Summability of a Tchebysheff system of functions

Z.T. Abdikalikova¹ and A.A. Kalybay

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Abstract. We consider a special type of Tchebysheff systems of functions $\{u_i(\cdot)\}_{i=0}^n$ and $\{v_i(\cdot)\}_{i=0}^n$ defined on the intervals $(0, 1]$ and $[1, +\infty)$, respectively, such that

$$u_i(t) = t^{\alpha_0} \int_t^1 t_1^{\alpha_1} \int_{t_1}^1 t_2^{\alpha_2} \dots \int_{t_{i-1}}^1 t_i^{\alpha_i} dt_i dt_{i-1} \dots dt_1$$

and

$$v_i(t) = t^{\beta_0} \int_1^t t_1^{\beta_1} \int_1^{t_1} t_2^{\beta_2} \dots \int_1^{t_{i-1}} t_i^{\beta_i} dt_i dt_{i-1} \dots dt_1.$$

We find necessary and sufficient conditions under which functions from the investigated systems belong to the corresponding Lebesgue spaces $L_p(0, 1)$ and $L_p(1, +\infty)$. In order to prove the main results we obtain lower and upper estimates of these functions that are of independent interest.

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1. Introduction

Let α_i and β_i , $i = 0, 1, \dots, n$, be real numbers. On the intervals $(0, 1]$ and $[1, +\infty)$, respectively, we consider the following systems of functions:

$$(1.1) \quad \begin{cases} u_0(t) = t^{\alpha_0} \\ u_1(t) = t^{\alpha_0} \int_t^1 t_1^{\alpha_1} dt_1 \\ u_2(t) = t^{\alpha_0} \int_t^1 t_1^{\alpha_1} \int_{t_1}^1 t_2^{\alpha_2} dt_2 dt_1 \\ \dots \\ u_n(t) = t^{\alpha_0} \int_t^1 t_1^{\alpha_1} \int_{t_1}^1 t_2^{\alpha_2} \dots \int_{t_{n-1}}^1 t_n^{\alpha_n} dt_n dt_{n-1} \dots dt_1 \end{cases}$$

and

$$(1.2) \quad \begin{cases} v_0(t) = t^{\beta_0} \\ v_1(t) = t^{\beta_0} \int_1^t t_1^{\beta_1} dt_1 \\ v_2(t) = t^{\beta_0} \int_1^t t_1^{\beta_1} \int_1^{t_1} t_2^{\beta_2} dt_2 dt_1 \\ \dots \\ v_n(t) = t^{\beta_0} \int_1^t t_1^{\beta_1} \int_1^{t_1} t_2^{\beta_2} \dots \int_1^{t_{n-1}} t_n^{\beta_n} dt_n dt_{n-1} \dots dt_1. \end{cases}$$

These functional systems $\{u_i(\cdot)\}_{i=0}^n$ and $\{v_i(\cdot)\}_{i=0}^n$ form Tchebysheff systems or T – systems on the intervals where they are defined. Moreover, the second system $\{v_i(\cdot)\}_{i=0}^n$ is an extended complete Tchebysheff system or ECT – system.

T – systems are very important in different areas of analysis, the theory of differential equations and statistics, e.g., in the theory of approximation (interpolation methods, cubature formulas), in boundary-value problems and problems with oscillation properties of zeros of the solutions of differential equations, and in the theory of statistical inequalities. In the monographs [1] and [2] we can find an almost complete presentation of applications of Tchebysheff systems.

Let us illustrate at least one classical problem with a solution given in terms of a Tchebysheff system.

Problem definition. Let $P_n(\cdot)$ be a generalized polynomial $P_n(\cdot) = \sum_{i=0}^n c_i u_i(\cdot)$, where $\{u_i(\cdot)\}_{i=0}^n$ are real functions from a function space $\Phi[a, b]$. In the case when $\Phi[a, b]$ is a space of continuous functions and $u(\cdot) \in$

$\Phi[a, b]$, it is required to find a polynomial $P_n^0(\cdot)$ such that $\|u - P_n^0\| = \inf_{P_n} \|u - P_n\|$. This polynomial is called the polynomial of the best uniform approximation.

Haar Theorem. For any continuous function $u(\cdot)$ there exists a unique polynomial of the best uniform approximation if and only if the system of functions $\{u_i(\cdot)\}_{i=0}^n$ is a Tchebysheff system on $[a, b]$.

In our turn we study T – systems $\{u_i(\cdot)\}_{i=0}^n$ and $\{v_i(\cdot)\}_{i=0}^n$ in connection with problems of correct posing of some boundary–value problems for $(n + 1)$:th order differential equations, which have singularities at zero and infinity. In the case when a solution of the differential equation and its derivatives do not have traces at these singular points, we consider the following generalized conditions: at zero

$$\lim_{t \rightarrow 0} D_i u(t) = D_i u(0), \quad i = 0, 1, \dots, n,$$

and at infinity

$$\lim_{t \rightarrow +\infty} D_i u(t) = D_i u(\infty), \quad i = 0, 1, \dots, n,$$

where

$$(1.3) \quad D_0 u(t) = \frac{u(t)}{t^{\alpha_0}}, \quad D_i u(t) = \frac{1}{t^{\alpha_i}} \frac{d}{dt} D_{i-1} u(t), \quad i = 1, 2, \dots, n.$$

It is obvious that the introduced differential operator (1.3) and the systems $\{u_i(\cdot)\}_{i=0}^n$ and $\{v_i(\cdot)\}_{i=0}^n$ have the following close connection: these two systems (here for the second system consider α instead of β) are the fundamental systems of the solutions of the equation:

$$D_{n+1} u(t) = 0.$$

In order to be able to solve the boundary–value problems for equations with singularities at zero and infinity, we are faced with the problem to find conditions under which the functions from (1.1) and (1.2) belong to the Lebesgue space $L_p(I)$, where $I = (0, 1)$ and $I = (1, +\infty)$, respectively, with the norm:

$$\|u\|_{L_p} := \left(\int_I |u(t)|^p dt \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

In this paper we present a complete solution of this problem. In particular, our result can be useful to solve the approximation problem given above or in many other applications of Tchebysheff systems.

