha}} \lim_{|a| \to 1} \int_{|\varphi| \leq r} |g(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) = 0.
\]
It follows that $C_\varphi^\beta : B_0^\alpha \to Q_{K,0}(p,q)$ is bounded.

(1) $\Rightarrow$ (2) Suppose that $C_\varphi^\beta : B_0^\alpha \to Q_{K,0}(p,q)$ is bounded, then $C_\varphi^\beta : B_0^\alpha \to Q_K(p,q)$ is bounded. By Theorem 3.1, we get that

$$\sup_{a \in D} \int_D \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}} K(g(z,a))dA(z) < \infty.$$ 

Let $f(z) = z \in B_0^\alpha$, the boundedness of $C_\varphi$ implies

$$\lim_{|z| \to 1} \int_D |g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) = 0.$$ 

\[\square\]

**Theorem 3.4.** Let $\alpha > 0, 0 < p < \infty, -2 < q < \infty, \varphi \in H(D)$ and $\varphi$ be an analytic self-map of $D$. Then following statements are equivalent:

1. $C_\varphi^\beta : B_0^\alpha \to Q_K(p,q)$ is compact;
2. $C_\varphi^\beta : B_0^\alpha \to Q_{K}(p,q)$ is compact;
3. $\sup_{a \in D} \int_D |g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) < \infty$ and

$$\lim_{r \to 1} \sup_{a \in D} \int_{|\varphi| > r} |g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) = 0.$$ 

**Proof.** (1) $\Rightarrow$ (2) It is obvious.

(2) $\Rightarrow$ (3) Let $\|f\|_{B_0^\alpha} \leq 1$ and $f_\delta(z) = f(tz)$, then $\|f_\delta\|_{B_0^\alpha} \leq 1$. Fixed $t \in (0,1)$, set $B_{B_0^\alpha}^t = \{f_\delta, f \in B_{B_0^\alpha}\}$. Then $B_{B_0^\alpha}^t \subset B_{B_0^\alpha}$. The compactness of $C_\varphi^\beta$ implies that $C_\varphi^\beta(B_{B_0^\alpha}^t)$ is a relative compact subset of $Q_K(p,q)$. By Lemma 2.8, we see that for any $\varepsilon > 0$ and $a \in D$ there is a $\delta \in (0,1)$ such that as $r \in [\delta,1)$,

$$\sup_{\|f\|_{B_0^\alpha} \leq 1} \int_{|\varphi| > r} |f_\delta'(\varphi(z))|^p |g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) < \varepsilon.$$ 

As in the proof of Theorem 3.1, there exist $f_1, f_2 \in B^\alpha$ such that

$$\int_{|\varphi| > r} |g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) \leq C \int_{|\varphi| > r} |f_1'(t\varphi(z))|^p |g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z)$$

$$+ C \int_{|\varphi| > r} |f_2'(t\varphi(z))|^p |g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z)$$

$$< 2C\varepsilon.$$
By Fatou’s lemma, we have
\[
\lim_{r \to 1} \sup_{a \in D} \int_{|\varphi| > r} \frac{|g(z)|^p(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{pa}} K(g(z, a))dA(z) = 0.
\]

On the other hand, by choosing \( f = z \in B_0^\alpha \) we obtain
\[
\sup_{a \in D} \int_D |g(z)|^p(1 - |z|^2)^q K(g(z, a))dA(z) < \infty.
\]

(3) \( \Rightarrow \) (1) Assume that \( \{f_n\} \subset B^\alpha, \|f_n\|_{B^\alpha} \leq 1 \) and \( f_n \to 0 \) uniformly on compact subsets of \( D \). By condition (3), for any \( \varepsilon > 0 \) there is a \( \delta \in (0, 1) \) such that as \( r \in [\delta, 1) \),
\[
\sup_{a \in D} \int_{|\varphi| > r} |f_n'(\varphi(z))|^p |g(z)|^p(1 - |z|^2)^q K(g(z, a))dA(z)
\leq \|f_n\|_{B^\alpha}^p \sup_{a \in D} \int_{|\varphi| > r} \frac{|g(z)|^p(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{pa}} K(g(z, a))dA(z) < \varepsilon.
\]

On the other hand, since \( f_n'(\varphi(z)) \to 0 \) uniformly on \( \{ z : |\varphi(z)| \leq r \} \), for the above \( \varepsilon > 0 \) there is an integer \( N > 1 \) such that as \( n \geq N \),
\[
\sup_{a \in D} \int_{|\varphi| \leq r} |f_n'(\varphi(z))|^p |g(z)|^p(1 - |z|^2)^q K(g(z, a))dA(z)
\leq \sup_{|\varphi| \leq r} |f_n'(\varphi(z))|^p \sup_{a \in D} \int_{|\varphi| \leq r} |g(z)|^p(1 - |z|^2)^q K(g(z, a))dA(z)
\leq \varepsilon \sup_{a \in D} \int_{|\varphi| \leq r} |g(z)|^p(1 - |z|^2)^q K(g(z, a))dA(z)
\leq C\varepsilon.
\]

Since \((C^g f)(0) = 0\), then we obtain \( \|C^g f_n\|_{Q_K(p, q)} \to 0 \), as \( n \to \infty \). Therefore \( C^g : B^\alpha \to Q_K(p, q) \) is compact. \( \square \)

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