If in the system (1.2) we make a change variables $t = \frac{1}{x}$, then we get the system of functions $\{\tilde{v}_i(x)\}_{i=0}^n$, which are defined on the interval $(0, 1]$ and have the same forms as functions from $\{u_i(t)\}_{i=0}^n$, i.e.:

$$(1.4) \quad \begin{cases} \tilde{v}_0(x) = x^{-\beta_0} \\ \tilde{v}_1(x) = x^{-\beta_0} \int_x^1 x_1^{-(\beta_1+2)} dx_1 \\ \tilde{v}_2(x) = x^{-\beta_0} \int_x^1 x_1^{-(\beta_1+2)} \int_{x_1}^1 x_2^{-(\beta_2+2)} dx_2 dx_1 \\ \dots \\ \tilde{v}_n(x) = x^{-\beta_0} \int_x^1 x_1^{-(\beta_1+2)} \int_{x_1}^1 x_2^{-(\beta_2+2)} \dots \int_{x_{n-1}}^1 x_n^{-(\beta_n+2)} dx_n dx_{n-1} \dots dx_1. \end{cases}$$

Therefore, we consider the system (1.1) as the main object of our investigation.

The paper consists of three sections. It is organized as follows: In Section 2 we state and prove our main result (Theorem 1) concerning a characterization of the functions in the studied system, which belong to the space $L_p(0, 1)$. Moreover, we present some lemmas concerning lower and upper estimates of functions from the investigated system. These lemmas are of independent interest but also necessary for the proof of main theorem. Finally, in Section 3 we present and discuss some further remarks and results.

Conventions. Here and in the sequel we suppose that $\sum_{j=i}^k = 0$ if $i > k$. The symbol $X \ll Y$ means $X \leq cY$ with some constant $c > 0$.

2. The main result

Our main result reads:

Theorem 1. *The functions u_i , $i = 0, 1, \dots, n$, from the system (1.1) belong to $L_p(0, 1)$, $1 \leq p < \infty$, if and only if*

$$\min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^s (\alpha_j + 1)) > -\frac{1}{p}.$$

Remark 1. A corresponding result for the functions v_i , $i = 0, 1, \dots, n$, from the system (1.2) is given in the next Section (see Theorem 2).

For the proof of Theorem 1 we need two Lemmas of independent interest.

Assume that $i_0 = \min_{k \in M_i} k$, where

$$M_i = \{k : 0 \leq k \leq i, \alpha_0 + \sum_{j=1}^k (\alpha_j + 1) = \min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^s (\alpha_j + 1))\}.$$

Lemma 1. *For the functions u_i , $i = 1, 2, \dots, n$, from the system (1.1) and for all $0 < \delta < 1$ there exists $0 < \delta_1 \leq \delta$ such that for any $t \in (0, \delta_1]$ the following estimate*

$$(2.1) \quad c_i(\delta) t^{\min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^s (\alpha_j + 1))} \leq u_i(t)$$

holds, where $c_i(\delta) \rightarrow 0$ when $\delta \rightarrow 1$, $i = 1, 2, \dots, n$.

Proof. It is obvious that $0 \leq i_0 \leq i$. We divide the proof into three cases: (1) $i_0 = i$, (2) $0 < i_0 < i$, and (3) $i_0 = 0$. Let $t \in (0, 1]$.

Case 1. $i_0 = i$. In this case, we have

$$u_i(t) = u_{i_0}(t) = t^{\alpha_0} \int_t^1 t_1^{\alpha_1} \int_{t_1}^1 t_2^{\alpha_2} \dots \int_{t_{i_0-1}}^1 t_{i_0}^{\alpha_{i_0}} dt_{i_0} dt_{i_0-1} \dots dt_1.$$

By changing the order of integration, we obtain that

$$u_i(t) = u_{i_0}(t) = t^{\alpha_0} \int_t^1 t_{i_0}^{\alpha_{i_0}} \int_t^{t_{i_0}} t_{i_0-1}^{\alpha_{i_0-1}} \dots \int_t^{t_2} t_1^{\alpha_1} dt_1 dt_2 \dots dt_{i_0}.$$

Moreover, by changing variables $t_j = t\tau_j$, $j = 1, 2, \dots, i_0$, in the obtained integrals, we have that

$$(2.2) \quad u_i(t) = u_{i_0}(t) = t^{\alpha_0 + \sum_{j=1}^{i_0} (\alpha_j + 1)} \int_1^{1/t} \tau_{i_0}^{\alpha_{i_0}} \int_1^{\tau_{i_0}} \tau_{i_0-1}^{\alpha_{i_0-1}} \dots \int_1^{\tau_2} \tau_1^{\alpha_1} d\tau_1 d\tau_2 \dots d\tau_{i_0}.$$

It is easy to see that the function

$$\int_1^{1/t} \tau_{i_0}^{\alpha_{i_0}} \int_1^{\tau_{i_0}} \tau_{i_0-1}^{\alpha_{i_0-1}} \dots \int_1^{\tau_2} \tau_1^{\alpha_1} d\tau_1 d\tau_2 \dots d\tau_{i_0}$$

is non-increasing with respect to the variable t , $0 < t < 1$. Hence, for any $t \in (0, \delta]$, where $0 < \delta < 1$, we get that

$$(2.3) \quad \int_1^{1/t} \tau_{i_0}^{\alpha_{i_0}} \int_1^{\tau_{i_0}} \tau_{i_0-1}^{\alpha_{i_0-1}} \dots \int_1^{\tau_2} \tau_1^{\alpha_1} d\tau_1 d\tau_2 \dots d\tau_{i_0} \geq \int_1^{1/\delta} \tau_{i_0}^{\alpha_{i_0}} \int_1^{\tau_{i_0}} \tau_{i_0-1}^{\alpha_{i_0-1}} \dots \int_1^{\tau_2} \tau_1^{\alpha_1} d\tau_1 d\tau_2 \dots d\tau_{i_0}.$$

By denoting the right-hand side of the last inequality by $c_i^{(1)}(\delta)$, from (2.2) and (2.3) for any $t \in (0, \delta]$ we obtain that

$$(2.4) \quad u_i(t) = u_{i_0}(t) \geq c_i^{(1)}(\delta) t^{\alpha_0 + \sum_{j=1}^{i_0} (\alpha_j + 1)} = c_i^{(1)}(\delta) t^{\min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^s (\alpha_j + 1))},$$

where it is obvious that $c_i^{(1)}(\delta) \rightarrow 0$, when $\delta \rightarrow 1$.

Case 2 $0 < i_0 < i$. For $0 < t \leq \delta < 1$ we have that

$$(2.5) \quad \begin{aligned} u_i(t) &= t^{\alpha_0} \int_t^1 t_1^{\alpha_1} \int_{t_1}^1 t_2^{\alpha_2} \dots \int_{t_{i_0-1}}^1 t_{i_0}^{\alpha_{i_0}} \int_{t_{i_0}}^1 t_{i_0+1}^{\alpha_{i_0+1}} \dots \int_{t_{i-1}}^1 t_i^{\alpha_i} dt_i dt_{i-1} \dots dt_1 \\ &\geq t^{\alpha_0} \int_t^\delta t_1^{\alpha_1} \int_{t_1}^\delta t_2^{\alpha_2} \dots \int_{t_{i_0-1}}^\delta t_{i_0}^{\alpha_{i_0}} dt_{i_0} dt_{i_0-1} \dots dt_1 \\ &\quad \times \int_\delta^1 t_{i_0+1}^{\alpha_{i_0+1}} \int_{t_{i_0+1}}^1 t_{i_0+2}^{\alpha_{i_0+2}} \dots \int_{t_{i-1}}^1 t_i^{\alpha_i} dt_i dt_{i-1} \dots dt_{i_0+1}. \end{aligned}$$

By denoting the last line by $c_i^{(2)}(\delta)$, from (2.5) we get that

$$(2.6) \quad u_i(t) \geq c_i^{(2)}(\delta) t^{\alpha_0} \int_t^\delta t_1^{\alpha_1} \int_{t_1}^\delta t_2^{\alpha_2} \dots \int_{t_{i_0-1}}^\delta t_{i_0}^{\alpha_{i_0}} dt_{i_0} dt_{i_0-1} \dots dt_1,$$

where $c_i^{(2)}(\delta) \rightarrow 0$, when $\delta \rightarrow 1$.

Arguing as before for the first case, in (2.6) we change the order of integration:

$$(2.7) \quad u_i(t) \geq c_i^{(2)}(\delta) t^{\alpha_0} \int_t^\delta t_{i_0}^{\alpha_{i_0}} \int_t^{t_{i_0}} t_{i_0-1}^{\alpha_{i_0-1}} \dots \int_t^{t_2} t_1^{\alpha_1} dt_1 dt_2 \dots dt_{i_0}.$$

Next we change variables $t_j = t\tau_j$, $j = 1, 2, \dots, i_0$, in the integrals from (2.7):

$$(2.8) \quad u_i(t) \geq c_i^{(2)}(\delta) t^{\alpha_0 + \sum_{j=1}^{i_0} (\alpha_j + 1)} \int_1^{\delta/t} \tau_{i_0}^{\alpha_{i_0}} \int_1^{\tau_{i_0}} \tau_{i_0-1}^{\alpha_{i_0-1}} \dots \int_1^{\tau_2} \tau_1^{\alpha_1} d\tau_1 d\tau_2 \dots d\tau_{i_0}.$$

Due to the fact that the function

$$\int_1^{\delta/t} \tau_{i_0}^{\alpha_{i_0}} \int_1^{\tau_{i_0}} \tau_{i_0-1}^{\alpha_{i_0-1}} \dots \int_1^{\tau_2} \tau_1^{\alpha_1} d\tau_1 d\tau_2 \dots d\tau_{i_0}$$

is non-increasing with respect to the variable t , $0 < t \leq \delta$, for any $t \in (0, \frac{1}{2}\delta]$, we have that

$$(2.9) \quad \int_1^{\delta/t} \tau_{i_0}^{\alpha_{i_0}} \int_1^{\tau_{i_0}} \tau_{i_0-1}^{\alpha_{i_0-1}} \dots \int_1^{\tau_2} \tau_1^{\alpha_1} d\tau_1 d\tau_2 \dots d\tau_{i_0} \geq \int_1^2 \tau_{i_0}^{\alpha_{i_0}} \int_1^{\tau_{i_0}} \tau_{i_0-1}^{\alpha_{i_0-1}} \dots \int_1^{\tau_2} \tau_1^{\alpha_1} d\tau_1 d\tau_2 \dots d\tau_{i_0}.$$

By denoting the last integral by $c_i^{(3)}$, from (2.8) and (2.9) for any $t \in (0, \frac{1}{2}\delta]$ we find that

$$(2.10) \quad u_i(t) \geq c_i^{(2)}(\delta) c_i^{(3)} t^{\alpha_0 + \sum_{j=1}^{i_0} (\alpha_j + 1)} = c_i^{(2)}(\delta) c_i^{(3)} t^{\min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^s (\alpha_j + 1))},$$

where $c_i^{(2)}(\delta) c_i^{(3)} \rightarrow 0$, when $\delta \rightarrow 1$.

Case 3 $i_0 = 0$. In this case, $\alpha_0 = \alpha_{i_0}$. Hence, the function u_i can be presented in the following form:

$$u_i(t) = t^{\alpha_{i_0}} \int_t^1 t_{i_0+1}^{\alpha_{i_0+1}} \int_{t_{i_0+1}}^1 t_{i_0+2}^{\alpha_{i_0+2}} \dots \int_{t_{i-1}}^1 t_i^{\alpha_i} dt_i dt_{i-1} \dots dt_{i_0+1}.$$

Arguing in the same way as in the previous cases, for t , $0 < t \leq \delta < 1$, we have that

$$\begin{aligned} & \int_t^1 t_{i_0+1}^{\alpha_{i_0+1}} \int_{t_{i_0+1}}^1 t_{i_0+2}^{\alpha_{i_0+2}} \dots \int_{t_{i-1}}^1 t_i^{\alpha_i} dt_i dt_{i-1} \dots dt_{i_0+1} \\ & \geq \int_\delta^1 t_{i_0+1}^{\alpha_{i_0+1}} \int_{t_{i_0+1}}^1 t_{i_0+2}^{\alpha_{i_0+2}} \dots \int_{t_{i-1}}^1 t_i^{\alpha_i} dt_i dt_{i-1} \dots dt_{i_0+1}. \end{aligned}$$

By denoting the last integral by $c_i^{(4)}(\delta)$, for any $t \in (0, \delta]$ we obtain that

$$(2.11) \quad u_i(t) \geq c_i^{(4)}(\delta) t^{\alpha_{i_0}} = c_i^{(4)}(\delta) t^{\min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^s (\alpha_j + 1))},$$

where $c_i^{(4)}(\delta) \rightarrow 0$, when $\delta \rightarrow 1$.

By combining (2.4), (2.10), and (2.11) we obtain (2.1). $\square$

Lemma 2. *For the functions u_i , $i = 1, 2, \dots, n$, from the system (1.1) for any $t \in (0, 1]$ the following estimate*

$$(2.12) \quad u_i(t) \ll t^{\min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^s (\alpha_j + 1))} |\ln t|^{l_i}$$

holds, where l_i is the number of k , $i_0 + 1 \leq k \leq i$, such that $\sum_{j=i_0+1}^k (\alpha_j + 1) = 0$, if $i_0 < i$, and $l_i = 0$ if $i_0 = i$.

Proof. By assumption, i_0 is the minimal index such that $0 \leq i_0 < i$ and

$$\alpha_0 + \sum_{j=1}^{i_0} (\alpha_j + 1) = \min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^s (\alpha_j + 1)).$$

Therefore, for $i_0 < s \leq i$ we have that

$$(2.13) \quad (\alpha_0 + \sum_{j=1}^{i_0} (\alpha_j + 1)) - (\alpha_0 + \sum_{j=1}^s (\alpha_j + 1)) = - \sum_{j=i_0+1}^s (\alpha_j + 1) \leq 0,$$

and, if $i_0 > 0$, for $0 \leq s \leq i_0 - 1$ we have that

$$(2.14) \quad (\alpha_0 + \sum_{j=1}^{i_0} (\alpha_j + 1)) - (\alpha_0 + \sum_{j=1}^s (\alpha_j + 1)) = \sum_{j=s+1}^{i_0} (\alpha_j + 1) < 0.$$

Let us again divide the proof of Lemma 2 into three cases: (1) $i_0 = i$, (2) $0 < i_0 < i$, and (3) $i_0 = 0$. Let $t \in (0, 1]$.

Case 1 $i_0 = i$. Here $l_i = l_{i_0} = 0$. According to (2.14) we have that

$$\begin{aligned} u_i(t) = u_{i_0}(t) &= t^{\alpha_0} \int_t^1 t_1^{\alpha_1} \int_{t_1}^1 t_2^{\alpha_2} \dots \int_{t_{i_0-1}}^1 t_{i_0}^{\alpha_{i_0}} dt_{i_0} dt_{i_0-1} \dots dt_1 \\ &\leq t^{\alpha_0} \int_t^\infty t_1^{\alpha_1} \int_{t_1}^\infty t_2^{\alpha_2} \dots \int_{t_{i_0-1}}^\infty t_{i_0}^{\alpha_{i_0}} dt_{i_0} dt_{i_0-1} \dots dt_1 \end{aligned}$$

$$(2.15) \quad \leq c_i^{(5)} t^{\alpha_0 + \sum_{j=1}^{i_0} (\alpha_j + 1)} = c_i^{(5)} t^{0 \leq s \leq i}^{\min(\alpha_0 + \sum_{j=1}^s (\alpha_j + 1))} |\ln t|^{l_i},$$

where

$$c_i^{(5)} = \left(\prod_{k=1}^{i_0} \left| \sum_{j=k}^{i_0} (\alpha_j + 1) \right| \right)^{-1}.$$

Case 2 $0 < i_0 < i$. By changing the order of integration in the inter integral, we can present u_i in the following form:

$$(2.16) \quad \begin{aligned} u_i(t) &= t^{\alpha_0} \int_t^1 t_1^{\alpha_1} \int_{t_1}^1 t_2^{\alpha_2} \dots \int_{t_{i_0-1}}^1 t_{i_0}^{\alpha_{i_0}} \int_{t_{i_0}}^1 t_{i_0+1}^{\alpha_{i_0+1}} \dots \int_{t_{i-1}}^1 t_i^{\alpha_i} dt_i dt_{i-1} \dots dt_1 \\ &= t^{\alpha_0} \int_t^1 t_1^{\alpha_1} \int_{t_1}^1 t_2^{\alpha_2} \dots \int_{t_{i_0-1}}^1 t_{i_0}^{\alpha_{i_0}} \\ &\quad \times \left(\int_{t_{i_0}}^1 t_i^{\alpha_i} \int_{t_{i_0}}^{t_i} t_{i-1}^{\alpha_{i-1}} \dots \int_{t_{i_0}}^{t_{i_0+2}} t_{i_0+1}^{\alpha_{i_0+1}} dt_{i_0+1} dt_{i_0+2} \dots dt_i \right) dt_{i_0} dt_{i_0-1} \dots dt_1. \end{aligned}$$

We separately consider the following integral:

$$(2.17) \quad I_{(i, i_0)}(t_{i_0}) = \int_{t_{i_0}}^1 t_i^{\alpha_i} \int_{t_{i_0}}^{t_i} t_{i-1}^{\alpha_{i-1}} \dots \int_{t_{i_0}}^{t_{i_0+2}} t_{i_0+1}^{\alpha_{i_0+1}} dt_{i_0+1} dt_{i_0+2} \dots dt_i.$$

According to (2.13) for $s = i_0 + 1$ we have that $\alpha_{i_0+1} + 1 \geq 0$. There are two possible cases: either $\alpha_{i_0+1} + 1 = 0$ or $\alpha_{i_0+1} + 1 > 0$. For the first case $\alpha_{i_0+1} + 1 = 0$, by taking into account that $t \leq t_{i_0} \leq t_{i_0+2} \leq 1$, we get that

$$\int_{t_{i_0}}^{t_{i_0+2}} t_{i_0+1}^{\alpha_{i_0+1}} dt_{i_0+1} = \int_{t_{i_0}}^{t_{i_0+2}} t_{i_0+1}^{-1} dt_{i_0+1} \leq \int_t^1 t_{i_0+1}^{-1} dt_{i_0+1} = |\ln t|.$$

This gives that

$$(2.18) \quad I_{(i, i_0+1)}(t_{i_0}) \leq |\ln t| \int_{t_{i_0}}^1 t_i^{\alpha_i} \int_{t_{i_0}}^{t_i} t_{i-1}^{\alpha_{i-1}} \dots \int_{t_{i_0}}^{t_{i_0+3}} t_{i_0+2}^{\alpha_{i_0+2} + \alpha_{i_0+1} + 1} dt_{i_0+2} dt_{i_0+3} \dots dt_i$$

for $0 < t \leq t_0 < 1$.

If $\alpha_{i_0+1} + 1 > 0$, then

$$\int_{t_{i_0}}^{t_{i_0}+2} t_{i_0+1}^{\alpha_{i_0+1}+1} dt_{i_0+1} \leq \int_0^{t_{i_0}+2} t_{i_0+1}^{\alpha_{i_0+1}+1} dt_{i_0+1} = \frac{1}{\alpha_{i_0+1}+1} t_{i_0+2}^{\alpha_{i_0+1}+1}.$$

Hence,

(2.19)

$$I_{(i,i_0+1)}(t_{i_0}) \leq \frac{1}{\alpha_{i_0+1}+1} \int_{t_{i_0}}^1 t_i^{\alpha_i} \int_{t_{i_0}}^{t_i} t_{i-1}^{\alpha_{i-1}} \dots \int_{t_{i_0}}^{t_{i_0}+3} t_{i_0+2}^{\alpha_{i_0+2}+\alpha_{i_0+1}+1} dt_{i_0+2} dt_{i_0+3} \dots dt_i.$$

In (2.18) and (2.19) the integral expressions are equal, and the number of iterated integrals is one less than in (2.17). From (2.13), by continuing to reduce the number of iterated integrals, for $0 < t \leq t_{i_0} \leq 1$, we have that

$$(2.20) \quad I_{(i,i_0)}(t_{i_0}) \leq c_i^{(6)} |\ln t|^{l_i},$$

where $c_i^{(6)}$, $i = 0, 1, \dots, n$, depend only on α_i , $i = 0, 1, \dots, n$.

Hence, from (2.16) and (2.20) it follows that

$$u_i(t) \leq c_i^{(6)} |\ln t|^{l_i} t^{\alpha_0} \int_t^1 t_1^{\alpha_1} \int_{t_1}^1 t_2^{\alpha_2} \dots \int_{t_{i_0-1}}^1 t_{i_0}^{\alpha_{i_0}} dt_{i_0} dt_{i_0-1} \dots dt_1 = c_i^{(6)} |\ln t|^{l_i} u_{i_0}(t).$$

In view of (2.15) this estimate yields

$$(2.21) \quad u_i(t) \leq c_i^{(5)} c_i^{(6)} t^{\min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^s (\alpha_j + 1))} |\ln t|^{l_i}.$$

Case 3 $i_0 = 0$. By changing the order of integration, the function u_i can be presented in the following form:

(2.22)

$$u_i(t) = t^{\alpha_{i_0}} \int_t^1 t_i^{\alpha_i} \int_t^{t_i} t_{i-1}^{\alpha_{i-1}} \dots \int_t^{t_2} t_1^{\alpha_1} dt_1 dt_2 \dots dt_i = t^{\min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^s (\alpha_j + 1))} I_{(i,0)}(t).$$

Since in this case, due to (2.13), $\sum_{j=1}^s (\alpha_j + 1) \geq 0$ for all $1 \leq s \leq i$, then in the same way as in the previous case it is easy to prove that $I_{(i,0)}(t) \leq c_i^{(7)} |\ln t|^{l_i}$, where $c_i^{(7)}$, $i = 0, 1, \dots, n$, depend only on α_i , $i = 0, 1, \dots, n$.

This estimate together with (2.22) give that

$$(2.23) \quad u_i(t) \leq c_i^{(7)} t^{\min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^s (\alpha_j + 1))} |\ln t|^{l_i}.$$

By combining (2.15), (2.21) and (2.23) we obtain (2.12). $\square$

Remark 2. The assumptions of Lemma 1 and Lemma 2 hold also for the functions u_0 , since in this case $i_0 = 0$ and $l_0 = 0$, and, thus,

$$u_0 = t^{\alpha_0} = t^{\min_{0 \leq s \leq 0} (\alpha_0 + \sum_{j=1}^s (\alpha_j + 1))} |\ln t|^{l_0}.$$

Proof of Theorem 1. Let $u_i \in L_p(0, 1)$. Then, due to Lemma 1 (see (2.1)) and Remark 2, we have that $t^{\min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^s (\alpha_j + 1))} \in L_p(0, 1)$. This yields that

$$\min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^s (\alpha_j + 1)) > -\frac{1}{p}.$$

Now suppose that $\min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^s (\alpha_j + 1)) > -\frac{1}{p}$. Then, for every i there exists $\delta_i > 0$ such that

$$(2.24) \quad \min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^s (\alpha_j + 1)) - \delta_i > -\frac{1}{p}.$$

Moreover, according to Lemma 2 (see (2.12)) and Remark 2, it yields that

$$(2.25) \quad u_i(t) \ll t^{\min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^s (\alpha_j + 1))} |\ln t|^{l_i} \ll t^{\min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^s (\alpha_j + 1)) - \delta_i} \sup_{0 \leq t \leq 1} t^{\delta_i} |\ln t|^{l_i}.$$

Since $\sup_{0 \leq t \leq 1} t^{\delta_i} |\ln t|^{l_i} < \infty$ for any $l_i \geq 0$, then from (2.25) and (2.24) we conclude that $u_i \in L_p(0, 1)$. $\square$

3. Further remarks and results

The following Corollary follows from Lemma 1 and Lemma 2:

Corollary 1. *If $\alpha_0 + \sum_{j=1}^m (\alpha_j + 1) \neq \alpha_0 + \sum_{j=1}^l (\alpha_j + 1)$ for $m \neq l$, $m, l = 0, 1, \dots, n$, then for the functions u_i , $i = 1, 2, \dots, n$, from the system*

(1.1) for $0 < \delta < 1$ the following estimate holds

$$u_i(t) \approx t^{\min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^s (\alpha_j + 1))}, \quad t \in (0, \delta).$$

Let us now turn to the problem of summability of functions from (1.2).

Theorem 2. *The functions v_i , $i = 0, 1, \dots, n$, from the system (1.2) belong to $L_p(1, +\infty)$, $1 \leq p < \infty$, if and only if*

$$\max_{0 \leq s \leq i} (\beta_0 + \sum_{j=1}^s (\beta_j + 1)) < -\frac{1}{p}.$$

It was mentioned above that if in (1.2) we change variables $t = \frac{1}{x}$, then we get the system (1.4) of functions defined on $(0, 1]$. Moreover, these functions have the same forms as functions from (1.1). By comparing powers of functions from (1.1) and (1.4), we have that $\alpha_0 = -\beta_0$, $\alpha_i = -(\beta_i + 2)$, $i = 1, 2, \dots, n$. Therefore, all conditions we introduced for (1.1) can easily be rewritten for the system (1.4). Thus,

$$\begin{aligned} \min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^s (\alpha_j + 1)) &= \min_{0 \leq s \leq i} (-\beta_0 + \sum_{j=1}^s (-(\beta_j + 2) + 1)) \\ (3.1) \qquad \qquad \qquad &= -\max_{0 \leq s \leq i} (\beta_0 + \sum_{j=1}^s (\beta_j + 1)). \end{aligned}$$

Now, $i_0 = \min_{k \in L_i} k$, where

$$L_i = \{k : 0 \leq k \leq i, \beta_0 + \sum_{j=1}^k (\beta_j + 1) = \max_{0 \leq s \leq i} (\beta_0 + \sum_{j=1}^s (\beta_j + 1))\}.$$

Moreover, the conditions $\sum_{j=i_0+1}^k (\alpha_j + 1) = 0$ and $\sum_{j=i_0+1}^k (\beta_j + 1) = 0$ are equivalent since

$$(3.2) \quad \sum_{j=i_0+1}^k (\alpha_j + 1) = \sum_{j=i_0+1}^k (-(\beta_j + 2) + 1) = -\sum_{j=i_0+1}^k (\beta_j + 1) = 0.$$

Finally, to prove Theorem 2 we state and prove the corresponding Lemmas (to Lemmas 1 and Lemma 2).

Lemma 3. *For the functions v_i , $i = 1, 2, \dots, n$, from the system (1.2) and for all $\lambda > 1$ there exists $\lambda_1 \geq \lambda > 1$ such that for any $t \in [\lambda_1, +\infty)$*

the estimate

$$c_i(\lambda)t^{\max_{0 \leq s \leq i}(\beta_0 + \sum_{j=1}^s(\beta_j+1))} \leq v_i(t)$$

holds, where $c_i(\lambda) \rightarrow 0$ when $\lambda \rightarrow 1$.

Proof. In view of Lemma 1 and (3.1) for functions from (1.4) and for any $0 < \delta < 1$ there exists $0 < \delta_1 \leq \delta$ such that

$$(3.3) \quad \tilde{c}_i(\delta)x^{-\max_{0 \leq s \leq i}(\beta_0 + \sum_{j=1}^s(\beta_j+1))} \leq \tilde{v}_i(x), \text{ for } x \in (0, \delta_1],$$

where $\tilde{c}_i(\delta) \rightarrow 0$ when $\delta \rightarrow 1$.

Let $\lambda = \frac{1}{\delta}$, $\lambda_1 = \frac{1}{\delta_1}$. Since $\tilde{v}_i(x) = v_i(t)$ for $x = \frac{1}{t}$, we can substitute them into (3.3) and then, for any $t \in [\lambda_1, +\infty)$, we get that

$$\tilde{c}_i\left(\frac{1}{\lambda}\right)\left(\frac{1}{t}\right)^{-\max_{0 \leq s \leq i}(\beta_0 + \sum_{j=1}^s(\beta_j+1))} \leq v_i(t),$$

where $\tilde{c}_i(\frac{1}{\lambda}) \rightarrow 0$ when $\lambda \rightarrow 1$, i.e. that

$$c_i(\lambda)t^{\max_{0 \leq s \leq i}(\beta_0 + \sum_{j=1}^s(\beta_j+1))} \leq v_i(t),$$

where $c_i(\lambda) \equiv \tilde{c}_i(\frac{1}{\lambda}) \rightarrow 0$ when $\lambda \rightarrow 1$. □

Lemma 4. For the functions v_i , $i = 1, 2, \dots, n$, from the system (1.2) for any $t \in [1, +\infty)$ the estimate

$$v_i(t) \ll t^{\max_{0 \leq s \leq i}(\beta_0 + \sum_{j=1}^s(\beta_j+1))} |\ln t|^{l_i}$$

holds, where l_i is the number of k , $i_0 + 1 \leq k \leq i$, such that $\sum_{j=i_0+1}^k (\beta_j + 1) = 0$, if $i > i_0$, and $l_i = 0$, if $i = i_0$.

Proof. Due to Lemma 2 and (3.1) for functions from (1.4) for any $x \in (0, 1]$ we find that

$$(3.4) \quad \tilde{v}_i(x) \ll x^{-\max_{0 \leq s \leq i}(\beta_0 + \sum_{j=1}^s(\beta_j+1))} |\ln x|^{l_i},$$

where, due to (3.2), l_i is the number of k , $i_0 + 1 \leq k \leq i$, such that $\sum_{j=i_0+1}^k (\beta_j + 1) = 0$.

In (3.4) we substitute $\tilde{v}_i(x) = v_i(t)$ and $x = \frac{1}{t}$ and then, for any $t \in [1, +\infty)$, we get that

$$v_i(t) \ll \left(\frac{1}{t}\right)^{-\max_{0 \leq s \leq i} (\beta_0 + \sum_{j=1}^s (\beta_j + 1))} \left|\ln \frac{1}{t}\right|^{l_i} = t^{\max_{0 \leq s \leq i} (\beta_0 + \sum_{j=1}^s (\beta_j + 1))} |\ln t|^{l_i}.$$

□

Remark 3. The assumptions of Lemma 3 and Lemma 4 hold also for the functions v_0 , since in this case $i_0 = 0$ and $l_0 = 0$, and, thus, $v_0 = t^{\beta_0} = t^{\max_{0 \leq s \leq 0} (\beta_0 + \sum_{j=1}^s (\beta_j + 1))} |\ln t|^{l_0}$.

Proof of Theorem 2. Assume that $\max_{0 \leq s \leq i} (\beta_0 + \sum_{j=1}^s (\beta_j + 1)) < -\frac{1}{p}$. Then, for every i there exists $\gamma_i > 0$ such that

$$(3.5) \quad \max_{0 \leq s \leq i} (\beta_0 + \sum_{j=1}^s (\beta_j + 1)) + \gamma_i < -\frac{1}{p}.$$

On the other hand from Lemma 4 and Remark 3 it follows that

$$(3.6) \quad v_i(t) \ll t^{\max_{0 \leq s \leq i} (\beta_0 + \sum_{j=1}^s (\beta_j + 1))} |\ln t|^{l_i} \ll t^{\max_{0 \leq s \leq i} (\beta_0 + \sum_{j=1}^s (\beta_j + 1)) + \gamma_i} \sup_{1 \leq t \leq \infty} t^{-\gamma_i} |\ln t|^{l_i}.$$

Since $\sup_{1 \leq t \leq \infty} t^{-\gamma_i} |\ln t|^{l_i} < \infty$ for any $l_i \geq 0$, then from (3.6) and (3.5) we have that $v_i \in L_p(1, +\infty)$.

Let now $v_i \in L_p(1, +\infty)$. Then, in view of Lemma 3 and Remark 3, we find that

$$t^{\max_{0 \leq s \leq i} (\beta_0 + \sum_{j=1}^s (\beta_j + 1))} \in L_p(1, +\infty).$$

This gives that $\max_{0 \leq s \leq i} (\beta_0 + \sum_{j=1}^s (\beta_j + 1)) < -\frac{1}{p}$. □

Next consider the following system:

$$(3.7) \quad \begin{cases} w_0(t) = t^{\alpha_0} \\ w_1(t) = t^{\alpha_0} \int_t^1 t_1^{\alpha_1} dt_1 \\ w_2(t) = t^{\alpha_0} \int_t^1 t_1^{\alpha_1} \int_t^{t_1} t_2^{\alpha_2} dt_2 dt_1 \\ \dots \\ w_n(t) = t^{\alpha_0} \int_t^1 t_1^{\alpha_1} \int_t^{t_1} t_2^{\alpha_2} \dots \int_t^{t_{n-1}} t_n^{\alpha_n} dt_n dt_{n-1} \dots dt_1. \end{cases}$$

In this system the first functions w_0 and w_1 coincide with the functions u_0 and u_1 from (1.1), respectively. But, as we can see, the lower and upper integral bounds of the functions w_i and u_i , $i = 2, 3, \dots, n$, are correspondingly different.

By changing the order of integration in each function from (3.7), for w_i we have that

$$w_i(t) = t^{\alpha_0} \int_t^1 t_i^{\alpha_i} \int_{t_i}^1 t_{i-1}^{\alpha_{i-1}} \dots \int_{t_2}^1 t_1^{\alpha_1} dt_1 dt_2 \dots dt_i.$$

Now the function w_i has the same form as the function u_i from (1.1) with the difference in order of powers of the functions in the integrals. Hence, it is easy to formulate lower and upper estimates for w_i and conditions of its summability to the power p on the interval $(0, 1)$.

Lemma 5. *For the functions w_i , $i = 1, 2, \dots, n$, from the system (3.7) and for all $0 < \delta < 1$ there exists $0 < \delta_1 \leq \delta$ such that for any $t \in (0, \delta_1]$ the estimate $c_i(\delta) t^{\min_{1 \leq s \leq i+1} (\alpha_0 + \sum_{j=s}^i (\alpha_j + 1))} \leq w_i(t)$ holds, where $c_i(\delta) \rightarrow 0$ when $\delta \rightarrow 1$, $i = 1, 2, \dots, n$.*

Lemma 6. *For the functions w_i , $i = 1, 2, \dots, n$, from the system (3.7) for any $t \in (0, 1]$ the following estimate*

$$w_i(t) \ll t^{\min_{1 \leq s \leq i+1} (\alpha_0 + \sum_{j=s}^i (\alpha_j + 1))} |\ln t|^{l_i}$$

holds, where l_i is the number of k , $1 \leq k \leq i_0 - 1$, such that $\sum_{j=k}^{i_0-1} (\alpha_j + 1) = 0$, when $1 < i_0$, and $l_i = 0$ when $i_0 = 1$.

Here $i_0 = \max_{k \in N_i} k$, where

$$N_i = \{k : 1 \leq k \leq i+1, \alpha_0 + \sum_{j=k}^i (\alpha_j + 1) = \min_{1 \leq s \leq i+1} (\alpha_0 + \sum_{j=s}^i (\alpha_j + 1))\}.$$

Remark 4. The assumptions of Lemma 5 and 6 hold also for the functions w_0 , since in this case $l_0 = 0$, and, thus,

$$w_0 = t^{\alpha_0} = t^{\min_{1 \leq s \leq 0+1} (\alpha_0 + \sum_{j=s}^0 (\alpha_j + 1))} |\ln t|^{l_0}.$$

Theorem 3. *The functions w_i , $i = 0, 1, \dots, n$, from the system (3.7) belong to $L_p(0, 1)$, $1 \leq p < \infty$, if and only if*

$$\min_{1 \leq s \leq i+1} (\alpha_0 + \sum_{j=s}^i (\alpha_j + 1)) > -\frac{1}{p}.$$

Remark 5. Consider the system $\{y_i(\cdot)\}_{i=0}^n$ with i :th function in the form:

$$y_i(t) = t^{\beta_0} \int_1^t t_1^{\beta_1} \int_{t_1}^t t_2^{\beta_2} \dots \int_{t_{i-1}}^t t_i^{\beta_i} dt_i dt_{i-1} \dots dt_1.$$

It is possible to formulate two main lemmas and a theorem as above also for this case. However, these results can be derived by only making a change of variables as discussed before so we leave out both the formulations and the proofs.

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L.N. Gumilyev Eurasian National University
Munaytpasov st., 5
010008 Astana
Kazakhstan
(E-mail : zamir-a-t@yandex.ru)

Kazakhstan Institute of Management, Economics and Strategic Research
Abai ave., 4
050010 Almaty
Kazakhstan
(E-mail : kalybay@kimep.kz)

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